Solutionbank S4
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

The random variable X is binomially distributed. A sample of 10 is taken, and it is desired to test $H_0 : p = 0.25$ against $H_1 : p > 0.25$, using a 5% level of significance.

a Calculate the critical region for this test.

b State the probability of a type I error for this test and, given that the true value of p was later found to be 0.30, calculate the probability of a type II error.

Solution:

$H_0 : p = 0.25 \qquad H_1 : p > 0.25$

a Seek c such that $P(X \geq c) < 0.05$ where $X \sim B(10, 0.25)$

Tables give:

$P(X \leq 5) = 0.9803$

$\therefore P(X \geq 6) = 0.0197$

$\therefore$ critical region is $X \geq 6$

b $P(\text{Type I error}) = P(X \geq 6) = 0.0197$

$P(\text{Type II error} \mid p = 0.3) = P(X \leq 5 \mid p = 0.3)$

$= 0.9527$

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Exercise A, Question 2

Question:

The random variable X is binomially distributed. A sample of 20 is taken, and it is desired to test $H_0 : p = 0.30$ against $H_1 : p < 0.30$, using a 1% level of significance.

a Calculate the critical region for this test.

b State the probability of a type I error for this test and, given that the true probability was later found to be 0.25, calculate the probability of a type II error.

Solution:

a $H_0 : p = 0.30 \quad H_1 : p < 0.30$

Seek c such that $P(X \leq c) < 0.01$ where $X \sim B(20, 0.30)$

From tables

$P(X \leq 1) = 0.0076$

and $P(X \leq 2) = 0.0355$

$\therefore$ critical region is $X \leq 1$

b

$$P(\text{Type I error}) = 0.0076$$

$$\begin{aligned} P(\text{Type II error}) &= P(X \geq 2 \mid p = 0.25) \\ &= 1 - P(X \leq 1 \mid p = 0.25) \\ &= 1 - 0.0243 \\ &= 0.9757 \end{aligned}$$

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Exercise A, Question 3

Question:

The random variable X is binomially distributed. A sample of 10 is taken, and it is desired to test $\mathrm{H}_0 : p = 0.45$ against $\mathrm{H}_1 : p \neq 0.45$, using a 5% level of significance.

a Calculate the critical region for this test.

b State the probability of a type I error for this test and, given that the true probability was later found to be 0.40, calculate the probability of a type II error.

Solution:

a $\mathrm{H}_0 : p = 0.45 \quad \mathrm{H}_1 : p \neq 0.45$

Seek c_1 and c_2 such that $\mathrm{P}(X \le c_1) < 0.025$ and $\mathrm{P}(X \ge c_2) < 0.025$ where

$X \sim \mathrm{B}(10, 0.45)$

From tables

$\mathrm{P}(X \le 1) = 0.0233$

$\mathrm{P}(X \le 7) = 0.9726 \Rightarrow \mathrm{P}(X \ge 8) = 0.0274$

$\mathrm{P}(X \le 8) = 0.9955 \Rightarrow \mathrm{P}(X \ge 9) = 0.0045$

$\therefore$ critical region is $\{X \le 1\} \cup \{X \ge 9\}$

b
$$\begin{aligned}\mathrm{P}(\text{Type I error}) &= \mathrm{P}(X \le 1) + \mathrm{P}(X \ge 9)\\ &= 0.0233 + 0.0045\\ &= 0.0278\end{aligned}$$

$$\begin{aligned}\mathrm{P}(\text{Type II error}) &= \mathrm{P}(2 \le X \le 8 \mid X \sim \mathrm{B}(10, 0.40))\\ &= \mathrm{P}(X \le 8) - \mathrm{P}(X \le 1)\\ &= 0.9983 - 0.0464\\ &= 0.9519\end{aligned}$$

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Exercise A, Question 4

Question:

The random variable X has a Poisson distribution. A sample is taken, and it is desired to test $H_0: \lambda = 6$ against $H_1: \lambda > 6$, using a 5% level of significance.

a Find the critical region for this test.

b Calculate the probability of a type I error and, given that the true value of λ was later found to be 7, calculate the probability of a type II error.

Solution:

$H_0: \lambda = 6 \qquad H_1: \lambda > 6$

a Seek c such that $P(X \geq c) < 0.05$ where $X \sim Po(6)$

From tables:

$P(X \leq 10) = 0.9574$

$\therefore P(X \geq 11) = 0.0426$

$\therefore$ critical region is $X \geq 11$

b $P(\text{Type I error}) = P(X \geq 11 \mid X \sim Po(6))$

$= 0.0426$

$P(\text{Type II error}) = P(X \leq 10 \mid \lambda = 7)$

$= 0.9015$

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Exercise A, Question 5

Question:

The random variable X has a Poisson distribution. A sample is taken, and it is desired to test $H_0: \lambda = 4.5$ against $H_1: \lambda < 4.5$, using a 5% level of significance.

a Find the critical region for this test.

b Calculate the probability of a type I error and, given that the true value of λ was later found to be 3.5, calculate the probability of a type II error.

Solution:

a $H_0: \lambda = 4.5 \quad H_1: \lambda < 4.5$

Seek c such that $P(X \le c) < 0.05$ where $X \sim Po(4.5)$

Tables give:

$P(X \le 1) = 0.0611$

$P(X = 0) = 0.0111$

$\therefore$ critical region is $X = 0$

b $P(\text{Type I error}) = 0.0111$

$$\begin{aligned} P(\text{Type II error}) &= P(X \ge 1 \mid \lambda = 3.5) \\ &= 1 - P(X = 0 \mid \lambda = 3.5) \\ &= 1 - 0.0302 \\ &= 0.9698 \end{aligned}$$

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Exercise A, Question 6

Question:

The random variable X has a Poisson distribution. A sample is taken, and it is desired to test $H_0: \lambda = 9$ against $H_1: \lambda \neq 9$, using a 5% level of significance.

a Find the critical region for this test.

b Calculate the probability of a type I error and, given that the true value of λ was later found to be 8, calculate the probability of a type II error.

Solution:

$H_0: \lambda = 9 \quad H_1: \lambda \neq 9$.

a Seek c_1 and c_2 such that $P(X \leq c_1) < 0.025$ and $P(X \geq c_2) < 0.025$

where $X \sim Po(9)$

From tables:

$P(X \leq 3) = 0.0212$

$P(X \leq 4) = 0.0550$

$P(X \leq 15) = 0.9780 \Rightarrow P(X \geq 16) = 0.0220$

$\therefore$ critical region is $\{X \leq 3\} \cup \{X \geq 16\}$

b $P(\text{Type I error}) = 0.0212 + 0.0220$

$= 0.0432$

$P(\text{Type II error}) = P(4 \leq X \leq 15 \mid \lambda = 8)$

$= P(X \leq 15) - P(X \leq 3)$

$= 0.9918 - 0.0424$

$= 0.9494$

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Exercise B, Question 1

Question:

The random variable $X \sim N(\mu, 3^2)$. A random sample of 20 observations of X is taken, and the sample mean $\bar{x}$ is taken to be the test statistic. It is desired to test $H_0: \mu = 50$ against $H_1: \mu > 50$, using a 1% level of significance.

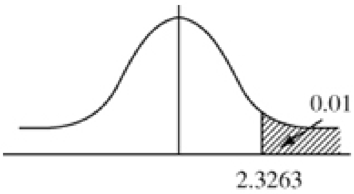
a Find the critical region for this test.
b State the probability of a type I error for this test.
Given that the true mean was later found to be 53,
c find the probability of a type II error.

Solution:

$H_0: \mu = 50 \qquad H_1: \mu > 50$

a Critical region when $Z = \dfrac{\bar{x} - 50}{\frac{3}{\sqrt{20}}} > 2.3263$

i.e. $\bar{X} > 51.5605\ldots$



b $P(\text{Type I error}) = \text{significance level} = 0.01$

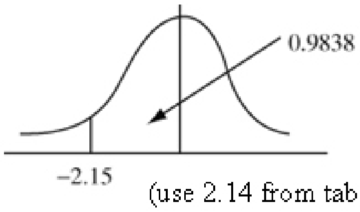
c $P(\text{Type II error}) = P(\bar{X} \le 51.5605\ldots \mid \mu = 53)$

$= P\left(Z < \dfrac{51.5605\ldots - 53}{\frac{3}{\sqrt{20}}}\right)$

$= P(Z < -2.1458\ldots)$

$= 1 - 0.9838$

$= 0.0162$



(use 2.14 from tables)

(Calculator gives $0.01594\ldots$ so accept awrt 0.016)

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Exercise B, Question 2

Question:

The random variable $X \sim N(\mu, 2^2)$. A random sample of 16 observations of X is taken, and the sample mean $\bar{x}$ is taken to be the test statistic. It is desired to test $H_0 : \mu = 30$ against $H_1 : \mu < 30$, using a 5% level of significance.

a Find the critical region for this test.

b State the probability of a type I error for this test.

Given that the true mean was later found to be 28.5,

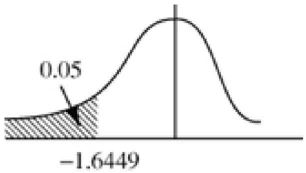
c find the probability of a type II error.

Solution:

$H_0 : \mu = 30 \qquad H_1 : \mu < 30$

a critical region when

$$Z = \frac{\bar{x} - 30}{\frac{2}{\sqrt{16}}} < -1.6449$$



i.e. $\bar{X} < 30 - 1.6449 \times \frac{1}{2} = 29.17755\ldots$

$\bar{X} < 29.178$

b $P(\text{Type I error}) = 0.05$

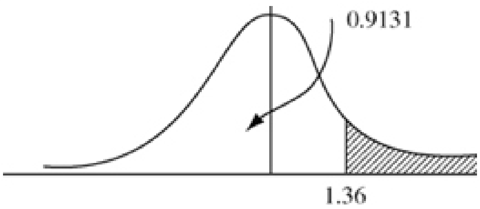
c $P(\text{Type II error}) = P(\bar{X} \geq 29.1775\ldots \mid \mu = 28.5)$

$$= P\left(Z > \frac{29.17755\ldots - 28.5}{\frac{2}{\sqrt{16}}}\right)$$

$= P(Z > 1.3551)$

$= 1 - 0.9131$

$= 0.0869$



(Calculator gives $0.08769\ldots$ so accept answer in range awrt $0.087 \sim 0.088$).

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Exercise B, Question 3

Question:

The random variable $X \sim N(\mu, 4^2)$. A random sample of 25 observations of X is taken, and the sample mean $\bar{x}$ is taken to be the test statistic. It is desired to test $H_0 : \mu = 40$ against $H_1 : \mu \neq 40$, using a 1% level of significance.

a Find the critical region for this test.
b State the probability of a type I error.
Given that the true mean was later found to be 42,
c find the probability of a type II error.

