Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 1

Question:

At time t seconds a particle P has position vector $\mathbf{r}$ metres, relative to a fixed origin O.
The particle moves so that

$$\frac{d\mathbf{r}}{dt} - \mathbf{r} = 2e^{-t}\mathbf{i}.$$

When $t = 0$, $\mathbf{r} = -\mathbf{i} + \mathbf{j}$.
Find $\mathbf{r}$ in terms of t. ***E***

Solution:

$$\frac{d\mathbf{r}}{dt} - \mathbf{r} = 2e^{-t}\mathbf{i}$$

Integrating factor $= e^{\int -dt} = e^{-t}$

$$e^{-t}\frac{d\mathbf{r}}{dt} - \mathbf{r}e^{-t} = 2e^{-2t}\mathbf{i}$$

Multiply through by the integrating factor.

$$\frac{d}{dt}\left(\mathbf{r}e^{-t}\right) = 2e^{-2t}\mathbf{i}$$

$$\mathbf{r}e^{-t} = -e^{-2t}\mathbf{i} + \mathbf{c}$$

Integrate with respect to t. Don't forget the constant!

$t = 0 \quad \mathbf{r} = -\mathbf{i} + \mathbf{j}$

$\Rightarrow -\mathbf{i} + \mathbf{j} = -\mathbf{i} + \mathbf{c}$

$\mathbf{c} = \mathbf{j}$

Use the initial conditions given in the question to find $\mathbf{c}$.

$\therefore \mathbf{r} = -e^{-t}\mathbf{i} + e^{t}\mathbf{j}$

Multiply through by e^{t} to obtain $\mathbf{r}$.

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Review Exercise 1
Exercise A, Question 2

Question:

With respect to a fixed origin O, the position vector, $\mathbf{r}$ metres, of a particle P at time t seconds satisfies

$$\frac{d\mathbf{r}}{dt}+\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}.$$

Given that P is at O when $t=0$, find

a $\mathbf{r}$ in terms of t,

b a cartesian equation of the path of P. ***E***

Solution:

a $\frac{d\mathbf{r}}{dt}+\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}$

Integrating factor $= e^{\int 1dt} = e^{t}$

$e^{t}\frac{d\mathbf{r}}{dt}+e^{t}\mathbf{r}=(\mathbf{i}-\mathbf{j})e^{-2t}\times e^{t}$ — Multiply through by the integrating factor.

$\frac{d}{dt}(\mathbf{r}e^{t})=(\mathbf{i}-\mathbf{j})e^{-t}$

$\therefore \mathbf{r}e^{t}=-(\mathbf{i}-\mathbf{j})e^{-t}+\mathbf{c}$ — Integrate with respect to t. Don't forget the constant!

$t=0, \mathbf{r}=\mathbf{0}\Rightarrow \mathbf{0}=-(\mathbf{i}-\mathbf{j})+\mathbf{c}$

$\mathbf{c}=\mathbf{i}-\mathbf{j}$ — P is at O when $t=0$.

$\therefore \mathbf{r}=-(\mathbf{i}-\mathbf{j})e^{-2t}+(\mathbf{i}-\mathbf{j})e^{-t}$ — Divide by e^{t} to obtain $\mathbf{r}$

b $\mathbf{r}=(-e^{-2t}+e^{-t})\mathbf{i}+(e^{-2t}-e^{-t})\mathbf{j}$

$x=-e^{-2t}+e^{-t}$ — Using the $\mathbf{i}$ component.

$y=e^{-2t}-e^{-t}$ — Using the $\mathbf{j}$ component.

$\therefore y=-x$ — Eliminate t (by observation).

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Review Exercise 1
Exercise A, Question 3

Question:

At time t seconds the position vector of a particle P relative to a fixed origin O is $\mathbf{r}$ metres. The position vector satisfies the vector differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}.$$

At time $t = \frac{1}{2}\ln 3$, $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

a Find $\mathbf{r}$ in terms of t.

b Find the greatest value of the magnitude of the acceleration of P for $t \geq 0$. ***E***

Solution:

a $\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}$

Integrating factor $= e^{\int 2\,dt} = e^{2t}$

$\therefore e^{2t}\frac{d\mathbf{r}}{dt} + 2e^{2t}\mathbf{r} = \mathbf{0}$

$\frac{d}{dt}\left(e^{2t}\mathbf{r}\right) = \mathbf{0}$

Multiply through by the integrating factor

$e^{2t}\mathbf{r} = \mathbf{A}$

Integrate with respect to t.

$t = \frac{1}{2}\ln 3, \quad \mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Use the initial conditions given in the question to find $\mathbf{A}$.

$e^{\ln 3}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = \mathbf{A}$

$\mathbf{A} = 3(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\therefore \mathbf{r} = 3e^{-2t}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

b $\dot{\mathbf{r}} = -6e^{-2t}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\ddot{\mathbf{r}} = 12e^{-2t}(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$|\ddot{\mathbf{r}}|_{max} = |12(\mathbf{i} - 2\mathbf{j} + \mathbf{k})|$

$|\ddot{\mathbf{r}}|$ will be maximum when $e^{-2t} = 1$

$= 12\sqrt{(1+4+1)}$

$= 12\sqrt{6}$

The greatest value of the magnitude of the acceleration is $12\sqrt{6}$ m s^{-2}.

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Review Exercise 1
Exercise A, Question 4

Question:

The position vector, $\mathbf{r}$ m, of a particle P is measured relative to a fixed origin O, and its velocity $\mathbf{v}$ m s^{-1} at time t seconds satisfies the differential equation

$$\frac{d\mathbf{v}}{dt} = -2\mathbf{v}.$$

When $t = 0$, P is at the point with position vector $(-2\mathbf{i} + \mathbf{j})$ m, and has velocity $(12\mathbf{i} + 8\mathbf{j})$ m s^{-1}. Find

a an expression for $\mathbf{v}$ in terms of t,

b the position vector of P when $t = \ln 2$. ***E***

Solution:

a $\frac{d\mathbf{v}}{dt} = -2\mathbf{v}$

$\frac{d\mathbf{v}}{dt} + 2\mathbf{v} = \mathbf{0}$

Integrating factor $= e^{\int 2dt} = e^{2t}$

$e^{2t}\frac{d\mathbf{v}}{dt} + 2e^{2t}\mathbf{v} = \mathbf{0}$ ← Multiply through by the integrating factor

$\frac{d}{dt}\left(e^{2t}\mathbf{v}\right) = \mathbf{0}$

$e^{2t}\mathbf{v} = \mathbf{A}$ ← Integrate.

$t = 0, \mathbf{v} = 12\mathbf{i} + 8\mathbf{j}$ ← Use the initial conditions given in the question to find $\mathbf{A}$.

$\Rightarrow 12\mathbf{i} + 8\mathbf{j} = \mathbf{A}$

$\therefore \mathbf{v} = \left(12\mathbf{i} + 8\mathbf{j}\right)e^{-2t}$

b $\frac{d\mathbf{r}}{dt} = \left(12\mathbf{i} + 8\mathbf{j}\right)e^{-2t}$

$\mathbf{r} = -\frac{1}{2}\left(12\mathbf{i} + 8\mathbf{j}\right)e^{-2t} + \mathbf{B}$

$t = 0, \mathbf{r} = -2\mathbf{i} + \mathbf{j}$ ← Use the initial conditions given in the question to find $\mathbf{B}$.

$\Rightarrow -2\mathbf{i} + \mathbf{j} = -\frac{1}{2}\left(12\mathbf{i} + 8\mathbf{j}\right) + \mathbf{B}$

$\mathbf{B} = 4\mathbf{i} + 5\mathbf{j}$

$\therefore \mathbf{r} = -(6\mathbf{i} + 4\mathbf{j})e^{-2t} + 4\mathbf{i} + 5\mathbf{j}$

$t = \ln 2 \quad \mathbf{r} = -\frac{1}{4}\left(6\mathbf{i} + 4\mathbf{j}\right) + 4\mathbf{i} + 5\mathbf{j}$ ← $e^{-2\ln 2} = e^{\ln\left(\frac{1}{4}\right)} = \frac{1}{4}$

$\mathbf{r} = \frac{5}{2}\mathbf{i} + 4\mathbf{j}$

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Review Exercise 1
Exercise A, Question 5

Question:

At time t seconds the position vector of a particle P, relative to a fixed origin O, is $\mathbf{r}$ metres, where $\mathbf{r}$ satisfies the differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = 3e^{-t}\mathbf{j}.$$

Given that $\mathbf{r} = 2\mathbf{i} - \mathbf{j}$ when $t = 0$, find $\mathbf{r}$ in terms of t. ***E***

Solution:

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = 3e^{-t}\mathbf{j}$$

Integrating factor $= e^{\int 2dt} = e^{2t}$

$\therefore e^{2t}\frac{d\mathbf{r}}{dt} + 2\mathbf{r}e^{2t} = 3e^{t}\mathbf{j}$ ← Multiply through by the integrating factor.

$\frac{d}{dt}\left(\mathbf{r}e^{2t}\right) = 3e^{t}\mathbf{j}$

$\mathbf{r}e^{2t} = 3e^{t}\mathbf{j} + \mathbf{c}$ ← Integrate with respect t. Don't forget the constant!

$t = 0 \quad \mathbf{r} = 2\mathbf{i} - \mathbf{j}$

$(2\mathbf{i} - \mathbf{j}) = 3\mathbf{j} + \mathbf{c}$ ← Use the initial conditions given in the question.

$\mathbf{c} = 2\mathbf{i} - 4\mathbf{j}$

$\therefore \mathbf{r} = 3e^{-t}\mathbf{j} + (2\mathbf{i} - 4\mathbf{j})e^{-2t}$ ← Divide by e^{2t} to obtain $\mathbf{r}$.

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Review Exercise 1
Exercise A, Question 6

Question:

The position vector $\mathbf{r}$ metres of a particle P, relative to a fixed origin O, at time t seconds, satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 4\mathbf{r} = \mathbf{0}.$$

When $t = 0, \mathbf{r} = 3\mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 4\mathbf{j}$.