Solution:

$H_0 : \mu = 40 \qquad H_1 : \mu \neq 40$

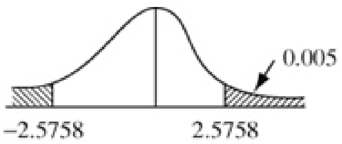
a Critical region $Z < -2.5758$ or $Z > 2.5758$

where $Z = \dfrac{\bar{x} - 40}{\frac{4}{\sqrt{25}}}$

$\therefore \bar{X} > 40 + 0.8 \times 2.5758 = 42.0606\ldots$

or $\bar{X} < 40 - 0.8 \times 2.5758 = 37.9393\ldots$

i.e. $\{\bar{X} < 37.939\} \cup \{\bar{X} > 42.061\}$



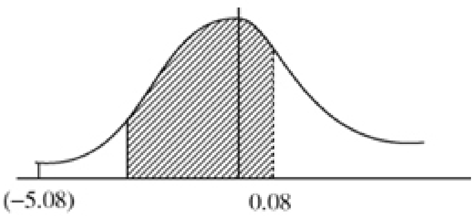
b $P(\text{Type I error}) = 0.01$

c $P(\text{Type II error}) = P(37.939 < \bar{X} < 42.061 \mid \mu = 42)$

$= P(-5.076\ldots < Z < 0.07625)$

$= 0.5319$

(Calculator gives $0.530389\ldots$ so accept awrt 0.53)



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Exercise B, Question 4

Question:

A manufacturer claims that the average outside diameter of a particular washer produced by his factory is 15 mm. The diameter is assumed to be normally distributed with a standard deviation of 1. The manufacturer decides to take a random sample of 25 washers from each day's production in order monitor any changes in the mean diameter.

a Using a significance level of 5% find the critical region to be used for this test.
Given that the average diameter had in fact increased to 15.6 mm

b find the probability that the day's production would be wrongly accepted.

Solution:

a $D \sim \mathrm{N}\left(\mu, 1^2\right)$

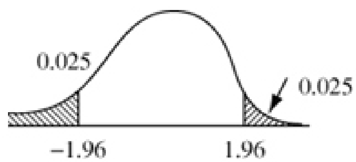
$\mathrm{H}_0 : \mu = 15$ (no change) $\mathrm{H}_1 : \mu \neq 15$ (change in D's mean)

$n = 25$

Critical region $|Z| > 1.96$

i.e. $\dfrac{\bar{X} - 15}{\frac{1}{5}} < -1.96$ or $\dfrac{\bar{X} - 15}{\frac{1}{5}} > 1.96$

i.e. $\bar{X} < 14.608$ or $\bar{X} > 15.392$



b $\mathrm{P}\left(\text{Accepting production} \mid \mu = 15.6\right)$

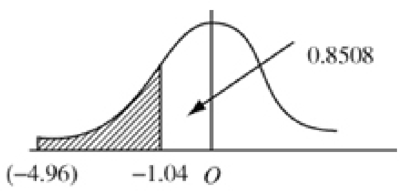
$= \mathrm{P}\left(14.608 < \bar{X} < 15.392 \mid \mu = 15.6\right)$

$= \mathrm{P}\left(-4.96 < Z < -1.04\right)$

$= 1 - 0.8508$

$= 0.1492$

(Calculator gives $0.1491699\ldots$ so accept awrt 0.1492)



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Exercise B, Question 5

Question:

The number of petrie dishes that a laboratory technician can deal with in one hour can be modelled by a normal distribution with mean 40 and standard deviation 8. A producer of glass pipettes claims that a new type of pipette will speed up the rate at which the technician works.
A random sample of 30 technicians tried out the new pipettes and the average number of petrie dishes they dealt with per hour $\bar{X}$ was recorded.

a Using a 5% significance level find the critical value of $\bar{X}$.

The average number of petrie dishes dealt with per hour using the new pipettes was in fact 42.

b Find the probability of making a type II error.

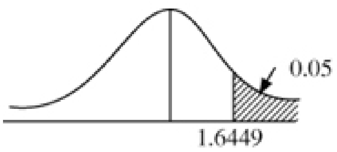
Solution:

a $X =$ number of dishes per hour $\sim \mathrm{N}(\mu, 8^2)$

$\mathrm{H}_0: \mu = 40 \qquad \mathrm{H}_1: \mu > 40$

critical region $Z = \dfrac{\bar{X} - 40}{\frac{8}{\sqrt{30}}} > 1.6449$

i.e. $\bar{X} > 42.4025\ldots$

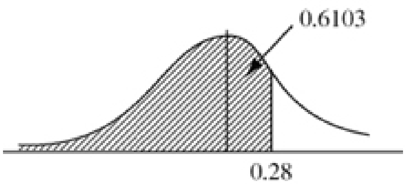


b $\mathrm{P}(\text{Type II error}) = \mathrm{P}(\bar{X} < 42.4025\ldots \mid \mu = 42)$

$= \mathrm{P}(Z < 0.2755\ldots)$

$= 0.6103$

(Calculator gives $0.60856\ldots$)
So accept awrt 0.61



c Increasing P(Type II error) will decrease P(Type I error)
Decreasing P(Type II error) will increase P(Type I error)
So only way of reducing P(Type II error) and changing significance level is to increase sample size.

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Exercise C, Question 1

Question:

The random variable $X \sim \mathrm{N}(\mu, 3^2)$. A random sample of 25 observations of X is taken and the sample mean $\bar{x}$ is taken as the test statistic. It is desired to test $\mathrm{H}_0 : \mu = 20$ against $\mathrm{H}_1 : \mu > 20$ using a 5% level of significance.

a Find the critical region for this test.

b Given that $\mu = 20.8$ find the power of this test.

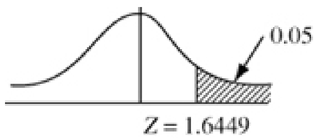
Solution:

a $\mathrm{H}_0 : \mu = 20 \qquad \mathrm{H}_1 : \mu > 20$

critical region $Z = \dfrac{\bar{x} - 20}{\frac{3}{\sqrt{25}}} > 1.6449$

$\therefore \bar{X} > 20 + 0.6 \times 1.6449$

$\bar{X} > 20.9869\ldots$



b Power $= 1 - \mathrm{P}(\text{Type II error})$

$= 1 - \mathrm{P}(\bar{X} \le 20.9869\ldots \mid \mu = 20.8)$

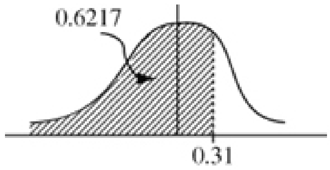
$= 1 - \mathrm{P}\left(Z \le \dfrac{20.9869\ldots - 20.8}{0.6}\right)$

$= 1 - \mathrm{P}(Z \le 0.3115\ldots)$

$= 1 - 0.6217$

$= 0.3783$

(Calculator gives $0.37768\ldots$ so accept awrt 0.378)



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Exercise C, Question 2

Question:

The random variable X is a binomial distribution. A sample of 20 is taken from it. It is desired to test $H_0 : p = 0.35$ against $H_1 : p > 0.35$ using a 5% level of significance.

a Calculate the size of this test.

b Given that $p = 0.36$ calculate the power of this test.

Solution:

a $H_0 : p = 0.35 \qquad H_1 : p > 0.35$

Seek c such that $P(X \geq c) < 0.05$ where $X \sim B(20, 0.35)$

Tables give

$P(X \leq 10) = 0.9468$

$P(X \leq 11) = 0.9804$

$\therefore P(X \geq 12) = 1 - 0.9804 = 0.0196$

$\therefore$ size of test is 0.0196

b Power $= 1 - P(\text{Type II error})$

critical region is $X \geq 12$

$\therefore \text{Power} = P(X \geq 12 \mid p = 0.36)$

$\therefore \text{Power} = 1 - P(X \leq 11 \mid p = 0.36)$

$= 1 - 0.9753$

$= 0.0247$

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Exercise C, Question 3

Question:

The random variable X has a Poisson distribution. A sample is taken and it is desired to test $H_0 : \lambda = 4.5$ against $H_1 : \lambda < 4.5$. If a 5% significance level is to be used,

a find the size of this test.

b Given that $\lambda = 4.1$ find the power of the test.

Solution:

a $H_0 : \lambda = 4.5 \quad H_1 : \lambda < 4.5$

critical region seek c such that

$P(X \le c) < 0.05$ where $X \sim Po(4.5)$

Tables give:

$P(X \le 1) = 0.0611$

$P(X = 0) = 0.0111$

$\therefore$ critical region is $X = 0$

Size is 0.0111

b $\text{Power} = P(X = 0 \mid \lambda = 4.1)$

$= e^{-4.1}$

$= 0.016572\ldots$

$= 0.0166 \quad (3\text{ s.f.})$

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Exercise C, Question 4

Question:

A manufacturer claims that a particular rivet produced in his factory has a diameter of 2 mm, and that the diameter is normally distributed with a variance of 0.004 mm^2. A random sample of 25 rivets is taken from a day's production to test whether the mean diameter had altered, up or down, from the stated figure. A 5% significance level is to be used for this test.
If the mean diameter had in fact altered to 2.02 mm, calculate the power of this test.

Solution:

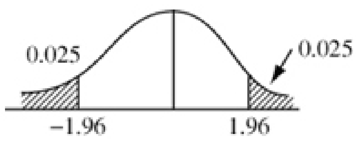
$D = \text{diameter} \sim \text{N}(\mu, 0.004)$

$\text{H}_0 : \mu = 2 \qquad \text{H}_1 : \mu \neq 2$

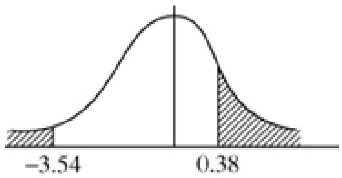
Critical region is $|Z| > 1.96$

$\therefore \dfrac{\bar{X}-2}{\sqrt{\dfrac{0.004}{25}}} > 1.96$ or $\dfrac{\bar{X}-2}{\sqrt{\dfrac{0.004}{25}}} < -1.96$

i.e. $\bar{X} < 1.9752\ldots$ or $\bar{X} > 2.0247\ldots$



$$
\begin{aligned}
\text{Power} &= \text{P}(\bar{X} < 1.9752\ldots \mid \mu = 2.02) + \text{P}(\bar{X} > 2.0247\ldots \mid \mu = 2.02) \\
&= \text{P}(Z < -3.54\ldots) + \text{P}(Z > 0.3788\ldots) \\
&= 0.0002 + (1 - 0.6480) \\
&= 0.0002 + 0.352 \\
&= 0.3522
\end{aligned}
$$



Calculator gives $0.352596\ldots$
so accept awrt $0.352 \sim 0.353$

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Exercise C, Question 5

Question:

In a binomial experiment consisting of 10 trials the random variable X represents the number of successes, and p is the probability of a success.
In a test of $H_0: p = 0.3$ against $H_1: p > 0.3$, a critical region of $X \geq 7$ is used.
Find the power of this test when
a $p = 0.4$,
b $p = 0.8$.
c Comment on your results. **[E]**

Solution:

$H_0: p = 0.3 \qquad H_1: p > 0.3$
Critical region is $X \geq 7 \quad n = 10$

a Power $= \mathrm{P}(X \geq 7 \mid p = 0.4)$
$= 1 - \mathrm{P}(X \leq 6)$
$= 1 - 0.9452$
$= 0.0548$

b Power $= \mathrm{P}(X \geq 7 \mid p = 0.8)$
Let $Y \sim \mathrm{B}(10, 0.2)$
$= \mathrm{P}(Y \leq 3)$
$= 0.8791$

c The test is more powerful for values of p further away from $p = 0.3$.