Find $\mathbf{r}$ in terms of t. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + 4\mathbf{r} = \mathbf{0}$$

Auxiliary equation : $m^2 + 4 = 0$

$$m = \pm 2\mathrm{i}$$

$$\therefore \mathbf{r} = \mathbf{A}\cos 2t + \mathbf{B}\sin 2t$$

$$t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \mathbf{A}$$

Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

$$\dot{\mathbf{r}} = -2\mathbf{A}\sin 2t + 2\mathbf{B}\cos 2t$$

$$t = 0, \dot{\mathbf{r}} = 2\mathbf{i} + 4\mathbf{j}$$

$$\Rightarrow 2\mathbf{i} + 4\mathbf{j} = 2\mathbf{B}$$

$$\mathbf{B} = \mathbf{i} + 2\mathbf{j}$$

$$\therefore \mathbf{r} = 3\mathbf{i}\cos 2t + (\mathbf{i} + 2\mathbf{j})\sin 2t$$

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Review Exercise 1
Exercise A, Question 7

Question:

A particle P moves in a horizontal plane containing a fixed origin O. At time t, $\overrightarrow{OP} = \mathbf{r}$, where $\mathbf{r}$ satisfies the vector differential equation

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + \omega^2\mathbf{r} = \mathbf{0}.$$

At time $t = 0$ the particle is at the point with position vector $a\mathbf{j}$, and has velocity $\omega b\mathbf{i}$, where a, b and ω are constants.
Solve the differential equation to find $\mathbf{r}$ and hence find the cartesian equation of the path of the particle. ***E***

Solution:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + \omega^2\mathbf{r} = \mathbf{0}$$

Auxiliary equation : $m^2 + \omega^2 = 0$

$$m = \pm i\omega$$

$$\therefore \mathbf{r} = \mathbf{A}\cos\omega t + \mathbf{B}\sin\omega t$$

$$t = 0, \mathbf{r} = a\mathbf{j} \Rightarrow a\mathbf{j} = \mathbf{A}$$

Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

$$\dot{\mathbf{r}} = -\mathbf{A}\omega\sin\omega t + \mathbf{B}\omega\cos\omega t$$

$$t = 0, \dot{\mathbf{r}} = \omega b\mathbf{i}$$

$$\Rightarrow \omega b\mathbf{i} = \mathbf{B}\omega$$

$$\mathbf{B} = b\mathbf{i}$$

$$\therefore \mathbf{r} = b\sin\omega t\mathbf{i} + a\cos\omega t\mathbf{j}$$

hence $x = b\sin\omega t$

$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

$$y = a\cos\omega t$$

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = \sin^2\omega t + \cos^2\omega t$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

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Review Exercise 1
Exercise A, Question 8

Question:

At time t seconds, the position vector of a particle P is $\mathbf{r}$ metres, relative to a fixed origin. The particle moves in such a way that

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

At $t = 0$, P is moving with velocity $(8\mathbf{i} - 6\mathbf{j})$ m s^{-1}.

Find the speed of P when $t = \frac{1}{2}\ln 2$. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = \mathbf{0}$$

$$\frac{d\mathbf{v}}{dt} - 4\mathbf{v} = \mathbf{0}$$

You need to find the speed of P when $t = \frac{1}{2}\ln 2$ and speed $= |$velocity$|$. You do not need to find $\mathbf{r}$.

$$\text{Integrating factor} = e^{\int -4\,dt} = e^{-4t}$$

$$\therefore e^{-4t}\frac{d\mathbf{v}}{dt} - 4e^{-4t}\mathbf{v} = \mathbf{0}$$

Multiply through by the integrating factor.

$$\frac{d}{dt}\left(e^{-4t}\mathbf{v}\right) = \mathbf{0}$$

$$e^{-4t}\mathbf{v} = \mathbf{A}$$

Integrate with respect to t.

$$t = 0,\ \mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$$

$$\mathbf{A} = 8\mathbf{i} - 6\mathbf{j}$$

$$\therefore \mathbf{v} = (8\mathbf{i} - 6\mathbf{j})e^{4t}$$

$$t = \frac{1}{2}\ln 2,\ \mathbf{v} = 4(8\mathbf{i} - 6\mathbf{j})$$

$e^{2\ln 2} = e^{\ln 4} = 4$

$$\text{speed} = |\mathbf{v}| = 4\sqrt{(64 + 36)} = 40$$

The speed is 40 m s^{-1}.

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Review Exercise 1
Exercise A, Question 9

Question:

A particle P moves in the x-y plane and has position vector $\mathbf{r}$ metres at time t seconds.
It is given that $\mathbf{r}$ satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}.$$

When $t = 0, P$ is at the point with position vector $3\mathbf{i}$ metres and is moving with velocity $\mathbf{j}$ m s^{-1}.

a Find $\mathbf{r}$ in terms of t.

b Describe the path of P, giving its cartesian equation. ***E***

Solution:

a $\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}$

Auxiliary equation:

$m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0$ or $m = 2$

$\therefore \mathbf{r} = \mathbf{A}e^0 + \mathbf{B}e^{2t}$

$\mathbf{r} = \mathbf{A} + \mathbf{B}e^{2t}$

$t = 0, \mathbf{r} = 3\mathbf{i}$

$\Rightarrow \mathbf{A} + \mathbf{B} = 3\mathbf{i}$

$\dot{\mathbf{r}} = 2\mathbf{B}e^{2t}$

$t = 0, \dot{\mathbf{r}} = \mathbf{j} \Rightarrow 2\mathbf{B} = \mathbf{j}$

$\therefore \mathbf{B} = \frac{1}{2}\mathbf{j}$

$\mathbf{A} = 3\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\therefore \mathbf{r} = 3\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{j}e^{2t}$

or $\mathbf{r} = 3\mathbf{i} + \frac{1}{2}\mathbf{j}\left(e^{2t} - 1\right)$

> Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

b The particle moves in a straight line.
The equation of the line is $x = 3$.

> The $\mathbf{i}$ component is constant.

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Review Exercise 1
Exercise A, Question 10

Question:

At time t seconds, the position vector $\mathbf{r}$ metres of a particle P, relative to a fixed origin O, satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+3\mathbf{r}=\mathbf{0}.$$

At time $t=0, P$ is at the point with position vector $2\mathbf{i}$ m and is moving with velocity $2\mathbf{j}$ m s^{-1}.

Find the position vector of P when $t=\ln 2$. ***E***

Solution:

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+3\mathbf{r}=\mathbf{0}$$

Auxiliary equation:

$m^2+4m+3 \quad =0$

$(m+3)(m+1)=0$

$\therefore m=-3$ or $m=-1$

$\mathbf{r}=\mathbf{A}e^{-t}+\mathbf{B}e^{-3t}$

$t=0, \mathbf{r}=2\mathbf{i} \Rightarrow 2\mathbf{i}=\mathbf{A}+\mathbf{B}$ ①

Use the initial conditions given in the question to find $\mathbf{A}$ and $\mathbf{B}$.

$\dot{\mathbf{r}}=-\mathbf{A}e^{-t}-3\mathbf{B}e^{-3t}$

$t=0, \dot{\mathbf{r}}=2\mathbf{j} \Rightarrow 2\mathbf{j}=-\mathbf{A}-3\mathbf{B}$ ②

Solve ① and ② simultaneously.

$2\mathbf{i}+2\mathbf{j} = -2\mathbf{B}$

$\mathbf{B} = -(\mathbf{i}+\mathbf{j})$

$\therefore \mathbf{A} = 2\mathbf{i}-\mathbf{B}=3\mathbf{i}+\mathbf{j}$

$\therefore \mathbf{r} = (3\mathbf{i}+\mathbf{j})e^{-t}-(\mathbf{i}+\mathbf{j})e^{-3t}$

$t = \ln 2 \Rightarrow e^{-t}=\frac{1}{2}$ and $e^{-3t}=\frac{1}{8}$

$e^{-\ln 2}=e^{\ln\left(\frac{1}{2}\right)}$

$\therefore$ When $t=\ln 2$

$\mathbf{r} = \frac{1}{2}(3\mathbf{i}+\mathbf{j})-\frac{1}{8}(\mathbf{i}+\mathbf{j})$

$\dot{\mathbf{r}} = \frac{11}{8}\mathbf{i}+\frac{3}{8}\mathbf{j}$

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Review Exercise 1
Exercise A, Question 11

Question:

A particle P of mass 2 kg moves in the x-y plane. At time t seconds its position vector is $\mathbf{r}$ metres. When $t = 0$, the position vector of P is $\mathbf{i}$ metres and the velocity of P is $(-\mathbf{i}+\mathbf{j})$ m s^{-1}.

The vector $\mathbf{r}$ satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+2\frac{d\mathbf{r}}{dt}+2\mathbf{r}=\mathbf{0}.$$

a Find $\mathbf{r}$ in terms of t.

b Show that the speed of P at time t is $e^{-t}\sqrt{2}$ m s^{-1}.

c Find, in terms of e, the loss of kinetic energy of P in the interval $t = 0$ to $t = 1$. ***E***

Solution:

a $\frac{d^2\mathbf{r}}{dt^2}+2\frac{d\mathbf{r}}{dt}+2\mathbf{r}=\mathbf{0}$

Auxiliary equation:

$m^2+2m+2=0$

$$m=\frac{-2\pm\sqrt{(4-8)}}{2}$$

$$m=-1\pm i$$

$\therefore \mathbf{r}=e^{-t}(\mathbf{A}\cos t+\mathbf{B}\sin t)$

$t=0,\ \mathbf{r}=\mathbf{i}\Rightarrow \mathbf{i}=\mathbf{A}$

Initial conditions are given in the question.