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Exercise C, Question 6

Question:

Explain briefly what you understand by

a a type I error,

b the size of a significance test.

A single observation is made on a random variable X, where $X \sim \mathrm{N}(\mu, 10)$.

The observation, x, is to be used to test $\mathrm{H}_0 : \mu = 20$ against $\mathrm{H}_1 : \mu > 20$. The critical region is chosen to be $X \geq 25$.

c Find the size of the test.

Solution:

a Type I error is when H_0 is rejected when H_0 is in fact true.

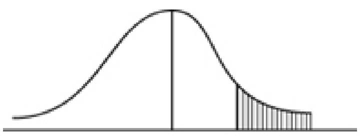
b Size $= \mathrm{P}(\text{Type I error})$

c $\mathrm{H}_0 : \mu = 20 \qquad \mathrm{H}_1 : \mu > 20$

Critical region is $X \geq 25 \qquad X \sim \mathrm{N}(\mu, 10)$

$$\begin{aligned}\text{Size} &= \mathrm{P}(\text{Reject } \mathrm{H}_0 \text{ when } \mathrm{H}_0 \text{ is true}) \\ &= \mathrm{P}(X \geq 25 \mid \mu = 20) \\ &= \mathrm{P}\left(Z > \frac{25-20}{\frac{\sqrt{10}}{\sqrt{1}}}\right) \\ &= \mathrm{P}(Z > 1.58\ldots) \\ &= 1 - 0.9429 \\ &= 0.0571\end{aligned}$$

$n = 1$ (single observation)



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Exercise D, Question 1

Question:

A single observation x is taken from a Poisson distribution with parameter λ. This observation is to be used to test $H_0: \lambda = 6.5$ against $H_1: \lambda < 6.5$. The critical region chosen was $X \leq 2$.

a Find the size of the test.

b Show that the power function of this test is given by $e^{-\lambda}\left(1+\lambda+\frac{1}{2}\lambda^2\right)$.

The table below gives the value of the power function to two decimal places.

λ	1	2	3	4	5	6
Power	0.92	s	0.42	0.24	t	0.06

c Calculate values for s and t.

d Draw a graph of the power function.

e Find the values of λ for which the test is more likely than not to come to the correct conclusion.

Solution:

$H_0: \lambda = 6.5 \qquad H_1: \lambda < 6.5$

Critical region $X \le 2$

a Size $= P(X \le 2 \mid \lambda = 6.5) = 0.0430$

b Power $= P(X \le 2 \mid \lambda)$

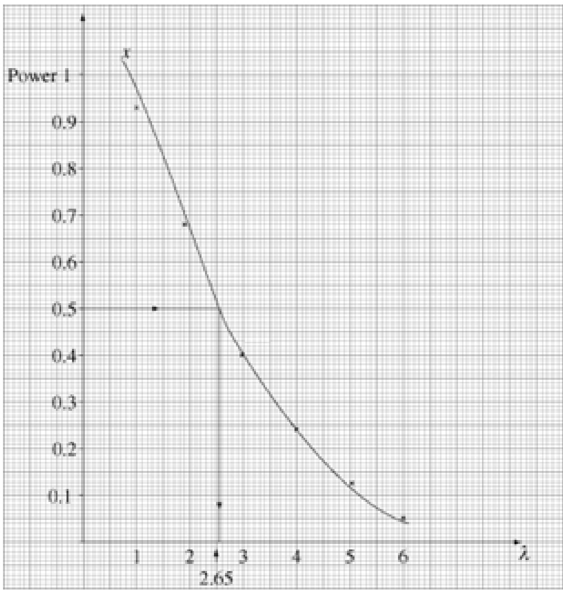
$$= e^{-\lambda} + \frac{e^{-\lambda} \cdot \lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!}$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{1}{2}\lambda^2\right)$$

c $\lambda = 2 \Rightarrow s = 0.6767$ (tables) $= 0.68$ (2 d.p.)

$\lambda = 5 \Rightarrow t = 0.1247$ (tables) $= 0.12$ (2 d.p.)

d See graph.



e Correct conclusion is arrived at when: $\lambda = 6.5$,
H_0 is accepted. So since size is 0.0430
probability of accepting $\lambda = 6.5$ is 0.957
$\therefore \lambda = 6.5$
or for $\lambda < 6.5$, correct conclusion is to reject H_0.
So require where power > 0.5 i.e. $\lambda < 2.65$
(from graph)

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Exercise D, Question 2

Question:

In a binomial experiment consisting of 12 trials X represents the number of successes and p the probability of a success.
In a test of $\text{H}_0 : p = 0.45$ against $\text{H}_1 : p < 0.45$ the null hypothesis is rejected if the number of successes is 2 or less.

a Find the size of this test.

b Show that the power function for this test is given by $(1-p)^{12} + 12p(1-p)^{11} + 66p^2(1-p)^{10}$.

c Find the power of this test when p is 0.3.

Solution:

$\text{H}_0 : p = 0.45 \quad \text{H}_1 : p < 0.45$

Critical region $X \le 2$, where $X \sim \text{B}(12, 0.45)$

a $\text{Size} = \text{P}(X \le 2)$
$= 0.0421$

b $\text{Power} = \text{P}(X \le 2 \mid X \sim \text{B}(12, p))$

$$= (1-p)^{12} + 12p(1-p)^{11} + \binom{12}{2} p^2 (1-p)^{10}$$

$$= (1-p)^{12} + 12p(1-p)^{11} + 66p^2(1-p)^{10}$$

c $p = 0.3$

$\text{Power} = 0.2528$

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Exercise D, Question 3

Question:

In a binomial experiment consisting of 10 trials the random variable X represents the number of successes and p the probability of a success.
In a test of $\mathrm{H}_0: p = 0.4$ against $\mathrm{H}_1: p > 0.4$, a critical region of $X \geq 8$ was used.
Find the power of this test when

a $p = 0.5$

b $p = 0.8$

c Comment on your results.

Solution:

$\mathrm{H}_0: p = 0.4 \quad \mathrm{H}_1: p > 0.4$

Critical region $X \geq 8$

a Power $= \mathrm{P}(X \geq 8 \mid X \sim \mathrm{B}(10, 0.5))$
$= 1 - \mathrm{P}(X \leq 7)$
$= 1 - 0.9453 = 0.0547$

b Power $= \mathrm{P}(X \geq 8 \mid X \sim \mathrm{B}(10, 0.8))$
Let $Y \sim \mathrm{B}(10, 0.2)$ then
Power $= \mathrm{P}(Y \leq 2 \mid Y \sim \mathrm{B}(10, 0.2))$
$= 0.6778$

c The test is more powerful for values of p further away from 0.4.

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Exercise D, Question 4

Question:

A certain gambler always calls heads when a coin is tossed. Before he uses a coin he tests it to see whether or not it is fair and uses the following hypotheses:

$H_0 : p = \frac{1}{2} \quad H_1 : p < \frac{1}{2}$

where p is the probability that the coin lands heads on a particular toss. Two tests are proposed.

In test A the coin is tossed 10 times and H_0 is rejected if the number of heads is 2 or fewer.

a Find the size of test A.

b Explain why the power of test A is given by

$(1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8$.

In test B the coin is first tossed 5 times. If no heads result H_0 is immediately rejected. Otherwise the coin is tossed a further 5 times and H_0 is rejected if no heads appear on this second occasion.

c Find the size of test B.

d Find an expression for the power of test B in terms of p.

The power for test A and the power for test B are given in the table for various values of p.

p	0.1	0.2	0.25	0.3	0.35	0.4
Power for test A	0.9298	0.6778		0.3828		0.1673
Power for test B	0.8323	0.5480	0.4183	0.3079	0.2186	0.1495

e Find the power for test A when p is 0.25 and 0.35.

f Giving a reason, advise the gambler about which test he should use. **[E]**

Solution:

$\text{H}_0 : p = \frac{1}{2} \quad \text{H}_1 : p < \frac{1}{2} \qquad (n = 10)$

Test A Critical region $X \leq 2$ where $X \sim \text{B}(10, p)$

a $\text{Size} = \text{P}(X \leq 2 \mid X \sim \text{B}(10, 0.5))$

$= 0.0547$

b $\text{Power} = \text{P}(X \leq 2 \mid X \sim \text{B}(10, p))$

$$= (1-p)^{10} + 10p(1-p)^9 + \binom{10}{2} p^2 (1-p)^8$$

$$= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8$$

Test B Let $Y \sim \text{B}(5, p)$.

c $\text{Size} = \text{P}(Y = 0) + [1 - \text{P}(Y = 0)]\,\text{P}(Y = 0)$ where $p = 0.5$

$= 0.0312 + [1 - 0.0312] \times 0.0312$

$= 0.06142$

NB calculator gives 0.06152

d $\text{Power} = (1-p)^5 + \left[1 - (1-p)^5\right](1-p)^5$

$$= (1-p)^5\left[2 - (1-p)^5\right]$$

e $p = 0.25 \Rightarrow \text{power}_\text{A} = 0.5256$
$p = 0.35 \Rightarrow \text{power}_\text{A} = 0.2616$ } from calculator

f Use test A as this is always more powerful

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Exercise E, Question 1

Question:

A random sample of size 3 is taken without replacement, from a population with mean μ and variance σ^2. Two unbiased estimators of the mean of the population are

$\hat{\mu}_1 = \frac{1}{3}(X_1 + X_2 + X_3)$ and $\hat{\mu}_2 = \frac{1}{4}(X_1 + 2X_2 + X_3)$.