$\dot{\mathbf{r}}=-e^{-t}(\mathbf{A}\cos t+\mathbf{B}\sin t)+e^{-t}(-\mathbf{A}\sin t+\mathbf{B}\cos t)$

$t=0,\ \dot{\mathbf{r}}=(-\mathbf{i}+\mathbf{j})$

$-\mathbf{i}+\mathbf{j}=-\mathbf{A}+\mathbf{B}\Rightarrow \mathbf{B}=\mathbf{j}$

$\therefore \mathbf{r}=e^{-t}(\cos t\mathbf{i}+\sin t\mathbf{j})$

b $\dot{\mathbf{r}}=-e^{-t}(\cos t\mathbf{i}+\sin t\mathbf{j})+e^{-t}(-\sin t\mathbf{i}+\cos t\mathbf{j})$

$=(-e^{-t}\cos t-e^{-t}\sin t)\mathbf{i}+(-e^{-t}\sin t+e^{-t}\cos t)\mathbf{j}$

speed $=|\dot{\mathbf{r}}|$

$=e^{-t}\sqrt{\left((-\cos t-\sin t)^2+(-\sin t+\cos t)^2\right)}$

$=e^{-t}\sqrt{\left(\cos^2 t+2\cos t\sin t+\sin^2 t+\sin^2 t-2\sin t\sin t+\cos^2 t\right)}$

$=e^{-t}\sqrt{2}\ \mathrm{m\ s^{-1}}$

Using $\sin^2 t+\cos^2 t=1$

c $t=0$ speed $=\sqrt{2}$

$t=1$ speed $=e^{-1}\sqrt{2}=\frac{\sqrt{2}}{e}$

$\therefore$ Loss of K.E. $=\frac{1}{2}\times2\times(\sqrt{2})^2-\frac{1}{2}\times2\times\left(\frac{\sqrt{2}}{e}\right)^2$

$=2-\frac{2}{e^2}$ J

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Review Exercise 1
Exercise A, Question 12

Question:

A particle of mass 0.5 kg is at rest at the point with position vector $(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$ m. The particle is then acted upon by two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$. These are the only two forces acting on the particle.
Subsequently, the particle passes through the point with position vector $(4\mathbf{i}+5\mathbf{j}-5\mathbf{k})$ m with speed 12 m s^{-1}. Given that $\mathbf{F}_1=(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ N, find $\mathbf{F}_2$. ***E***

Solution:

$\mathbf{d}=(4\mathbf{i}+5\mathbf{j}-5\mathbf{k})-(2\mathbf{i}+3\mathbf{j}-4\mathbf{k})$

$\mathbf{d}=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$

$\mathbf{F}\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=\frac{1}{2}\times\frac{1}{2}\times12^2=36$

Work done $(\mathbf{F}\cdot\mathbf{d})$ = gain in K.E.

$\mathbf{F}=\lambda(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

$\therefore \lambda(2\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=36$

The particle starts at rest and $\therefore$ the resultant force acts along its path

$\lambda(4+4+1)=36$

$\lambda=4$

$\therefore \mathbf{F}=4(2\mathbf{i}+2\mathbf{j}-\mathbf{k})$

But $\mathbf{F}=\mathbf{F}_1+\mathbf{F}_2$

and $\mathbf{F}_1=\mathbf{i}+2\mathbf{j}-\mathbf{k}$ ← Given in the question.

$\therefore \mathbf{F}_2=8\mathbf{i}+8\mathbf{j}-4\mathbf{k}-\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\mathbf{F}_2=7\mathbf{i}+6\mathbf{j}-3\mathbf{k}$

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Review Exercise 1
Exercise A, Question 13

Question:

Two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are the only forces acting on a particle. $\mathbf{F}_1$ has magnitude 9 N and acts in the direction of $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$. $\mathbf{F}_2$ has magnitude 18 N and acts in the direction of $\mathbf{i}+8\mathbf{j}-4\mathbf{k}$.
Find the total work done by the two forces in moving the particle from the point with position vector $(\mathbf{i}+\mathbf{j}+\mathbf{k})$ m to the point with position vector $(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$ m. *E*

Solution:

$$\mathbf{F}_1 = \lambda(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$$
$$|\mathbf{F}_1| = 9$$
$$|2\mathbf{i}+\mathbf{j}+2\mathbf{k}| = \sqrt{(4+1+4)} = 3$$
$$\therefore \lambda = 3$$
$$\therefore \mathbf{F}_1 = 6\mathbf{i}+3\mathbf{j}+6\mathbf{k}$$

$\mathbf{F}_1$ is a scalar multiple of $(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$. Use this fact and the magnitudes of the vectors to find $\mathbf{F}_1$.

$$\mathbf{F}_2 = \mu(\mathbf{i}+8\mathbf{j}-4\mathbf{k})$$
$$|\mathbf{F}_2| = 18$$
$$|\mathbf{i}+8\mathbf{j}-4\mathbf{k}| = \sqrt{(1+64+16)} = 9$$
$$\therefore \mu = 2$$
$$\therefore \mathbf{F}_2 = 2\mathbf{i}+16\mathbf{j}-8\mathbf{k}$$

Now find $\mathbf{F}_2$ in the same way.

$$\mathbf{F}_1+\mathbf{F}_2 = 8\mathbf{i}+19\mathbf{j}-2\mathbf{k}$$
$$\therefore \text{work done} = (8\mathbf{i}+19\mathbf{j}-2\mathbf{k})\cdot[3\mathbf{i}+2\mathbf{j}-\mathbf{k}-(\mathbf{i}+\mathbf{j}+\mathbf{k})]$$
$$= (8\mathbf{i}+19\mathbf{j}-2\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-2\mathbf{k})$$
$$= 16+19+4$$
$$= 39$$

work done $= \mathbf{F}\cdot\mathbf{d}$

The work done is 39 J

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Review Exercise 1
Exercise A, Question 14

Question:

[*In this question* $\mathbf{i}$ *and* $\mathbf{j}$ *are horizontal unit vectors.*]
A small smooth ring of mass 0.5 kg moves along a smooth horizontal wire. The only forces acting on the ring are its weight, the normal reaction from the wire, and a constant force $(5\mathbf{i}+\mathbf{j}-3\mathbf{k})$ N. The ring is initially at rest at the point with position vector $(\mathbf{i}+\mathbf{j}+\mathbf{k})$ m, relative to a fixed origin.
Find the speed of the ring as it passes through the point with position vector $(3\mathbf{i}+\mathbf{k})$ m. ***E***

Solution:

$\mathbf{d} = (3\mathbf{i}+\mathbf{k})-(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$= 2\mathbf{i}-\mathbf{j}$
$\text{work done} = (5\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j})$ ← The ring moves horizontally so its weight and the normal reaction from the wire do no work.
$= 10-1$
$= 9$

$\text{Gain of K.E.} = \frac{1}{2}\times 0.5v^2$ ← The ring starts from rest.

$\therefore \frac{1}{2}\times 0.5v^2 = 9$ ← Gain of K.E. = work done.
$v^2 = 36$
$v = 6$

The speed is 6 m s^{-1}.

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Review Exercise 1
Exercise A, Question 15

Question:

A smooth wire connects $A(0, 3, 0)$ to $B(2, 1, 4)$. The units of length on the x, y, and z axes are metres. A ring is threaded on the wire and a constant force is applied to the ring. The resultant of this force and the weight of the ring is $(\mathbf{i}-\mathbf{j}+\mathbf{k})$ N.
Find the increase in kinetic energy of the ring as it is moved from A to B. ***E***

Solution:

$\mathbf{d} = (2\mathbf{i}+\mathbf{j}+4\mathbf{k})-(0\mathbf{i}+3\mathbf{j}+0\mathbf{k})$
$\mathbf{d} = 2\mathbf{i}-2\mathbf{j}+4\mathbf{k}$
Work done $= \mathbf{F}\cdot\mathbf{d}$
$= (\mathbf{i}-\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-2\mathbf{j}+4\mathbf{k})$
$= 2+2+4$
$= 8$
Work done = increase in K.E.
$\therefore$ increase in kinetic energy is 8 J

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Review Exercise 1
Exercise A, Question 16

Question:

In this question $\mathbf{i}$ *and* $\mathbf{j}$ *are perpendicular horizontal unit vectors and* $\mathbf{k}$ *is a vertical unit vector.*

A bead of mass 0.125 kg moves along a smooth straight wire in the direction $\mathbf{i}+2\mathbf{j}$, from rest at the point A with position vector $(\mathbf{i}+3\mathbf{k})$ m, relative to a fixed origin O. The bead is acted on by three forces. These are a constant force $(-2\mathbf{i}+2\mathbf{j})$ N, the force exerted by the wire and its own weight. Given that the speed of the bead when it reaches the point B on the wire is 2 m s^{-1}, find the position vector of B relative to O.

E

Solution:

$\mathbf{d} = \lambda(\mathbf{i}+2\mathbf{j})$ — The direction of travel is $\mathbf{i}+2\mathbf{j}$

$$\text{work done} = \mathbf{F}\cdot\mathbf{d}$$
$$= (-2\mathbf{i}+2\mathbf{j})\cdot\lambda(\mathbf{i}+2\mathbf{j})$$

The bead moves horizontally so the force exerted by the wire and the weight of the bead do no work.

$$= -2\lambda+4\lambda$$
$$= 2\lambda$$

$$\text{K.E. gained} = \frac{1}{2}\times 0.125\times 2^2$$

The bead starts from rest.

work done = gain of K.E.

$$= \frac{1}{4}$$
$$\therefore 2\lambda = \frac{1}{4}$$
$$\lambda = \frac{1}{8}$$
$$\therefore \mathbf{d} = \frac{1}{8}(\mathbf{i}+2\mathbf{j})$$
$$\overrightarrow{OB} = \overrightarrow{OA}+\mathbf{d}$$
$$\overrightarrow{OB} = \mathbf{i}+3\mathbf{k}+\frac{1}{8}(\mathbf{i}+2\mathbf{j})$$
$$\overrightarrow{OB} = \frac{9}{8}\mathbf{i}+\frac{1}{4}\mathbf{j}+3\mathbf{k}$$

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Review Exercise 1
Exercise A, Question 17

Question:

A bead of mass 0.5 kg is threaded on a smooth straight wire. The forces acting on the bead are a constant force $(2\mathbf{i}+3\mathbf{j}+x\mathbf{k})$ N, its weight $(-4.9\mathbf{k})$ N, and the reaction on the bead from the wire.

a Explain why the reaction on the bead from the wire does no work as the bead moves along the wire.