a Calculate $\text{Var}(\hat{\mu}_1)$ and $\text{Var}(\hat{\mu}_2)$.

b Hence state, giving a reason, which estimator you would recommend. **[E]**

Solution:

a

$$\text{Var}(\hat{\mu}_1) = \frac{1}{9}\left[\text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3)\right]$$

$$= \frac{\sigma^2 + \sigma^2 + \sigma^2}{9} = \frac{\sigma^2}{3}$$

$$\text{Var}(\hat{\mu}_2) = \frac{1}{16}\left[\text{Var}(X_1) + 2^2\text{Var}(X_2) + \text{Var}(X_3)\right]$$

$$= \frac{\sigma^2 + 4\sigma^2 + \sigma^2}{16} = \frac{3\sigma^2}{8}$$

b Recommend $\hat{\mu}_1 \because \text{Var}(\hat{\mu}_1) < \text{Var}(\hat{\mu}_2)$

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Exercise E, Question 2

Question:

If X_1, X_2, X_3, is a random sample from a population with mean μ and variance σ^2, find which of the following estimators of μ are unbiased. If any are biased find an expression for the bias.

a $\frac{1}{8}X_1 + \frac{3}{8}X_2 + \frac{1}{2}X_3$

b $\frac{1}{4}X_1 + \frac{1}{2}X_2$

c $\frac{1}{3}X_1 + \frac{2}{3}X_2$

d $\frac{1}{3}(X_1 + X_2 + X_3)$

e $\frac{1}{5}X_1 + \frac{2}{5}X_2 + \frac{3}{5}X_3$

Solution:

a $\mathrm{E}\left(\frac{1}{8}X_1 + \frac{3}{8}X_2 + \frac{1}{2}X_3\right) = \frac{1}{8}\mu + \frac{3}{8}\mu + \frac{1}{2}\mu$

$= \frac{8}{8}\mu = \mu \therefore$ unbiased

b $\mathrm{E}\left(\frac{1}{4}X_1 + \frac{1}{2}X_2\right) = \frac{1}{4}\mu + \frac{1}{2}\mu = \frac{3}{4}\mu$

$\therefore$ bias $= \frac{3}{4}\mu - \mu = -\frac{1}{4}\mu$

c $\mathrm{E}\left(\frac{1}{3}X_1 + \frac{2}{3}X_2\right) = \frac{1}{3}\mu + \frac{2}{3}\mu = \mu$

$\therefore$ unbiased

d $\mathrm{E}\left\{\frac{1}{3}(X_1 + X_2 + X_3)\right\} = \frac{1}{3}(\mu + \mu + \mu) = \mu$

$\therefore$ unbiased

e $\mathrm{E}\left(\frac{1}{5}X_1 + \frac{2}{5}X_2 + \frac{3}{5}X_3\right) = \frac{1}{5}\mu + \frac{2}{5}\mu + \frac{3}{5}\mu$

$= \frac{6}{5}\mu$

$\therefore$ bias $= \frac{1}{5}\mu$

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Exercise E, Question 3

Question:

Find which one of the estimators in question 2 is the best.

Solution:

a $\text{Var}\left(\frac{1}{8}X_1+\frac{3}{8}X_2+\frac{1}{2}X_3\right)$

$=\frac{1}{64}\text{Var}(X_1)+\frac{9}{64}\text{Var}(X_2)+\frac{1}{4}\text{Var}(X_3)$

$=\frac{26}{64}\sigma^2$ or $\frac{13}{32}\sigma^2$ $(=0.40625\sigma^2)$

b estimator is biased so would not prefer

c $\text{Var}\left(\frac{1}{3}X_1+\frac{2}{3}X_2\right)=\frac{1}{9}\sigma^2+\frac{4}{9}\sigma^2=\frac{5}{9}\sigma^2$ or $0.555\sigma^2$

d $\text{Var}\left(\frac{1}{3}[X_1+X_2+X_3]\right)=\frac{1}{9}(\sigma^2+\sigma^2+\sigma^2)=\frac{3}{9}\sigma^2$ or $\frac{1}{3}\sigma^2$ or $0.333\sigma^2$

e estimator is biased

Best estimator is unbiased with smallest variance.

Since $\frac{1}{3}\sigma^2<\frac{13}{32}\sigma^2<\frac{5}{9}\sigma^2$

$\therefore$ Choose $\frac{1}{3}(X_1+X_2+X_3)$

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Exercise E, Question 4

Question:

A uniform distribution on the interval $[0, a]$ has a mean of $\frac{a}{2}$, and a variance of $\frac{a^2}{12}$.
Three single samples X_1, X_2 and X_3 are taken from this distribution, and are to be used to estimate a. The following estimators are proposed.

i $X_1 + X_2 + X_3$

ii $\frac{2}{3}(X_1 + X_2 + X_3)$

iii $2(X_1 + 2X_2 + X_3)$

a Determine the bias, if any of each of these estimators.
b Find the variance of each of these estimators.
c State, giving reasons, which of these estimators you would use.
d If $x_1 = 2, x_2 = 2.5$ and $x_3 = 3.2$, calculate the best estimate of a.

Solution:

a i $\mathrm{E}\left(X_1+X_2+X_3\right)=\frac{a}{2}+\frac{a}{2}+\frac{a}{2}=\frac{3a}{2} \therefore \text{ bias}=\frac{a}{2}$

ii $\mathrm{E}\left(\frac{2}{3}[X_1+X_2+X_3]\right)=\frac{2}{3}\left[\frac{3a}{2}\right]=a \therefore \text{ unbiased}$

iii $\mathrm{E}\left(2[X_1+2X_2+X_3]\right)=2\left[\frac{a}{2}+2\frac{a}{2}+\frac{a}{2}\right]=4a \therefore \text{ bias}=3a$

b i $\mathrm{Var}\left(X_1+X_2+X_3\right)=\frac{a^2}{12}+\frac{a^2}{12}+\frac{a^2}{12}=\frac{a^2}{4}$

ii $\mathrm{Var}\left(\frac{2}{3}[X_1+X_2+X_3]\right)=\frac{4}{9}\left[\frac{a^2}{12}+\frac{a^2}{12}+\frac{a^2}{12}\right]=\frac{a^2}{9}$

iii $\mathrm{Var}\left[2(X_1+2X_2+X_3)\right]=4\left[\frac{a^2}{12}+4\frac{a^2}{12}+\frac{a^2}{12}\right]=2a^2$

c Use $\frac{2}{3}(X_1+X_2+X_3)$ since it is unbiased (and has the smallest variance)

d $x_1 = 2, x_2 = 2.5, x_3 = 3.2$

$$\Rightarrow \frac{2}{3}(x_1+x_2+x_3) = \frac{2}{3}(2+2.5+3.2) = \frac{2}{3}(7.7)$$

$$\hat{a} = 5.13$$

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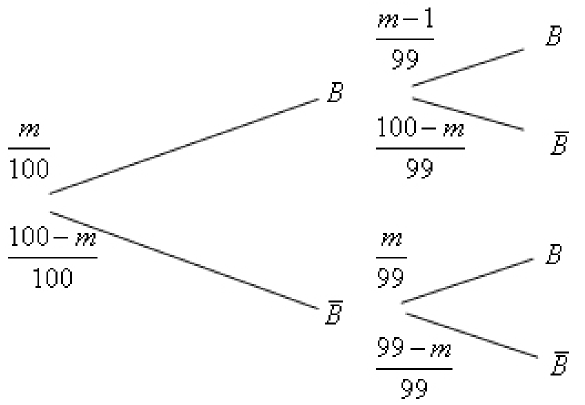
Exercise E, Question 5

Question:

A bag contains 100 counters of which an unknown number m are blue. It is known that $2 \leq m \leq 98$. Two discs are drawn simultaneously from the bag and the number n of blue ones counted. It is desired to estimate m by $\hat{m} = cn$ where c is an unknown constant. Find the value of c given that the estimate is unbiased.

Solution:



$X =$ number of blue ones chosen

n	0	1	2
$\mathrm{P}(X = n)$	$\frac{(100-m)(99-m)}{100 \times 99}$	$\frac{200m - 2m^2}{100 \times 99}$	$\frac{m(m-1)}{100 \times 99}$

$$\therefore \mathrm{E}(X) = \frac{200m - 2m^2 + 2m^2 - 2m}{100 \times 99}$$

$$= \frac{198m}{100 \times 99}$$

$$= \frac{m}{50}$$

Using $\hat{m} = cX$

$$\mathrm{E}(\hat{m}) = c\mathrm{E}(X) = c \times \frac{m}{50}$$

$\therefore$ for $\hat{m}$ to be unbiased you need $c = 50$

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Exercise E, Question 6

Question:

A sample of size n is taken from a population with a mean of μ and variance of σ^2.

a Show that the sample mean $\bar{X}$ is an unbiased estimator of μ.

b Show that as n increases $\text{Var}(\bar{X})$ decreases.

c Show that $S^2 = \dfrac{\sum X_i^2 - n\bar{X}^2}{n-1}$ is an unbiased estimator of σ^2, but that $T = \dfrac{\sum X_i^2 - n\bar{X}^2}{n}$ is a biased estimator of σ^2.

Solution:

a $\text{E}(\bar{X}) = \frac{1}{n}\text{E}(X_1 + X_2 + \cdots + X_n) = \frac{1}{n}(\mu + \mu + \cdots + \mu) = \frac{n\mu}{n} = \mu$

$\therefore \bar{X}$ is unbiased estimator of μ.

b $\text{Var}(\bar{X}) = \frac{1}{n^2}\text{Var}(X_1 + \cdots + X_n) = \frac{1}{n^2}(\sigma^2 + \cdots + \sigma^2) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$\therefore$ as $n \to \infty$ $\text{Var}(\bar{X}) \to 0$.

c $\text{E}(S^2) = \frac{1}{n-1}\left\{\text{E}(X_1^2 + X_2^2 + \cdots + X_n^2) - n\text{E}(\bar{X}^2)\right\}$

NB $\sigma^2 = \text{E}(X^2) - \mu^2 \quad \therefore \text{E}(X^2) = \mu^2 + \sigma^2$

$\frac{\sigma^2}{n} = \text{E}(\bar{X}^2) - \mu^2 \quad \therefore \text{E}(\bar{X}^2) = \mu^2 + \frac{\sigma^2}{n}$

So $\text{E}(S^2) = \frac{1}{n-1}\left\{n\left[\mu^2 + \sigma^2\right] - n\left[\mu^2 + \frac{\sigma^2}{n}\right]\right\}$

$= \frac{1}{n-1}\left\{n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right\}$

$= \frac{(n-1)}{n-1}\sigma^2 = \sigma^2$

$\text{E}(T) = \frac{1}{n} \times (n-1)\sigma^2 \neq \sigma^2$

$\therefore T$ is not unbiased for σ^2.