The bead moves from the point A with position vector $(\mathbf{i}+\mathbf{j}-3\mathbf{k})$ m relative to a fixed origin O to the point B with position vector $(3\mathbf{i}-\mathbf{j}+2\mathbf{k})$ m. The speed of the bead at A is 2 m s^{-1} and the speed of the bead at B is 4 m s^{-1}.

b Find the value of x. ***E***

Solution:

a The wire is smooth so the reaction is perpendicular to the wire and so does no work.

b $\mathbf{d} = \overrightarrow{OB}-\overrightarrow{OA}$

$= (3\mathbf{i}-\mathbf{j}+2\mathbf{k})-(\mathbf{i}+\mathbf{j}-3\mathbf{k})$

$= 2\mathbf{i}-2\mathbf{j}+5\mathbf{k}$

$\mathbf{F} = 2\mathbf{i}+3\mathbf{j}+x\mathbf{k}+(-4.9\mathbf{k})$

$= 2\mathbf{i}+3\mathbf{j}+(x-4.9)\mathbf{k}$

$\mathbf{F}\cdot\mathbf{d} = (2\mathbf{i}+3\mathbf{j}+(x-4.9)\mathbf{k})\cdot(2\mathbf{i}-2\mathbf{j}+5\mathbf{k})$ ← $\mathbf{F}\cdot\mathbf{d}$ = work done

$= 4-6+5(x-4.9)$

$= 5x-26.5$

Gain of K.E. $= \frac{1}{2}\times 0.5\times 4^2 - \frac{1}{2}\times 0.5\times 2^2$

$= 3$

$\therefore 5x-26.5 = 3$

$5x = 29.5$ ← work done = gain of K.E.

$x = 5.9$

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Review Exercise 1
Exercise A, Question 18

Question:

In this question **i** *and* **j** *are perpendicular unit vectors in a horizontal plane and* **k** *is a unit vector vertically upwards.*
A small smooth ring of mass 0.1 kg is threaded onto a smooth horizontal wire which is parallel to $(\mathbf{i}+2\mathbf{j})$. The only forces acting on the ring are its weight, the normal reaction from the wire and a constant force $(\mathbf{i}+2\mathbf{j}-2\mathbf{k})$ N. The ring starts from rest at the point A on the wire, whose position vector relative to a fixed origin is $(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})$ m, and passes through the point B with speed 5 m s^{-1}. Find the position vector of B. ***E***

Solution:

$$\text{work done} = (\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot\overrightarrow{AB}$$

The ring moves horizontally so the reaction from the wire and the weight do no work.

$$\text{K.E. gained} = \frac{1}{2}\times 0.1\times 5^2 = 1.25$$

The ring starts from rest.

$$\overrightarrow{AB} = \lambda(\mathbf{i}+2\mathbf{j})$$

The wire is parallel to $(\mathbf{i}+2\mathbf{j})$.

$$\therefore (\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot\lambda(\mathbf{i}+2\mathbf{j}) = 1.25$$

work done = gain of K.E.

$$\lambda(1+4) = 1.25$$

$$\lambda = \frac{1.25}{5} = 0.25$$

$$\therefore \overrightarrow{AB} = \frac{1}{4}(\mathbf{i}+2\mathbf{j})$$

$$\overrightarrow{OB} = \overrightarrow{OA}+\overrightarrow{AB}$$

$$\overrightarrow{OB} = (2\mathbf{i}-2\mathbf{j}-3\mathbf{k})+\frac{1}{4}(\mathbf{i}+2\mathbf{j})$$

$$\overrightarrow{OB} = \frac{9}{4}\mathbf{i}-\frac{3}{2}\mathbf{j}-3\mathbf{k}$$

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Review Exercise 1
Exercise A, Question 19

Question:

A particle P of mass 4 kg is acted upon by the constant force $\mathbf{F} = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ N. The force $\mathbf{F}$ is the resultant of all the forces acting on P, including its weight. Initially P is at rest at the point A with position vector $(\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ m, relative to a fixed origin O. Under the action of $\mathbf{F}$, P moves to the point B with position vector $(7\mathbf{i} + 8\mathbf{j})$ m.

a Find the speed of P when it reaches B.

b Find the vector moment of $\mathbf{F}$ about the origin. ***E***

Solution:

a $\mathbf{d} = 7\mathbf{i} + 8\mathbf{j} - (\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

$= 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$

work done $= (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$ ← work done $= \mathbf{F} \cdot \mathbf{d}$

$= 12 + 27 + 3$

$= 42$

K.E. gained $= \frac{1}{2} \times 4v^2$ ← The particle starts from rest.

$\therefore 2v^2 = 42$

$v^2 = 21$

$v = \sqrt{21}$

The speed of P at B is $\sqrt{21}$ m s^{-1} (or 4.6 m s^{-1})

b Vector moment $= \mathbf{r} \times \mathbf{F}$

$= (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 2 & 3 & -1 \end{vmatrix}$ ← The determinant method makes calculation of the vector product easier.

$= \mathbf{i}(1 - 9) - \mathbf{j}(-1 - 6) + \mathbf{k}(3 + 2)$

$= -8\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$

The vector moment is $(-8\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})$ Nm.

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Review Exercise 1
Exercise A, Question 20

Question:

Two constant forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are the only forces acting on a particle P of mass 2 kg. The particle is initially at rest at the point A with position vector $(-2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$ m. Four seconds later, P is at the point B with position vector $(6\mathbf{i} - 5\mathbf{j} + 8\mathbf{k})$ m.

Given that $\mathbf{F}_1 = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})$ N, find

a $\mathbf{F}_2$,

b the work done on P as it moves from A to B. ***E***

Solution:

a $\mathbf{d} = (6\mathbf{i} - 5\mathbf{j} + 8\mathbf{k}) - (-2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$

$= 8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

Use $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ to find the acceleration.

$8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k} = \mathbf{0} + \frac{1}{2}\mathbf{a} \times 4^2$

$\mathbf{a} = \mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}$

$\therefore (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) + \mathbf{F}_2 = 2\left(\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\right)$

Using $\mathbf{F} = m\mathbf{a}$ where $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

$\therefore \mathbf{F}_2 = (-10\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$N

b Work done $= (\mathbf{F}_1 + \mathbf{F}_2) \cdot \mathbf{d}$

$= 2\left(\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\right) \cdot (8\mathbf{i} - 4\mathbf{j} + 12\mathbf{k})$

$= 2(8 + 2 + 18)$

$= 56$

The work done is 56 J.

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Review Exercise 1
Exercise A, Question 21

Question:

A particle P of mass 4 kg is constrained to move along a smooth straight horizontal wire. Relative to a fixed origin, the vector equation of the wire is $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j})$ where $\mathbf{r}$ is measured in metres. The particle moves under the action of a constant force $(12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ N, from the point A where $\lambda = 1$, to the point B where $\lambda = 3$. Given that the speed of P at B is 6 m s^{-1}, find the speed of P at A.

E

Solution:

$\mathbf{r}_A = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + (3\mathbf{i} - 4\mathbf{j})$

$\mathbf{r}_B = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + 3(3\mathbf{i} - 4\mathbf{j})$

$\therefore \mathbf{d} = 2(3\mathbf{i} - 4\mathbf{j}) = 6\mathbf{i} - 8\mathbf{j}$

$\text{work done} = (12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \cdot (6\mathbf{i} - 8\mathbf{j})$ ← work done $= \mathbf{F} \cdot \mathbf{d}$

$= 72 - 32$

$= 40$

$\text{Gain of K.E.} = \frac{1}{2} \times 4 \times 6^2 - \frac{1}{2} \times 4 \times v^2$ ← work done = gain of K.E.

$\therefore 72 - 2v^2 = 40$

$2v^2 = 32$

$v^2 = 16$

$v = 4$

The speed of P at A is 4 m s^{-1}.

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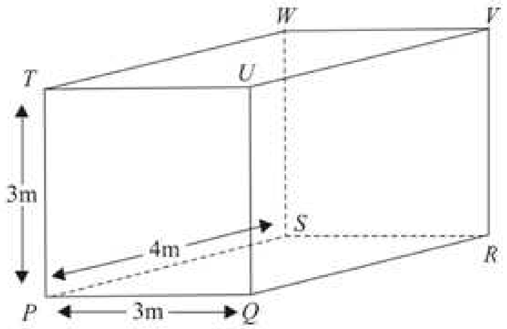
Review Exercise 1
Exercise A, Question 22

Question:



The diagram shows a box in the shape of a cuboid *PQRSTUVW* where $\overrightarrow{PQ} = 3\mathbf{i}$ metres, $\overrightarrow{PS} = 4\mathbf{j}$ metres and $\overrightarrow{PT} = 3\mathbf{k}$ metres. A force $(4\mathbf{i} - 2\mathbf{j})$ N acts at *Q*, a force $(4\mathbf{i} + 2\mathbf{j})$ N acts at *R*, a force $(-2\mathbf{j} + \mathbf{k})$ N acts at *T*, and a force $(2\mathbf{j} + \mathbf{k})$ N acts at *W*. Given that these are the only forces acting on the box, find

a the resultant force acting on the box,

b the resultant vector moment about *P* of the four forces acting on the box.

When an additional force **F** acts on the box at a point *X* on the edge *PS*, the box is in equilibrium.

c Find **F**.

d Find the length of *PX*. ***E***

Solution:

> Moment = $\mathbf{r} \times \mathbf{F}$
> $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$
> $\overrightarrow{PW} = \overrightarrow{PT} + \overrightarrow{TW}$

a $\mathbf{R} = (4\mathbf{i} - 2\mathbf{j}) + (4\mathbf{i} + 2\mathbf{j}) + (-2\mathbf{j} + \mathbf{k}) + (2\mathbf{j} + \mathbf{k})$

$= 8\mathbf{i} + 2\mathbf{k}$

The resultant is $= (8\mathbf{i} + 2\mathbf{k})$ N

b Vector moment about P

$= 3\mathbf{i} \times (4\mathbf{i} - 2\mathbf{j}) + (3\mathbf{i} + 4\mathbf{j}) \times (4\mathbf{i} + 2\mathbf{j}) + 3\mathbf{k} \times (-2\mathbf{j} + \mathbf{k}) + (4\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{j} + \mathbf{k})$

$= -6\mathbf{k} + (6\mathbf{k} - 16\mathbf{k}) + 6\mathbf{i} + (4\mathbf{i} - 6\mathbf{i})$

$= 4\mathbf{i} - 16\mathbf{k}$

> Moment = $\mathbf{r} \times \mathbf{F}$
> $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$
> $\overrightarrow{PW} = \overrightarrow{PT} + \overrightarrow{TW}$

The vector moment is $(4\mathbf{i} - 16\mathbf{k})$ Nm

c For equilibrium, $\mathbf{F} + \mathbf{R} = 0$

$\therefore \mathbf{F} = -8\mathbf{i} - 2\mathbf{k}$

$\mathbf{E} = (-8\mathbf{i} - 2\mathbf{k})$N

d For equilibrium, $\mathbf{r} \times \mathbf{F} = -4\mathbf{i} + 16\mathbf{k}$

> $\mathbf{r} = \overrightarrow{PX}$ and moment of $\mathbf{F}$ + vector moment from part **b** must $= \mathbf{0}$ for equilibrium.

$\mathbf{r} \times (-8\mathbf{i} - 2\mathbf{k}) = -4\mathbf{i} + 16\mathbf{k}$

$\overrightarrow{PX} = \lambda\mathbf{j}$

> X is on PS

$\therefore \lambda\mathbf{j} \times (-8\mathbf{i} - 2\mathbf{k}) = -(4\mathbf{i} - 16\mathbf{k})$

$\lambda(8\mathbf{k} - 2\mathbf{i}) = -(4\mathbf{i} - 16\mathbf{k})$

$\lambda = 2$

$\therefore$ length $PX = 2$ m

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Review Exercise 1
Exercise A, Question 23

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$, and a couple $\mathbf{G}$ act on a rigid body. The force $\mathbf{F}_1 = (3\mathbf{i} + 4\mathbf{j})$ N acts through the point with position vector $2\mathbf{i}$ m and the force $\mathbf{F}_2 = (2\mathbf{i} - \mathbf{j} + \mathbf{k})$ N acts through the point with position vector $(\mathbf{i} + \mathbf{j})$ m, relative to a fixed origin O. The forces and couple are equivalent to a single force $\mathbf{F}$ acting through O.

a Find the force $\mathbf{F}$.

b Find $\mathbf{G}$ and show that it has magnitude $3\sqrt{3}$ Nm. ***E***

Solution:

a $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$

$= (3\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= 5\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

b Vector moment of $\mathbf{F}_1$ and $\mathbf{F}_2$ about O ← Vector moment $= \mathbf{r} \times \mathbf{F}$

$= 2\mathbf{i} \times (3\mathbf{i} + 4\mathbf{j}) + (\mathbf{i} + \mathbf{j}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{vmatrix}$$

$= 8\mathbf{k} + (\mathbf{i} - \mathbf{j}(1) + \mathbf{k}(-1 - 2))$

$= 8\mathbf{k} + \mathbf{i} - \mathbf{j} - 3\mathbf{k}$

$= \mathbf{i} - \mathbf{j} + 5\mathbf{k}$

The forces and the couple are equivalent to a single force $\mathbf{F}$ acting through O.

$\therefore \mathbf{i} - \mathbf{j} + 5\mathbf{k} + \mathbf{G} = \mathbf{0}$

$\mathbf{G} = -\mathbf{i} + \mathbf{j} - 5\mathbf{k}$

$|\mathbf{G}| = \sqrt{(1 + 1 + 25)}$

$= \sqrt{27}$

$= 3\sqrt{3}$

$\therefore \mathbf{G}$ is $(-\mathbf{i} + \mathbf{j} - 5\mathbf{k})$ and it has magnitude $3\sqrt{3}$ Nm.

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Review Exercise 1
Exercise A, Question 24

Question:

Two forces $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ N and $(3\mathbf{i}-\mathbf{k})$ N act through a point O of a rigid body, which is also acted upon by a couple of moment $(\mathbf{i}+\mathbf{j}+3\mathbf{k})$ Nm .

a Show that the couple and forces are equivalent to a single resultant force **F**.

b Find a vector equation for the line of action of **F** in the form $\mathbf{r} = \mathbf{a}+\lambda\mathbf{b}$, where **a** and **b** are constant vectors and λ is a parameter. ***E***

Solution:

a $\mathbf{F}_1 = (\mathbf{i}+2\mathbf{j}-\mathbf{k})\,\text{N}$

$$\mathbf{F}_2 = (3\mathbf{i}-\mathbf{k})\,\text{N}$$

$$\mathbf{G} = (\mathbf{i}+\mathbf{j}+3\mathbf{k})\,\text{Nm}$$

$$(\mathbf{F}_1+\mathbf{F}_2)\cdot\mathbf{G} = (4\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$= 4+2-6$$

$$= 0$$

$\therefore$ The forces and the couple are equivalent to a single resultant force.

b $\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2 = (4\mathbf{i}+2\mathbf{j}-2\mathbf{k})\,\text{N}$

So **F** is parallel to the vector $(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

Let **F** pass through the point with position vector $\mathbf{r} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ relative to O

Then $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(4\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = (\mathbf{i}+\mathbf{j}+3\mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 4 & 2 & -2 \end{vmatrix} = \mathbf{i}+\mathbf{j}+3\mathbf{k}$$

$\mathbf{r}\times\mathbf{F}$ is the vector moment of **F** about O and so must equal the given couple.

$$(-2y-2z)\mathbf{i}-(-2x-4z)\mathbf{j}+(2x-4y)\mathbf{k} = \mathbf{i}+\mathbf{j}+3\mathbf{k}$$

$$-2y-2z = 1 \quad ①$$

$$2x+4z = 1 \quad ②$$

$$2x-4y = 3 \quad ③$$

Equate coefficients of **i**, **j** and **k**.

$$③-②: -4y-4z = 2$$

$$-2y-2z = 1$$

This is the same as ①, so y can be any value

Let $y = 0$

then $z = \frac{-1}{2}$

$x = \frac{3}{2}$

F passes through $\left(\frac{3}{2}, 0, \frac{-1}{2}\right)$

$\therefore$ An equation of the line of action is $\mathbf{r} = \left(\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{k}\right)+\lambda(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

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Review Exercise 1
Exercise A, Question 25

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a rigid body. $\mathbf{F}_1 = (21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k})\text{N}$ and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j} + r\mathbf{k})\text{N}$, where p, q and r are constants. $\mathbf{F}_1$ acts through the point A with position vector $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\text{m}$, relative to a fixed origin O. $\mathbf{F}_2$ acts through the point B with position vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})\text{m}$ relative to O.

The two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are equivalent to a single force $(25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})\text{N}$, acting through O, together with a couple $\mathbf{G}$.

a Find the values of p, q and r.

b Find the magnitude of $\mathbf{G}$. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})\text{N}$

$(21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k}) + (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})$

$(21 + p)\mathbf{i} + (q - 12)\mathbf{j} + (12 + r)\mathbf{k} = (25\mathbf{i} - 14\mathbf{j} + 12\mathbf{k})$

$\therefore p = 4 \quad q = -2 \quad r = 0$ ← Equating coefficients of **i**, **j** and **k**

b $\mathbf{G} = \sum \mathbf{r} \times \mathbf{F} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (21\mathbf{i} - 12\mathbf{j} + 12\mathbf{k}) + (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} - 2\mathbf{j})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 21 & -12 & 12 \end{vmatrix} = -12\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 4 & -2 & 0 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$$

$\therefore \mathbf{G} = (-10\mathbf{i} - 11\mathbf{j})\text{ Nm}$

$|\mathbf{G}| = \sqrt{(10^2 + 11^2)} = \sqrt{221}$

The magnitude of $\mathbf{G}$ is $\sqrt{221}\text{ Nm}$. ← Remember to complete the work by finding the magnitude of **G**.

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Review Exercise 1
Exercise A, Question 26

Question:

A system of forces consists of a force $(\mathbf{i}+2\mathbf{k})$N acting at the point with position vector $(-\mathbf{i}+3\mathbf{j})$m and a force $(-\mathbf{j}+\mathbf{k})$N acting at the point with position vector $(2\mathbf{i}+\mathbf{j}+\mathbf{k})$m . This system is equivalent to a single force $\mathbf{F}$ N acting at the point with position vector $(\mathbf{j}+2\mathbf{k})$m together with a couple $\mathbf{G}$ Nm.

a Find $\mathbf{F}$.

b Find $\mathbf{G}$.

c Give a reason why the system cannot be reduced to a single force without a couple.

E

Solution:

a $\mathbf{F}_1 = (\mathbf{i}+2\mathbf{k})$N

$\mathbf{F}_2 = (-\mathbf{j}+\mathbf{k})$N

$\mathbf{F} = \mathbf{F}_1+\mathbf{F}_2 = (\mathbf{i}-\mathbf{j}+3\mathbf{k})$N

b $\therefore \Sigma\mathbf{r}_i \times \mathbf{F}_i = (-\mathbf{i}+3\mathbf{j})\times(\mathbf{i}+2\mathbf{k})+(2\mathbf{i}+\mathbf{j}+\mathbf{k})\times(-\mathbf{j}+\mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 6\mathbf{i}+2\mathbf{j}-3\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 2\mathbf{i}-2\mathbf{j}-2\mathbf{k}$$

$\Sigma\mathbf{r}\times\mathbf{F} = 8\mathbf{i}-5\mathbf{k}$

Vector moment of resultant

$= (\mathbf{j}+2\mathbf{k})\times(\mathbf{i}-\mathbf{j}+3\mathbf{k})$ ← $\mathbf{F}$ acts at the point with position vector $(\mathbf{j}+2\mathbf{k})$.

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = 5\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

$\therefore 5\mathbf{i}+2\mathbf{j}-\mathbf{k}+\mathbf{G} = 8\mathbf{i}-5\mathbf{k}$ ← Moment of resultant force + couple $= \Sigma\mathbf{r}_i\times\mathbf{F}_i$

$\Rightarrow \mathbf{G} = 3\mathbf{i}-2\mathbf{j}-4\mathbf{k}$

c $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\cdot(8\mathbf{i}-5\mathbf{j}) = 8+5$ ← The resultant force and the couple must be perpendicular if the system is to be reduced to a single force without a couple.

$= 13 \neq 0$

$\therefore$ The system cannot be reduced to a single force without a couple.