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Exercise E, Question 7

Question:

A six-sided die has some of its faces showing the number 0 and the rest showing the number 1 so that p is the probability of getting a 1 when the die is thrown and q is the probability of getting a 0. If the random variable X is the value showing when the die is rolled,

a find E(X) and Var(X).

A random sample is now taken by rolling the die three times in order to get an estimate for p.

b Show that if $a_1X_1 + a_2X_2 + a_3X_3$ is to be an unbiased estimator of p then $a_1 + a_2 + a_3 = 1$.

c Find the variance of this estimator.

The following estimators of p are proposed.

i $\frac{1}{5}X_1 + \frac{2}{5}X_2 + \frac{2}{5}X_3$

ii $\frac{1}{4}X_1 + \frac{3}{8}X_2 + \frac{1}{4}X_3$

iii $\frac{4}{9}X_1 + \frac{5}{9}X_3$

d Find which of these is the best unbiased estimator.

Solution:

x	0	1
$P(X=x)$	q	p

$p+q=1$

a $E(X)=p$

$\text{Var}(X)=0+p-p^2=p(1-p)$ or pq

b $E(a_1X_1+a_2X_2+a_3X_3)=a_1p+a_2p+a_3p$

$\therefore$ if unbiased $a_1+a_2+a_3=1$

c $\text{Var}(a_1X_1+a_2X_2+a_3X_3)=a_1^2pq+a_2^2pq+a_3^2pq$

$=pq\left(a_1^2+a_2^2+a_3^2\right)$

d i $E\left(\frac{1}{5}X_1+\frac{2}{5}X_2+\frac{2}{5}X_3\right)=\frac{p+2p+2p}{5}=p\ \therefore$ unbiased

$\text{Var}\left(\frac{1}{5}X_1+\frac{2}{5}X_2+\frac{2}{5}X_3\right)=\frac{pq}{25}(1+4+4)=\frac{9pq}{25}$

ii $\frac{1}{4}+\frac{3}{8}+\frac{1}{4}=\frac{7}{8}\neq 1\ \therefore$ not unbiased

iii $\frac{4}{9}+\frac{5}{9}=\frac{9}{9}=1\ \therefore$ unbiased

$\text{Var}\left(\frac{4}{9}X_1+\frac{5}{9}X_3\right)=\frac{pq}{81}(16+25)=\frac{41pq}{81}$

$\therefore$ Best estimator is $\frac{1}{5}X_1+\frac{2}{5}X_2+\frac{2}{5}X_3$ since it is unbiased and has smaller variance, $\left(\frac{9}{25}<\frac{41}{81}\right)$

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Exercise F, Question 1

Question:

A biased die has probability of a six equal to p. The die is rolled n times and the number of sixes recorded. The die is then rolled a further n times and the number of sixes recorded. The proportion of the $2n$ rolls that were sixes is called R.

a Show that R is a consistent estimator of p.

The die is rolled a total of 50 times and 18 sixes are recorded.

b Find an estimate of p.

Solution:

$X =$ number of sixes in n rolls $\quad X \sim \text{B}(n, p)$

$R = \dfrac{X_1 + X_2}{2n} \qquad \text{E}(X) = np, \text{var}(X) = np(1-p)$

a $\text{E}(R) = \dfrac{1}{2n}\text{E}(X_1 + X_2)$

$= \dfrac{1}{2n}[np + np] = \dfrac{2np}{2n} = p$

$\therefore$ R is an unbiased estimator of p.

$\text{Var}(R) = \dfrac{1}{4n^2}\text{Var}(X_1 + X_2)$

$= \dfrac{1}{4n^2}[np(1-p) + np(1-p)]$

$= \dfrac{2np(1-p)}{4n^2}$

$= \dfrac{p(1-p)}{2n}$

$\therefore$ as $n \to \infty \quad \text{Var}(R) \to 0$

$\therefore$ R is a consistent estimator for p.

b $\hat{p} = \dfrac{18}{50}$ or $\dfrac{9}{25}$

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Exercise F, Question 2

Question:

The continuous random variable $X \sim \text{U}[0, a]$.

a Show that $2\bar{X}$ is an unbiased estimator of a.

b Determine whether or not $2\bar{X}$ is a consistent estimator of a.

Solution:

$X \sim \text{U}[0, a]$

$\text{E}(X) = \mu = \frac{a}{2} \quad \text{Var}(X) = \sigma^2 = \frac{a^2}{12}$

a $\text{E}(2\bar{X}) = 2\text{E}(\bar{X}) = 2\mu = 2 \times \frac{a}{2} = a$

$\therefore 2\bar{X}$ is an unbiased estimator of a.

b $\text{Var}(2\bar{X}) = 4\text{Var}(\bar{X}) = 4\frac{\sigma^2}{n}$

$= \frac{4a^2}{12n} = \frac{a^2}{3n}$

$\therefore$ as $n \to \infty \quad \text{Var}(2\bar{X}) \to 0$

$\therefore 2\bar{X}$ is a consistent estimator of a

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Exercise F, Question 3

Question:

Using the information and results from Example 16 show that M is a consistent estimator of a.

Solution:

From Example 16 $M = \max\{X_1 \ldots X_n\}$

$$\mathrm{E}(M) = \frac{n}{n+1}a = \left(\frac{n+1}{n+1}\right)a - \left(\frac{1}{n+1}\right)a$$

$$= a - \left(\frac{1}{n+1}\right)a$$

as $n \to \infty$ $\mathrm{E}(M) \to a$

$\therefore M$ is an asymptotically unbiased estimator of a.

$$\mathrm{Var}(M) = \frac{na^2}{(n+2)(n+1)^2}$$

as $n \to \infty$ $\mathrm{Var}(M) \to 0$

$\therefore M$ is a consistent estimator of a.

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Exercise F, Question 4

Question:

If a random sample $X_1, X_2, X_3, \ldots, X_n$, is taken from a population with mean μ and standard deviation σ, show that both,

$\frac{1}{n}(X_1 + X_2 + \ldots + X_{n-1} + X_n)$, and

$2\frac{(nX_1 + (n-1)X_2 + \ldots + 2X_{n-1} + 1X_n)}{n(n+1)}$

are unbiased and consistent estimators for μ.

You may use $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

Solution:

a $\frac{1}{n}(X_1 + X_2 + \cdots + X_n) = \bar{X}$

$\text{E}(\bar{X}) = \mu$ and $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

$\therefore \bar{X}$ is an unbiased estimator of μ and $\because \text{Var}(\bar{X}) \to 0$ an $n \to \infty, \bar{X}$ is a consistent estimator of μ

b Let $Y = \frac{2(nX_1 + (n-1)X_2 + \cdots + 1X_n)}{n(n+1)}$

$$\text{E}(Y) = \frac{2}{n(n+1)}\left[n\text{E}(X_1) + (n-1)\text{E}(X_2) + \cdots + \text{E}(X_n)\right]$$

$$= \frac{2\mu}{n(n+1)}\left[n + (n-1) + \cdots + 1\right]$$

But $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$$\therefore \text{E}(Y) = \frac{2\mu}{n(n+1)} \times \frac{n(n+1)}{2} = \mu$$

$\therefore Y$ is an unbiased estimator of μ

$$\text{Var}(Y) = \frac{4}{n^2(n+1)^2}\left[n^2\sigma^2 + (n-1)^2\sigma^2 + \cdots + 1^2\sigma^2\right]$$

$$= \frac{4\sigma^2}{n^2(n+1)^2} \times \frac{n}{6}(n+1)(2n+1) \qquad \because \sum_{1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{4\sigma^2(2n+1)}{6n(n+1)}$$

As $n \to \infty$ $\text{Var}(Y) \to 0$ Y is in consistent estimator of μ.

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Exercise F, Question 5

Question:

The random variable $X \sim U[0, a]$.

a Show that $E(X^n) = \frac{a^n}{n+1}$

A random sample of 3 readings is taken from X and the statistic $S = X_1^2 + X_2^2 + X_3^2$ is calculated.

b Show that S is an unbiased estimator of a^2.

c Show that $\text{Var}(X^2) = \frac{4}{45}a^4$

A random sample of size n is taken of X.

d Show that $T = \frac{3}{n}(X_1^2 + X_2^2 + \ldots + X_n^2)$ is a consistent estimator of a^2.

Solution:

$X \sim \text{U}[0, a]$

a $\text{E}(X^n) = \int_0^a x^n \times \frac{1}{a}\,\text{d}x = \left[\frac{x^{n+1}}{a(n+1)}\right]_0^a = \left(\frac{a^{n+1}}{a(n+1)}\right) - (0)$

$= \frac{a^n}{n+1}$

b $S = X_1^2 + X_2^2 + X_3^2$

$\text{E}(X^2) = \frac{a^2}{3}$ (by **a**)

$\therefore \text{E}(S) = \frac{a^2}{3} + \frac{a^2}{3} + \frac{a^2}{3} = a^2$

$\therefore S$ is an unbiased estimator of a^2

c $\text{Var}(X^2) = \text{E}(X^4) - [\text{E}(X^2)]^2 = \frac{a^4}{5} - \left[\frac{a^2}{3}\right]^2$

$= \frac{9a^4 - 5a^4}{45} = \frac{4a^4}{45}$

d $\text{E}(T) = \frac{3}{n}\text{E}[X_1^2 + \cdots + X_n^2] = \frac{3}{n}\left[\frac{a^2}{3} + \frac{a^2}{3} + \cdots + \frac{a^2}{3}\right]$

$= \frac{3}{n} \times \frac{na^2}{3} = a^2$

$\therefore T$ is an unbiased estimator of a^2

$\text{Var}(T) = \frac{9}{n^2}\left[\text{Var}(X_1^2) + \text{Var}(X_2^2) + \cdots + \text{Var}(X_n^2)\right]$

$= \frac{9}{n^2}\left[\frac{4a^4}{45} \times n\right] = \frac{4a^4}{5n}$

$\therefore$ as $n \to \infty$ $\text{Var}(T) \to 0$

$\therefore T$ is a consistent estimator for a^2.

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Edexcel AS and A Level Modular Mathematics

Exercise F, Question 6

Question:

When a die is rolled the probability of obtaining a six is an unknown constant p. In order to estimate p the die is rolled n times and the number, X, of sixes is recorded. A second trial is then done with the die being rolled the same number of times, and the number of sixes, Y, is again recorded. Show that

a $\hat{p}_1 = \frac{3\bar{X} + 4\bar{Y}}{7n}$, and $\hat{p}_2 = \frac{\bar{X} + \bar{Y}}{2n}$, are unbiased and consistent estimators of p.

b State, giving reasons, which of the two estimators is the better one.