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Review Exercise 1
Exercise A, Question 27

Question:

The three forces $\mathbf{F}_1 = (q\mathbf{j} + r\mathbf{k})\text{N}$, $\mathbf{F}_2 = (p\mathbf{i} + r\mathbf{k})\text{N}$ and $\mathbf{F}_3 = (p\mathbf{i} + q\mathbf{j})\text{N}$, where p, q and r are non-zero constants, act on a rigid body. $\mathbf{F}_1$ acts at the point with position vector $p\mathbf{i}$ m relative to a fixed origin O. $\mathbf{F}_2$ acts at the point with position vector $q\mathbf{j}$ m relative to O. $\mathbf{F}_3$ acts at the point with position vector $r\mathbf{k}$ m relative to O.

a Show that the three forces are equivalent to a single non-zero force acting at O.

b Find the magnitude of this single force. ***E***

Solution:

a $\Sigma\mathbf{F}_i = (q\mathbf{j} + r\mathbf{k}) + (p\mathbf{i} + r\mathbf{k}) + (p\mathbf{i} + q\mathbf{j})$

$= 2(p\mathbf{i} + q\mathbf{j} + r\mathbf{k})$ ← This is the resultant of the three forces.

Vector moment of system about 0

$= \Sigma\mathbf{r}_i \times \mathbf{F}_i$

$= p\mathbf{i} \times (q\mathbf{j} + r\mathbf{k}) + q\mathbf{j} \times (p\mathbf{i} + r\mathbf{k}) + r\mathbf{k} \times (p\mathbf{i} + q\mathbf{j})$

$= (pq\mathbf{k} - pr\mathbf{j}) + (-pq\mathbf{k} + qr\mathbf{i}) + (pr\mathbf{j} - qr\mathbf{i})$

$= \mathbf{0}$

No moment about O. ← Resultant must act through O.

So system is equivalent to force $2(p\mathbf{i} + q\mathbf{j} + r\mathbf{k})\text{N}$ through O.

b Magnitude of resultant $= 2\sqrt{(p^2 + q^2 + r^2)}\text{N}$.

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Review Exercise 1
Exercise A, Question 28

Question:

Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a rigid body, where $\mathbf{F}_1 = (2\mathbf{j} + 3\mathbf{k})\text{N}$ and $\mathbf{F}_2 = (\mathbf{i} + 4\mathbf{k})\text{N}$. The force $\mathbf{F}_1$ acts through the point with position vector $(\mathbf{i} + \mathbf{k})\text{m}$ relative to a fixed origin O. The force $\mathbf{F}_2$ acts through the point with position vector $(2\mathbf{j})\text{m}$. The two forces are equivalent to a single force $\mathbf{F}$.

a Find the magnitude of $\mathbf{F}$.

b Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, a vector equation of the line of action of $\mathbf{F}$.

E

Solution:

a $\mathbf{F} = \Sigma\mathbf{F}_i$

$= (2\mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + 4\mathbf{k})$

$= (\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})\text{N}$

$|\mathbf{F}| = \sqrt{(1 + 4 + 49)} = \sqrt{54}\text{ N}$

$= 3\sqrt{6}\text{ N}$

Remember to finish this part of the question by finding the magnitude of $\mathbf{F}$.

b Let $\mathbf{F}$ act through the point with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + 2\mathbf{k}$.

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$= (\mathbf{i} + \mathbf{k}) \times (2\mathbf{j} + 3\mathbf{k}) + 2\mathbf{j} \times (\mathbf{i} + 4\mathbf{k})$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 2 & 7 \end{vmatrix} = (7y - 2z)\mathbf{i} - (7x - z)\mathbf{j} + (2x - y)\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 2 & 3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 8\mathbf{i} - 2\mathbf{k}$$

$\therefore 7y - 2z = -2 + 8 = 6$ ①

$-7x + z = -3$ ②

$2x - y = 2 - 2 = 0$ ③

Equating coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$

① + 2 ×②: $7y - 14x = 0$

$y = 2x$

Same as ③

$\therefore$ A suitable point is $(0, 0, -3)$

$\mathbf{F}$ is parallel to $(\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

$\therefore$ An equation for the line of action of $\mathbf{F}$ is $\mathbf{r} = -3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 7\mathbf{k})$

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Review Exercise 1
Exercise A, Question 29

Question:

Three forces, $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ act on a rigid body. $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\text{N}$, $\mathbf{F}_2 = (\mathbf{i} + \mathbf{j} - 4\mathbf{k})\text{N}$ and $\mathbf{F}_3 = (p\mathbf{i} + q\mathbf{j} + r\mathbf{k})\text{N}$, where p, q and r are constants. All three forces act through the point with position vector $(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})\text{m}$, relative to a fixed origin. The three forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$ are equivalent to a single force $(5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})\text{N}$, acting at the origin, together with a couple $\mathbf{G}$.

a Find the values of p, q and r.

b Find $\mathbf{G}$. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ ← $\Sigma\mathbf{F}_i = \mathbf{F}$

$(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 4\mathbf{k})$
$+ (p\mathbf{i} + q\mathbf{j} + r\mathbf{k}) = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$(3 + p)\mathbf{i} + q\mathbf{j} + (r - 1)\mathbf{k} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$\Rightarrow p = 2, q = -4, r = 3$ ← Equate coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

b $\Sigma\mathbf{r}_i \times \mathbf{F}_i = \mathbf{r} \times \mathbf{F}$

$= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ ← All three forces act through the same point. $\mathbf{F} = \Sigma\mathbf{F}_i$.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 5 & -4 & 2 \end{vmatrix} = 0\mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

$\therefore \mathbf{G} = (-\mathbf{j} - 2\mathbf{k})\text{Nm}$

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Review Exercise 1
Exercise A, Question 30

Question:

A force system consists of three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ acting on a rigid body.

$\mathbf{F}_1 = (\mathbf{i} + 2\mathbf{j})\text{N}$ and acts at the point with position vector $(-\mathbf{i} + 4\mathbf{j})\text{m}$.

$\mathbf{F}_2 = (-\mathbf{j} + \mathbf{k})\text{N}$ and acts at the point with position vector $(2\mathbf{i} + \mathbf{j} + \mathbf{k})\text{m}$.

$\mathbf{F}_3 = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\text{N}$ and acts at the point with position vector $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{m}$.

It is given that this system can be reduced to a single force $\mathbf{R}$.

a Find $\mathbf{R}$.

b Find a vector equation of the line of action of $\mathbf{R}$, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and λ is a parameter. ***E***

Solution:

a $\mathbf{R} = \Sigma\mathbf{F}_i$

$= (\mathbf{i}+2\mathbf{j})+(-\mathbf{j}+\mathbf{k})+(3\mathbf{i}-\mathbf{j}+\mathbf{k})$

$= (4\mathbf{i}+2\mathbf{k})\text{ N}$

b Let $\mathbf{R}$ act through a point with position vector $\mathbf{r} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

$\mathbf{r}\times\mathbf{R} = \Sigma\mathbf{r}_i\times\mathbf{F}_i$

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(4\mathbf{i}+2\mathbf{k})$

$=(-\mathbf{i}+4\mathbf{j})\times(\mathbf{i}+2\mathbf{j})+(2\mathbf{i}+\mathbf{j}+\mathbf{k})\times(-\mathbf{j}+\mathbf{k})+(\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}-\mathbf{j}+\mathbf{k})$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ 4 & 0 & 2\end{vmatrix} = 2y\mathbf{i}-(2x-4z)\mathbf{j}-4y\mathbf{k}$$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 4 & 0\\ 1 & 2 & 0\end{vmatrix} = -6\mathbf{k}$$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 1 & 1\\ 0 & -1 & 1\end{vmatrix} = 2\mathbf{i}-2\mathbf{j}-2\mathbf{k}$$

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 2\\ 3 & -1 & 1\end{vmatrix} = \mathbf{i}+5\mathbf{j}+2\mathbf{k}$$

$\therefore 2y\mathbf{i}-(2x-4z)\mathbf{j}-4y\mathbf{k} = 3\mathbf{i}+3\mathbf{j}-6\mathbf{k}$

$\therefore 2y = 3$ ①

$-2x+4z = 3$ ②

$-4y = -6$ ③

$\therefore y = \frac{3}{2}$

Make $z=0$ in ②, $x = -\frac{3}{2}$ $\therefore \left(-\frac{3}{2},\frac{3}{2},0\right)$ lies on the line of action of $\mathbf{R}$.

$\mathbf{R} = (4\mathbf{i}+2\mathbf{k})\text{ N}$

$\therefore$ $\mathbf{R}$ is parallel to $2\mathbf{i}+\mathbf{k}$.

$\therefore$ An equation of the line of action of $\mathbf{R}$ is

$\mathbf{r} = \left(-\frac{3}{2}\mathbf{i}+\frac{3}{2}\mathbf{j}\right)+\lambda(2\mathbf{i}+\mathbf{k})$.

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Review Exercise 1
Exercise A, Question 31

Question:

Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a rigid body. $\mathbf{F}_1 = (12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k})\text{N}$ and acts at the point with position vector $(2\mathbf{i} - 3\mathbf{j})\text{m}$, $\mathbf{F}_2 = (-3\mathbf{j} + 2\mathbf{k})\text{N}$ and acts at the point with position vector $(\mathbf{i} + \mathbf{j} + \mathbf{k})\text{m}$. The force $\mathbf{F}_3$ acts at the point with position vector $(2\mathbf{i} - \mathbf{k})\text{m}$.

Given that this set of forces is equivalent to a couple, find

a $\mathbf{F}_3$,

b the magnitude of the couple. ***E***

Solution:

a $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$

$$\mathbf{F}_3 = -(12\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) - (-3\mathbf{j} + 2\mathbf{k})$$
$$\mathbf{F}_3 = (-12\mathbf{i} + 7\mathbf{j} - 8\mathbf{k})\ \text{N}$$

b $\mathbf{G} = \Sigma \mathbf{r}_i \times \mathbf{F}_i$

$$\mathbf{r}_1 \times \mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 12 & -4 & 6 \end{vmatrix}$$
$$= -18\mathbf{i} - 12\mathbf{j} + 28\mathbf{k}$$

$$\mathbf{r}_2 \times \mathbf{F}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 0 & -3 & 2 \end{vmatrix}$$
$$= 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{r}_3 \times \mathbf{F}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ -12 & 7 & -8 \end{vmatrix}$$
$$= 7\mathbf{i} + 28\mathbf{j} + 14\mathbf{k}$$

$$\therefore \mathbf{G} = (-6\mathbf{i} + 14\mathbf{j} + 39\mathbf{k})\ \text{Nm}$$
$$|\mathbf{G}| = \sqrt{(6^2 + 14^2 + 39^2)}$$
$$= 41.9\ \text{Nm} \quad (3\text{ s.f.})$$

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Review Exercise 1
Exercise A, Question 32

Question:

A spaceship is moving in a straight line in deep space and needs to reduce its speed from U to V. This is done by ejecting fuel from the front of the spaceship at a constant speed k relative to the spaceship. When the speed of the spaceship is v, its mass is m.

a Show that, while the spaceship is ejecting fuel, $\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m}{k}$.