Solution:

$X \sim \mathrm{B}(n, p) \quad Y \sim \mathrm{B}(n, p)$

$\mathrm{E}(X) = \mu = np \quad \mathrm{Var}(X) = \sigma^2 = np(1-p)$

a $\mathrm{E}(\hat{p}_1) = \frac{3np + 4np}{7n} = \frac{7np}{7n} = p$

$$\mathrm{Var}(\hat{p}_1) = \frac{9\frac{np(1-p)}{n} + 16\frac{np(1-p)}{n}}{49n^2} = \frac{25\frac{np(1-p)}{n}}{49n^2} = \frac{25p(1-p)}{49n^2}$$

$\therefore \hat{p}_1$ is unbiased and $\mathrm{Var}(\hat{p}_1) \to 0$ an $n \to \infty \therefore \hat{p}_1$ is consistent for p

$\mathrm{E}(\hat{p}_2) = \frac{np + np}{2n} = \frac{2np}{2n} = p$

$$\mathrm{Var}(\hat{p}_2) = \frac{\frac{np(1-p)}{n} + \frac{np(1-p)}{n}}{4n^2} = \frac{p(1-p)}{2n^2}$$

$\therefore \hat{p}_2$ is unbiased and $\mathrm{Var}(\hat{p}_2) \to 0$ as $n \to \infty \therefore \hat{p}_2$ is consistent for p.

b $\because \frac{25}{49} > \frac{1}{2}$

$\therefore$ Choose $\frac{\bar{X} + \bar{Y}}{2n}$ since it has smaller variance.

Solutionbank S4

Edexcel AS and A Level Modular Mathematics

Exercise G, Question 1

Question:

The random variable X is binomially distributed. A sample of 15 observations is taken and it is desired to test $H_0 : p = 0.35$ against $H_1 : p > 0.35$ using a 5% significance level.

a Find the critical region for this test.

b State the probability of making a type I error for this test.

The true value of p was found later to be 0.5.

c Calculate the power of this test.

Solution:

$H_0 : p = 0.35 \quad H_1 : p > 0.35 \quad X \sim B(15, p)$

a Seek c such that $P(X \geq c) < 0.05$

Tables give: $P(X \leq 8) = 0.9578$

$\therefore P(X \geq 9) = 0.0422$

So critical region is $X \geq 9$

b $P(\text{Type I error}) = 0.0422$

c $\text{Power} = P(X \geq 9 \mid p = 0.5)$

$= 1 - P(X \leq 8 \mid p = 0.5)$

$= 1 - 0.6964$

$= 0.3036$

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Exercise G, Question 2

Question:

The random variable X has a Poisson distribution. A sample is taken and it is desired to test $H_0: \lambda = 3.5$ against $H_1: \lambda < 3.5$ using a 5% significance level.

a Find the critical region for this test.

b State the probability of committing a type I error for this test.

Given that the true value of λ is 3.0,

c find the power of this test.

Solution:

$H_0: \lambda = 3.5 \quad H_1: \lambda < 3.5$

a Seek c: $P(X \le c) < 0.05$ where $X \sim Po(\lambda)\ (\lambda = 3.5)$

Tables: $P(X \le 1) = 0.1359 > 0.05$

$P(X \le 0) = 0.0302 < 0.05$

$\therefore$ critical region $X = 0$

b $P(\text{Type I error}) = 0.0302$

c $\text{Power} = P(X = 0 \mid \lambda = 3.0)$

$= 0.0498$

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Exercise G, Question 3

Question:

The random variable $X \sim N(\mu, 9)$. A random sample of 18 observations is taken, and it is desired to test $H_0: \mu = 8$ against $H_1: \mu \neq 8$, at the 5% significance level. The test statistic to be used is $Z = \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$.

a Find the critical region for this test.
b State the probability of a type I error for this test.
Given that μ was later found to be 7,
c find the probability of making a type II error.
d State the power of this test.

Solution:

$X \sim N(\mu, 3^2)$ $\quad n = 18$

$H_0: \mu = 8 \quad H_1: \mu \neq 8$

a critical region when $|Z| > 1.96$

i.e. $\dfrac{\bar{X} - 8}{\frac{3}{\sqrt{18}}} < -1.96$ or $\dfrac{\bar{X} - 8}{\frac{3}{\sqrt{18}}} > 1.96$

$\Rightarrow \bar{X} < 6.614\ldots$ or $\bar{X} > 9.3859\ldots$

b $P(\text{Type I error}) = \text{significance level} = 0.05$

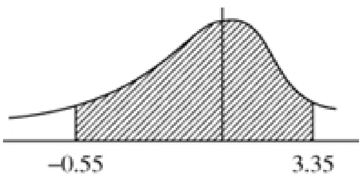
c $P(\text{Type II error}) = P(6.614\ldots < \bar{X} < 9.3859\ldots \mid \mu = 7)$

$$= P\left(\frac{6.614\ldots - 7}{\frac{3}{\sqrt{18}}} < Z < \frac{9.3859\ldots - 7}{\frac{3}{\sqrt{18}}}\right)$$

$$= P(-0.545\ldots < Z < 3.374\ldots)$$

$$= 0.9996 - (1 - 0.7088)$$

$$= 0.7084$$



Calculator gives 0.707023 so accept awrt $0.707 \sim 0.708$

d $\text{Power} = 1 - P(\text{Type II error})$

$= 0.293 \sim 0.292$

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Exercise G, Question 4

Question:

A single observation, x, is taken from a Poisson distribution with parameter λ. The observation is used to test $H_0 : \lambda = 4.5$ against $H_1 : \lambda > 4.5$. The critical region chosen for this test was $x \geq 8$.

a Find the size of this test.

b The table below gives the power of the test for different values of λ.

λ	1	2	3	4	5	6	7	8	9	10
Power	0	0.0011	0.0019	r	0.1334	s	0.4013	0.5470	t	0.7798

i Find values for r, s and t.

ii Using graph paper, plot the power function against λ.

Solution:

$H_0 : \lambda = 4.5 \quad H_1 : \lambda > 4.5$

Critical region $X \geq 8$

a Size $= P(X \geq 8 \mid \lambda = 4.5)$

$= 1 - P(X \leq 7 \mid \lambda = 4.5) = 1 - 0.9134$

$= 0.0866$

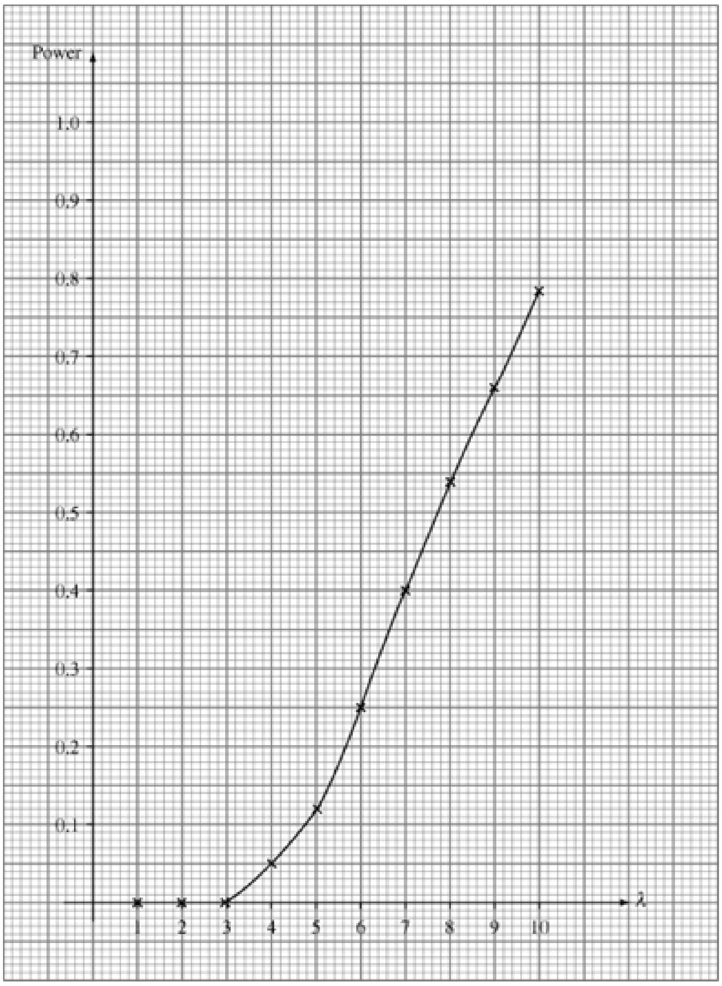
b i Power $= P(X \geq 8 \mid \lambda)$

$\therefore r = 1 - 0.9489 = 0.0511$

$s = 1 - 0.7440 = 0.2560$

$t = 1 - 0.3239 = 0.6761$

ii See graph.



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Exercise G, Question 5

Question:

In a binomial experiment consisting of 15 trials X represents the number of successes and p the probability of success.

In a test of $H_0 : p = 0.45$ against $H_1 : p < 0.45$ the critical region for the test was $X \leq 3$.

a Find the size of the test.

b Use the table of the binomial cumulative distribution function to complete the table given below.

p	0.1	0.2	0.3	0.4	0.5
Power	0.944	s	0.2969	t	0.0176

c Draw the graph of the power function for this test

Solution:

$H_0: p = 0.45 \quad H_1: p < 0.45$

Critical region $X \le 3$

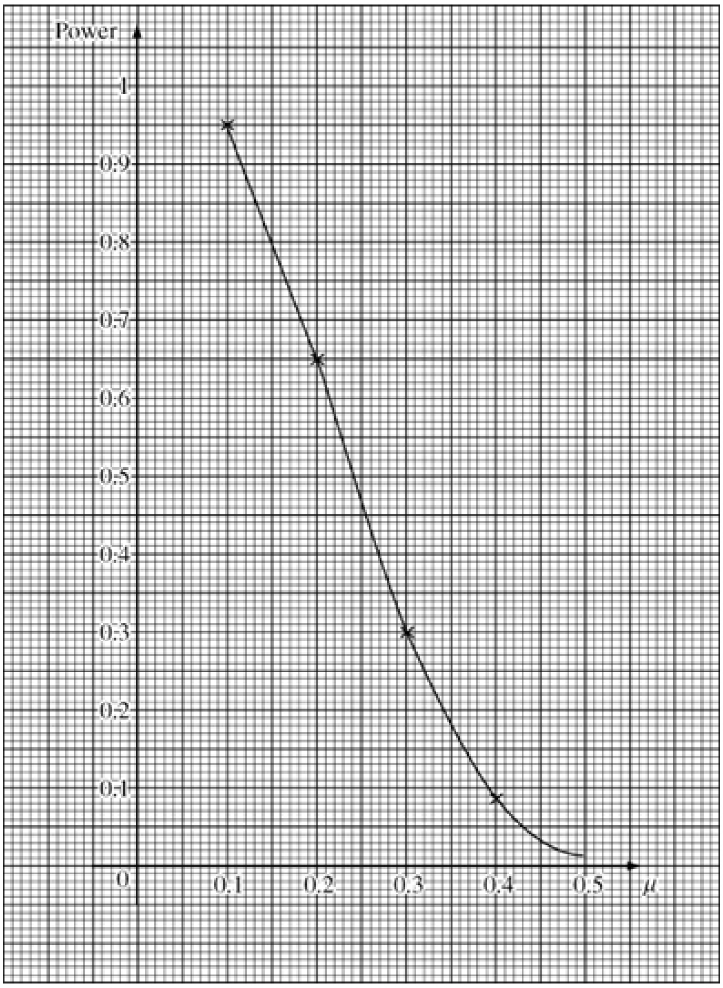
a Size $= P(X \le 3 \mid X \sim B(15, 0.45)) = 0.0424$

b Power $= P(X \le 3 \mid X \sim B(15, p))$

$p = 0.2 \Rightarrow s = 0.6482$

$p = 0.4 \Rightarrow t = 0.0905$

c See Graph



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Exercise G, Question 6

Question:

A bag contains 25 balls of which an unknown number, m, are coloured red $(3 < m \leq 22)$. Two of the balls are drawn from the bag and the number of red balls, X, is noted. It is desired to estimate m by $\hat{m} = cX$.

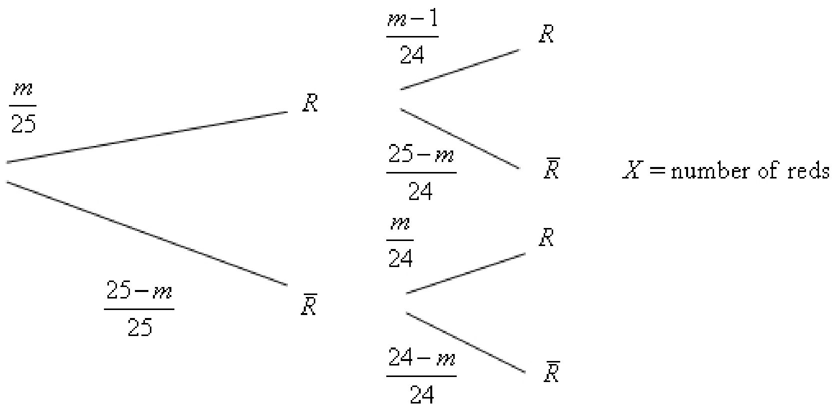
a Calculate a value for c if the estimate is to be unbiased.

The balls are replaced and a second draw is made and the number of red balls, Y, is noted.

b Write down $\mathrm{E}(Y)$.

c Show that $Z = (5X + 7.5Y)$ is an unbiased estimator of m.

Solution:



x	0	1	2
$\mathrm{P}(X = x)$	$\frac{(25-m)(24-m)}{25 \times 24}$	$\frac{m(25-m)+(25-m)m}{25 \times 24}$	$\frac{m(m-1)}{25 \times 24}$

$$\mathrm{E}(X) = \frac{50m - 2m^2}{25 \times 24} + \frac{2m^2 - 2m}{25 \times 24} = \frac{48m}{25 \times 24} = \frac{2m}{25}$$

a $\mathrm{E}(\hat{m}) = c\mathrm{E}(X) = c \times \frac{2m}{25}$

$\therefore$ for $\hat{m}$ to be unbiased for m, we need $c = \frac{25}{2}$

b $\mathrm{E}(Y) = \mathrm{E}(X) = \frac{2m}{25}$

c $\mathrm{E}(Z) = 5\mathrm{E}(X) + 7.5\,\mathrm{E}(Y)$

$$= 5 \times \frac{2m}{25} + \frac{7.5 \times 2m}{25}$$

$$= \frac{25m}{25} = m$$

$\therefore Z$ is an unbiased estimater of m.

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Exercise G, Question 7

Question:

A bag contains 25 balls of which an unknown number, m, are green, $(4 < m \leq 21)$. Three balls are drawn from the bag and the number, X, of green balls is recorded. The balls are replaced and four balls are drawn with the number, Y, of green balls noted. Three estimators of p, the probability of getting a green ball, are proposed

i $\dfrac{X+Y}{7}$

ii $\dfrac{3X+4Y}{25}$

iii $\dfrac{4X+3Y}{24}$.

a Show that all three are unbiased estimators of p.

b Find which is the best estimator.

Solution:

If 3 balls are selected $\mathrm{E}(X) = \frac{3m}{25}$ (compare with np for a binomial)

If 4 balls are selected $\mathrm{E}(Y) = \frac{4m}{25}$

a i $\mathrm{E}\left(\frac{X+Y}{7}\right) = \frac{\mathrm{E}(X)+\mathrm{E}(Y)}{7} = \frac{\frac{3m}{25}+\frac{4m}{25}}{7} = \frac{m}{25} = p$

ii $\mathrm{E}\left(\frac{3X+4Y}{25}\right) = \frac{\frac{9m}{25}+\frac{16m}{25}}{25} = \frac{m}{25} = p$

iii $\mathrm{E}\left(\frac{4X+3Y}{24}\right) = \frac{\frac{12m}{25}+\frac{12m}{25}}{24} = \frac{\frac{24m}{25}}{24} = \frac{m}{25} = p$

$\therefore$ all 3 are unbiased estimators of p

Similarly $\mathrm{Var}(X) = \frac{3m}{25}\frac{(25-m)}{25} = 3p(1-p)$

$\mathrm{Var}(Y) = \frac{4m}{25}\left(\frac{25-m}{25}\right) = 4p(1-p)$

b $\mathrm{Var}\left(\frac{X+Y}{7}\right) = \frac{\mathrm{Var}(X)+\mathrm{Var}(Y)}{49} = \frac{p(1-p)}{7} = 0.142p(1-p)$

$\mathrm{Var}\left(\frac{3X+4Y}{25}\right) = \frac{9\mathrm{Var}(X)+16\mathrm{Var}(Y)}{25^2} = \frac{(27+64)}{625}p(1-p) = \frac{91}{625}p(1-p)$

$= 0.1456p(1-p)$

$\mathrm{Var}\left(\frac{4X+3Y}{24}\right) = \frac{16\mathrm{Var}(X)+9\mathrm{Var}(Y)}{24^2} = \frac{48+36}{576}p(1-p) = \frac{84}{576}p(1-p)$

$= 0.1458p(1-p)$

$\because \frac{1}{7} < \frac{91}{625} < \frac{84}{576}$

$\therefore$ Choose $\frac{X+Y}{7}$ $\because$ variance is smallest.

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Exercise G, Question 8

Question:

A company buys rope from Bindings Ltd and it is known that the number of faults per 100 m of their rope follows a Poisson distribution with mean 2. The company is offered 100 m of rope by Tieup, a newly established rope manufacturer. The company is concerned that the rope from Tieup might be of poor quality.

a Write down the null and alternative hypotheses appropriate for testing that rope from Tieup is in fact as reliable as that from Bindings Ltd.

b Derive a critical region to test your null hypothesis with a size of approximately 0.05.

c Calculate the power of this test if rope from Tieup contains an average of 4 faults per 100 m. **[E]**

Solution:

a $H_0: \lambda = 2$ $H_1: \lambda > 2$
(Quality the same) (Quality is poorer)

b Seek c such that $P(X \geq c) \approx 0.05$ where $X \sim Po(2)$

Tables $P(X \leq 4) = 0.9473 \Rightarrow P(X \geq 5) = 0.0527$

$P(X \leq 5) = 0.9834 \Rightarrow P(X \geq 6) = 0.0166$

Nearest to 0.05 is $X \geq 5$

critical region is $X \geq 5$

c Power $= P(X \geq 5 | \lambda = 4)$

$= 1 - P(X \leq 4 | \lambda = 4)$

$= 1 - 0.6288$

$= 0.3712$

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Exercise G, Question 9

Question:

The number of faulty garments produced per day by machinists in a clothing factory has a Poisson distribution with mean 2. A new machinist is trained and the number of faulty garments made in one day by the new machinist is counted.

a Write down the appropriate null and alternative hypotheses involved in testing the theory that the new machinist is at least as reliable as the other machinists.

b Derive a critical region, of size approximately 0.05, to test the null hypothesis.

c Calculate the power of this test if the new machinist produces an average of 3 faulty garments per day.

The number of faulty garments produced by the new machinist over three randomly selected days is counted.

d Derive a critical region, of approximately the same size as in part **b**, to test the null hypothesis.

e Calculate the power of this test if the machinist produces an average of 3 faulty garments per day.

f Comment briefly on the difference between the two tests. **[E]**

Solution:

a $H_0 : \lambda = 2$ $H_1 : \lambda > 2$
(as good) (worse)

b Seek c such that $P(X \geq c) \approx 0.05$ where $X \sim Po(2)$

Tables $P(X \leq 4) = 0.9473 \Rightarrow P(X \geq 5) = 0.0527$ closest to 0.05

$P(X \leq 5) = 0.9834 \Rightarrow P(X \geq 6) = 0.0166$

$\therefore$ critical region is $X \geq 5$

c Power $= P(X \geq 5 \mid \lambda = 3)$

$= 1 - P(X \leq 4 \mid \lambda = 3) = 1 - 0.8153 = 0.1847$

d Seek d such that $P(X \geq d) \approx 0.05$ where $X \sim Po(6)$

Tables $P(X \leq 10) = 0.9574 \Rightarrow P(X \geq 11) = 0.0426$

$\therefore$ critical region is $X \geq 11$

e Power $= P(X \geq 11 \mid \lambda = 9)$ [3 days has mean $= 3 \times 3 = 9$]

$= 1 - P(X \leq 10 \mid \lambda = 9)$

$= 1 - 0.7060$

$= 0.294$

f Second test is more powerful as it uses more days.

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Exercise G, Question 10

Question:

A single observation, x, is to be taken from a Poisson distribution with parameter μ. This observation is to be used to test $H_0 : \mu = 6$ against $H_1 : \mu < 6$. The critical region is chosen to be $x \leq 2$.

a Find the size of the critical region.

b Show that the power function for this test is given by $\frac{1}{2}e^{-\mu}(2+2\mu+\mu^2)$

The table below gives the values of the power function to 2 decimal places.