The initial mass of the spaceship is M.

b Find, in terms of U, V, k and M, the amount of fuel which needs to be used to reduce the speed of the spaceship from U to V. ***E***

Solution:

a Conservation of momentum:

$$mv \approx (m+\delta m)(v+\delta v)+(-\delta m)(k+v+\delta v)$$

$$mv \approx mv + v\delta m + m\delta v + \delta m\delta v - k\delta m - v\delta m - \delta m\delta v$$

$$0 \approx m\delta v - k\delta m$$

$$k\delta m \approx m\delta v$$

In the limit, as $\delta t \to 0$

$$\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m}{k}$$

The fuel is ejected at a constant speed k relative to the space-ship. Its actual speed is therefore $(k+v+\delta v)$.

b $$\int_M^{m_1} \frac{\mathrm{d}m}{m} = \int_U^V \frac{\mathrm{d}v}{k}$$

m_1 is the final mass of the space-ship

$$[\ln m]_M^{m_1} = \left[\frac{v}{k}\right]_U^V$$

$$\ln m_1 - \ln M = \frac{1}{k}(V-U)$$

$$\ln\left(\frac{m_1}{M}\right) = \frac{1}{k}(V-U)$$

$$m_1 = M\mathrm{e}^{\frac{1}{k}(V-U)}$$

$$\text{Amount of fuel} = M - m_1 = M\left(1-\mathrm{e}^{\frac{1}{k}(V-U)}\right)$$

The difference between the initial and final masses is the mass of the fuel ejected.

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Review Exercise 1
Exercise A, Question 33

Question:

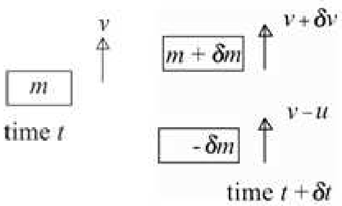
A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upward by ejecting burnt fuel vertically downwards at a constant speed u relative to the rocket. The initial mass of the rocket, including fuel, is M. At time t, before all the fuel has been used up, the mass of the rocket, including fuel, is $M(1-kt)$ and the speed of the rocket is v.

a Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{ku}{1-kt} - g$.

b Hence find the speed of the rocket when $t = \frac{1}{3k}$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv = -mg\,\delta t$$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m+u\,\delta m-mv = -mg\,\delta t$$ ← Change in momentum = impulse

Let $\delta t \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$ ← $\frac{\delta m}{\delta t}\delta v \to 0$ when $\delta t \to 0$

$$m = M(1-kt) \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = -kM$$ ← Given in the question.

$$\therefore M(1-kt)\frac{\mathrm{d}v}{\mathrm{d}t}+u(-kM) = -M(1-kt)g$$

$$(1-kt)\frac{\mathrm{d}v}{\mathrm{d}t}-uk = -(1-kt)g$$

$$\therefore \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{ku}{1-kt}-g$$

b $$v = \int_0^{\frac{1}{3k}} \left(\frac{ku}{1-kt}-g\right)\mathrm{d}t$$ ← Rocket starts from rest.

$$= \left[-u\ln(1-kt)-gt\right]_0^{\frac{1}{3k}}$$

$$= -u\ln\left(1-\frac{1}{3}\right)-\frac{g}{3k}$$

The speed is $u\ln\left(\frac{3}{2}\right)-\frac{g}{3k}$

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Review Exercise 1
Exercise A, Question 34

Question:

A raindrop falls vertically under gravity through a cloud which is at rest. As it falls, water from the cloud condenses onto the drop in such a way that the mass of the drop increases at a constant rate of 0.02 g s^{-1}. At time t seconds, the speed of the drop is $v \text{ m s}^{-1}$, and when $t = 0$ the mass of the drop is 0.06 g. It is assumed that the only external force acting on the drop is gravity.

a Show that v satisfies the differential equation

$$(3+t)\frac{dv}{dt}+v=9.8(3+t).$$

Given that when $t = 0$, the raindrop is at rest,

b find the speed of the raindrop when its mass is twice its initial mass. ***E***

Solution:

a $(m+\delta m)(v+\delta v)-mv = mg\,\delta t$

$mv + v\delta m + m\delta v + \delta m \delta v - mv = mg\,\delta t$

Change in momentum = impulse

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$$v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t} = mg$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = 0.02$$

The mass of the raindrop increases at a constant rate of 0.02 g s^{-1} (see question). *Note* It is OK to work in grams here as the mass units on each side of the differential equation cancel out. Had kg been used an extra factor of 10^{-3} would have been introduced on each side of the equation and this would have cancelled out.

$$\Rightarrow m = 0.02t + c$$

$$t = 0,\ m = 0.06$$

$$m = 0.02t + 0.06$$

$$\therefore 0.02v + (0.02t + 0.06)\frac{\mathrm{d}v}{\mathrm{d}t} = (0.02t + 0.06)g$$

$$v + (t+3)\frac{\mathrm{d}v}{\mathrm{d}t} = (t+3)g$$

$$\text{or } (3+t)\frac{\mathrm{d}v}{\mathrm{d}t} + v = 9.8(3+t)$$

b $\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{3+t} = 9.8$

Integrating factor $= \mathrm{e}^{\int \frac{1}{3+t}\mathrm{d}t}$

$= \mathrm{e}^{\ln(3+t)} = (3+t)$

$\therefore (3+t)\frac{\mathrm{d}v}{\mathrm{d}t} + v = 9.8(3+t)$

$\frac{\mathrm{d}}{\mathrm{d}t}[v(3+t)] = 9.8(3+t)$

$v(3+t) = 29.4t + 4.9t^2 + c$

Integrating both sides of the equation.

Initial mass = 0.06 g

$\frac{\mathrm{d}m}{\mathrm{d}t} = 0.02\text{ g s}^{-1}$

$\therefore$ Mass doubled when $t = 3$

$t = 0,\ v = 0\ \therefore\ c = 0$

$t = 3,\ 6v = 29.4\times 3 + 4.9\times 9$

$v = 22.05$

$\therefore$ The speed is 22.1 m s^{-1} (3 s.f.)

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Review Exercise 1
Exercise A, Question 35

Question:

A rocket has total initial mass M. It propels itself by burning fuel and ejecting the burnt matter at a uniform rate with constant speed u relative to the rocket. The total mass of fuel in the rocket is initially $\frac{1}{2}M$, and the fuel is all burnt up after a time T. The rocket is launched from rest vertically upwards from the surface of the Earth. It may be assumed that the acceleration due to gravity remains constant throughout the flight of the rocket, and that air resistance is negligible. At time t, the speed of the rocket is v.

a Show that, while the fuel is being burnt,

$$(2T-t)\frac{\mathrm{d}v}{\mathrm{d}t}=u-g(2T-t).$$

b Hence find the speed of the rocket at the instant when all the fuel has been burnt. ***E***

Solution:

a $m = M - kt$

m = mass of rocket at time t. The burnt matter is ejected at a uniform rate.

$t = T,\ m = \frac{1}{2}M$

$\frac{1}{2}M = M - kT$

At time T all the fuel has been burnt.

$k = \frac{M}{2T}$

$\therefore\ m = M - \frac{M}{2T}t$

Change of momentum = impulse

$(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv = -mg\,\delta t$

$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + u\delta m - mv = -mg\,\delta t$

$m\delta v + u\delta m + \delta m\delta v = -mg\,\delta t$

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$

$M\left(1-\frac{t}{2T}\right)\frac{\mathrm{d}v}{\mathrm{d}t} - \frac{M}{2T}u = -Mg\left(1-\frac{t}{2T}\right)$

$m = M\left(1-\frac{t}{2T}\right)$ from above $\Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = \frac{-M}{2T}$

$\therefore (2T-t)\frac{\mathrm{d}v}{\mathrm{d}t} = u - g(2T-t)$

Cancel M and multiply through by $2T$ to obtain the required result.

b $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{u}{(2T-t)} - g$

$v = \int_0^T \left(\frac{u}{2T-t} - g\right)\mathrm{d}t$

The fuel is all burnt when $t = T$. (see question) and the rocket starts from rest.

$v = \left[-u\ln(2T-t) - gt\right]_0^T$

$v = -u\ln T - gT - (-u\ln 2T)$

$= u\ln\frac{2T}{T} - gT$

$= u\ln 2 - gT$

The speed at the instant when all the fuel has been burnt is $u\ln 2 - gT$.

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Review Exercise 1
Exercise A, Question 36

Question:

A rocket initially has total mass M. It propels itself by its motor ejecting burnt fuel. When all of its fuel has been burned its mass is $kM, k < 1$. It is moving with speed U when its motor is started. The burnt fuel is ejected with constant speed c, relative to the rocket, in a direction opposite to that of the rocket's motion. Assuming that the only force acting on the rocket is that due to the motor, find the speed of the rocket when all of its fuel has been burned. ***E***

Solution:

Actual speed of fuel ejected $= v - c$

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-c) = mv$$

Momentum is conserved.

$$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + c\,\delta m = mv$$

$$m\delta v + c\,\delta m + \delta m\delta v = 0$$

Let $\delta t \to 0$

$$m + c\frac{\mathrm{d}m}{\mathrm{d}v} = 0$$

$\dfrac{\delta m\delta v}{\delta v} = \delta m \to 0$ when $\delta t \to 0$

$$c\int\frac{\mathrm{d}m}{m} = -\int \mathrm{d}v$$

$$c\ln m = -v + A$$

A is the constant of integration.

$$t = 0, m = M, v = U$$

$$A = U + c\ln M$$

$$\therefore v + c\ln m = U + c\ln M$$

When all the fuel is burned, $m = kM$

$$\therefore v = -c\ln kM + U + c\ln M$$

$$= -c\ln\frac{kM}{M} + U$$

The speed is $U - c\ln k$.