μ	1.0	1.5	2.0	4.0	5.0	6.0	7.0
Power	0.92	0.81	s	0.24	t	0.06	0.03

c Calculate the values of s and t.

d Draw a graph of the power function.

e Find the range of values of μ for which the power of this test is greater than 0.8.

[E]

Solution:

$H_0: \mu = 6 \quad H_1: \mu < 6$

Critical region is $X \leq 2$

a Size $= P(X \leq 2 \mid X \sim Po(6))$

$= 0.0620$

b Power $= P(X \leq 2 \mid X \sim Po(6))$

$$= e^{-\mu} + \mu e^{-\mu} + \frac{\mu^2}{2} e^{-\mu}$$

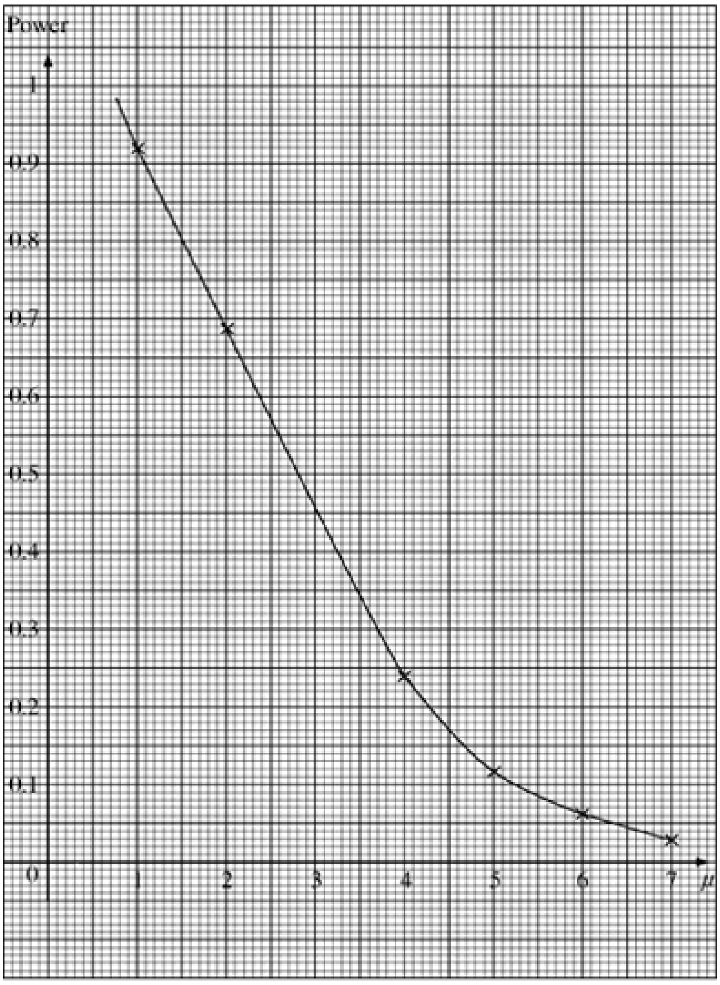
$$= e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2}\right)$$

$$= \frac{e^{-\mu}}{2}\left(2 + 2\mu + \mu^2\right)$$

c $s = 0.6767$

$t = 0.1247$

d See Graph.



e From Graph

$\mu < 1.55$

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Exercise G, Question 11

Question:

The random variable X has the following distribution:

x	0	1
$P(X = x)$	q	p

a Find $E(X)$ and $Var(X)$.

A random sample X_1, X_2, X_3, is taken from the distribution in order to estimate p.

b Find the condition which must be satisfied by the constants a_1, a_2, a_3, if $a_1X_1 + a_2X_2 + a_3X_3$ is to be an unbiased estimator of p.

c Find the variance of this estimator.

The following estimators are proposed:

i $\frac{1}{6}X_1 + \frac{1}{3}X_2 + \frac{1}{2}X_3$

ii $\frac{1}{3}X_1 + \frac{1}{6}X_2 + \frac{5}{12}X_3$

iii $\frac{7}{12}X_1 + \frac{5}{12}X_2$

d Of these three estimators, find the best unbiased estimator. **[E]**

Solution:

a $\text{E}(X) = 0 + p = p$

$\text{E}(X^2) = 0 + 1^2 p = p \quad \therefore \text{Var}(X) = p - p^2 = p(1-p)$

b $Y = a_1X_1 + a_2X_2 + a_3X_3$

$\text{E}(Y) = (a_1 + a_2 + a_3)p \quad \therefore$ for Y to be unbiased estimator of p

You need $a_1 + a_2 + a_3 = 1$

c $\text{Var}(Y) = a_1^2 p(1-p) + a_2^2 p(1-p) + a_3^2 p(1-p)$

$= (a_1^2 + a_2^2 + a_2^2) pq$

d **i** $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1 \therefore$ unbiased

Variance $= \left(\frac{1}{36} + \frac{1}{9} + \frac{1}{4}\right) pq = \left(\frac{1+4+9}{36}\right) pq = \frac{14}{36} pq = \frac{28}{72} pq$

ii $\frac{1}{3} + \frac{1}{6} + \frac{5}{12} \neq 1 \therefore$ biased

iii $\frac{7}{12} + \frac{5}{12} = 1 \therefore$ unbiased

Variance $= \left(\frac{49}{144} + \frac{25}{144}\right) pq = \frac{74}{144} pq = \frac{37}{72} pq$

$\therefore$ best estimator is $\frac{1}{6}X_1 + \frac{1}{3}X_2 + \frac{1}{2}X_3$ as it has the smallest variance

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Exercise G, Question 12

Question:

Two sets of binomial trials were carried out and in both sets the probability of success is p. In the first set there were X successes out of n trials and in the second set there were Y successes out of m trials.

Possible estimators for p are $\hat{p}_1 = \frac{1}{2}\left(\frac{X}{n} + \frac{Y}{m}\right)$ and $\hat{p}_2 = \frac{X+Y}{n+m}$

a Show that both $\hat{p}_1$ and $\hat{p}_2$ are unbiased estimators of p.

b Find the variances of $\hat{p}_1$ and $\hat{p}_2$

c If $n = 10$ and $m = 20$ state, giving a reason, which estimator you would use. **[E]**

Solution:

$X \sim \text{B}(n, p) \Rightarrow \mu_x = np \quad \sigma_x^2 = np(1-p)$

$Y \sim \text{B}(m, p) \Rightarrow \mu_y = mp \quad \sigma_y^2 = mp(1-p)$

a $\text{E}(\hat{p}_1) = \frac{1}{2}\left[\frac{\text{E}(X)}{n} + \frac{\text{E}(Y)}{m}\right] = \frac{1}{2}\left[\frac{np}{n} + \frac{mp}{m}\right] = p$

$\text{E}(\hat{p}_2) = \frac{\text{E}(X) + \text{E}(Y)}{n+m} = \frac{np + mp}{n+m} = \frac{(n+m)p}{n+m} = p$

$\therefore$ both $\hat{p}_1$ and $\hat{p}_2$ are unbiased estimators of p

b $\text{Var}(\hat{p}_1) = \frac{1}{4}\left[\frac{\text{Var}(X)}{n^2} + \frac{\text{Var}(Y)}{m^2}\right] = \frac{1}{4}\left[\frac{np(1-p)}{n^2} + \frac{mp(1-p)}{m^2}\right] = \frac{(m+n)p(1-p)}{4mn}$

$\text{Var}(\hat{p}_2) = \frac{\text{Var}(X) + \text{Var}(Y)}{(n+m)^2} = \frac{np(1-p) + mp(1-p)}{(n+m)^2} = \frac{p(1-p)}{n+m}$

c $n = 10, m = 20 \Rightarrow \text{Var}(\hat{p}_1) = \frac{(20+10)p(1-p)}{4(20)(10)}$

$= \frac{3p(1-p)}{80}$

and $\text{Var}(\hat{p}_2) = \frac{p(1-p)}{30}$

$\because \frac{1}{30} < \frac{3}{80} \quad (\because 80 < 90)$

$\therefore$ use $\hat{p}_2 \because$ unbiased and has smaller variance.

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Exercise G, Question 13

Question:

(In this question $\max(a, b)$ = the greater of the two values a and b.)
A palaeontologist was attempting to estimate the length of time, T, in years, during which a small herbivorous dinosaur existed on Earth. He believed from other evidence that the earliest existence of the animal had been at the start of the Jurassic period.

Two examples of the animal had been discovered in the fossil record, at times t_1 and t_2 after the start of the Jurassic period. His model assumed that these times were values of two independent random variables T_1 and T_2 each having a continuous uniform distribution on the interval $[0, \tau]$. He considered three estimators for τ:

$\tau_1 = T_1 + T_2$, $\tau_2 = \sqrt{3}|T_2 - T_1|$, $\tau_3 = 1.5\max(T_1, T_2)$

He used appropriate probability theory and calculated the results shown in the table.

Variable	Expectation	Variance
T_1	$\frac{\tau}{2}$	$\frac{\tau^2}{12}$
$\|T_2 - T_1\|$	$\frac{\tau}{3}$	$\frac{\tau^2}{18}$
$\max(T_1, T_2)$	$\frac{2\tau}{3}$	$\frac{\tau^2}{18}$

Using these results,

a determine the bias of each of his estimators,

b find the variance of each of his estimators.

Using your results from **a** and **b**, state, giving a reason,

c which estimator is the best of the three,

d which estimator is the worst.

Solution:

a $E(\tau_1) = E(T_1 + T_2) = \frac{\tau}{2} + \frac{\tau}{2} = \tau \therefore$ unbiased

$E(\tau_2) = \sqrt{3}E\,|T_2 - T_1| = \sqrt{3} \cdot \frac{\tau}{3} \therefore \text{bias} = \frac{\sqrt{3}}{3}\tau - \tau = \tau\left(\frac{\sqrt{3}}{3} - 1\right)$

$E(\tau_3) = 1.5E(\max(T_1, T_2)) = 1.5 \cdot \frac{2\tau}{3} = \tau \therefore$ unbiased.

$\text{Var}(\tau_1) = \text{Var}(T_1) + \text{Var}(T_2) = \frac{\tau^2}{12} + \frac{\tau^2}{12} = \frac{\tau^2}{6}$

$\text{Var}(\tau_2) = \sqrt{3}^2 \times \frac{\tau^2}{18} = \frac{\tau^2}{6}$

$\text{Var}(\tau_3) = \frac{9}{4} \times \frac{\tau^2}{18} = \frac{\tau^2}{8}$

c τ_3 is best since it is unbiased and it has the smallest variance.

d τ_2 is worst since it is biased (and variance is just the same as τ_1)