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Review Exercise 1
Exercise A, Question 37

Question:

A rocket is launched vertically upwards from rest. Initially, the total mass of the rocket and its fuel is 1000 kg. The rocket burns fuel at a rate of 10 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after one minute. At time t seconds, $t \leq 60$, after the launch, the speed of the rocket is v m s^{-1}. Air resistance is assumed to be negligible.

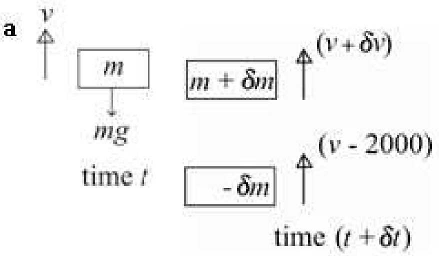
a Show that

$$-9.8(100-t) = (100-t)\frac{dv}{dt} - 2000.$$

b Find the speed of the rocket when burning stops. ***E***

Solution:

a



The fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket.

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv = -mg\delta t$$

$$mv+v\delta m+m\delta v+\delta m\delta v-v\delta m+2000\delta m-mv = -mg\delta t$$

Change of momentum = impulse

Let $\delta t \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$

$\frac{\delta m}{\delta t}\delta v \to 0$

But $m = 1000-10t$

$$\therefore \frac{\mathrm{d}m}{\mathrm{d}t} = -10$$

The initial total mass of the rocket and its fuel is 1000 kg and the fuel burns at 10 kg s^{-1}.

$$(1000-10t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times(-10) = -(1000-10t)g$$

$$(100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000 = -(100-t)\times 9.8$$

$$\text{or } -9.8(100-t) = (100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000$$

b $\frac{\mathrm{d}v}{\mathrm{d}t} = -9.8+\frac{2000}{(100-t)}$

$$v = -9.8t-2000\ln(100-t)+c$$

$t = 0, \quad v = 0$

$$\therefore \quad c = 2000\ln 100$$

$$v = 2000\ln 100-9.8t-2000\ln(100-t)$$

$$t = 60,\ v = 2000\ln 100-9.8\times 60-2000\ln 40$$

Burning stops after 1 minute (see question).

$$v = 2000\ln\frac{100}{40}-588$$

$$v = 2000\ln 2.5-588$$

$$v = 1244$$

When burning stops, the speed of the rocket is 1200 m s^{-1} (2 s.f.)

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Review Exercise 1
Exercise A, Question 38

Question:

A spherical raindrop falls under gravity through a stationary cloud. Initially the drop is at rest and its radius is a. As it falls, water from the cloud condenses on the drop in such a way that the radius of the drop increases at a constant rate k.
At time t, the speed of the drop is v.

a Show that

$$(a+kt)\frac{\mathrm{d}v}{\mathrm{d}t}+3kv=g(a+kt).$$

b Hence show that, when the drop has doubled its radius, it speed is $\frac{15ga}{32k}$. ***E***

Solution:

a $r = a + kt$ — The initial radius is a and the radius increases at a constant rate k.

$(m+\delta m)(v+\delta v) - mv = mg\,\delta t$ — Change of momentum = impulse.

$m = \frac{4}{3}\pi r^3 \rho$ — The raindrop is spherical.

$\Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = \frac{4}{3}\pi\rho \times 3r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$

$= 4\pi\rho r^2 k$ — $\frac{\mathrm{d}r}{\mathrm{d}t} = k$ (see question).

$\therefore mv + v\delta m + m\delta v + \delta m\delta v - mv = mg\,\delta t$

Let $\delta t \to 0$

$v\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}v}{\mathrm{d}t} = mg$ — $\frac{\delta m}{\delta t}\delta v \to 0$

$v \times 4\pi\rho r^2 k + \frac{4}{3}\pi r^3 \rho \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{4}{3}\pi r^3 \rho g$ — Substitute for m and $\frac{\mathrm{d}m}{\mathrm{d}t}$.

$3vk + r\frac{\mathrm{d}v}{\mathrm{d}t} = rg$

$(a+kt)\frac{\mathrm{d}v}{\mathrm{d}t} + 3vk = g(a+kt)$ — Substitute for r.

b $\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3vk}{(a+kt)} = g$

Integrating factor $= \mathrm{e}^{3k\int \frac{\mathrm{d}t}{(a+kt)}}$

$= \mathrm{e}^{3\ln(a+kt)} = (a+kt)^3$

$(a+kt)^3\frac{\mathrm{d}v}{\mathrm{d}t} + 3vk(a+kt)^2 = g(a+kt)^3$

$\frac{\mathrm{d}}{\mathrm{d}t}\left[v(a+kt)^3\right] = g(a+kt)^3$

$v(a+kt)^3 = \frac{g}{4k}(a+kt)^4 + c$

$t = 0, v = 0 \Rightarrow 0 = \frac{ga^4}{4k} + c$

$\therefore v(a+kt)^3 = \frac{g}{4k}(a+kt)^4 - \frac{ga^4}{4k}$

radius doubled $\Rightarrow kt = a$

$\therefore v(2a)^3 = \frac{g}{4k}(2a)^4 - \frac{ga^4}{4k}$

$8a^3 v = \frac{4ga^4}{k} - \frac{ga^4}{4k}$

$8v = \frac{15ga}{4k}$

The speed is $\frac{15ga}{32k}$

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Review Exercise 1
Exercise A, Question 39

Question:

A hailstone falls under gravity in still air and as it falls its mass increases. Its initial mass is m_0. The rate of increase of its mass is proportional to its speed v.

a Show that, when the hailstone has fallen a distance x, it mass m is given by
$m = m_0(1+\lambda x)$, where λ is a constant.

Assuming that there is no air resistance,

b Show that

$$\frac{d}{dx}(v^2) + \frac{2\lambda}{1+\lambda x}(v^2) = 2g.$$

Given that $v = 0$ when $x = 0$,

c find an expression for v^2 in terms of x, λ and g. ***E***

Solution:

a $\frac{dm}{dt} = kv = k\frac{dx}{dt}$

The rate of increase of the hailstone's mass is proportional to its speed.

$$\int dm = k\int dx$$

$$m = kx + c$$

$$x = 0,\ m = m_0 \Rightarrow c = m_0$$

$$\therefore m = kx + m_0$$

$$m = m_0\left(1 + \frac{k}{m_0}x\right)$$

writing $\lambda = \frac{k}{m_0}$ gives

$$m = m_0(1 + \lambda x)$$

b $(m + \delta m)(v + \delta v) - mv = mg\,\delta t$

Change in momentum = impulse

$$mv + v\delta m + m\delta v + \delta m\delta v - mv = mg\,\delta t$$

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$$v\frac{dm}{dt} + m\frac{dv}{dt} = mg$$

$$v\frac{dm}{dt} + mv\frac{dv}{dx} = mg$$

The required result contains $\frac{d}{dx}(v^2)$, not $\frac{dv}{dt}$

But $m = m_0(1 + \lambda x)$

$$\Rightarrow \frac{dm}{dt} = m_0\lambda\frac{dx}{dt} = m_0\lambda v$$

$$\therefore v m_0\lambda v + mv\frac{dv}{dx} = mg$$

$$v^2\lambda\frac{m}{(1 + \lambda x)} + \frac{1}{2}m\frac{d(v^2)}{dx} = mg$$

From **a** $m_0 = \frac{m}{(1 + \lambda x)}$

$$\therefore \frac{d(v^2)}{dx} + \frac{2\lambda}{1 + \lambda x}(v^2) = 2g$$

c Let $v^2 = Y$

$$\frac{dY}{dx} + \frac{2\lambda Y}{1+\lambda x} = 2g$$

Use the substitution if you need to. You can solve the equation keeping the v^2 if you wish.

$$\text{Integrating factor} = e^{\int \frac{2\lambda}{1+\lambda x} dx}$$

$$= e^{2\ln(1+\lambda x)}$$

$$= e^{\ln(1+\lambda x)^2} = (1+\lambda x)^2$$

$$\therefore (1+\lambda x)^2 \frac{dY}{dx} + (1+\lambda x) 2\lambda Y = 2(1+\lambda x)^2 g$$

$$\frac{d}{dx}\left[(1+\lambda x)^2 Y\right] = 2(1+\lambda x)^2 g$$

$$(1+\lambda x)^2 Y = \frac{2}{3}(1+\lambda x)^3 \times \frac{g}{\lambda} + c$$

$$v^2 = \frac{2g}{3\lambda}(1+\lambda x) + \frac{c}{(1+\lambda x)^2}$$

$Y = v^2$

$$x = 0, v = 0 \Rightarrow 0 = \frac{2g}{3\lambda} + c$$

$$\therefore v^2 = \frac{2g}{3\lambda}(1+\lambda x) - \frac{2g}{3\lambda(1+\lambda x)^2}$$

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Review Exercise 1
Exercise A, Question 40

Question:

A rocket-driven car propels itself forwards in a straight line on a horizontal track by ejecting burnt fuel backwards at a constant rate λ kg s^{-1} and at a constant speed U m s^{-1} relative to the car. At time t seconds, the speed of the car is v m s^{-1} and the total resistance to the motion of the car has magnitude kv N, where k is a positive constant. When $t = 0$ the total mass of the car, including fuel, is M kg. Assuming that at time t seconds some fuel remains in the car,

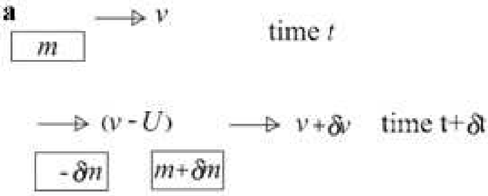
a show that

$$\frac{dv}{dt} = \frac{\lambda U - kv}{M - \lambda t},$$

b find the speed of the car at time t seconds, given that it starts from rest when $t = 0$ and that $\lambda = k = 10$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-U)-mv = -kv\delta t$$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m+U\delta m-mv = -kv\delta t$$

Change of momentum = impulse from resistance

Let $\delta t \to 0$

$\frac{\delta m}{\delta t}\delta v \to 0$

$$m\frac{dv}{dt}+U\frac{dm}{dt} = -kv$$

$$t=0, m=M \therefore m = M-\lambda t$$

The fuel is ejected at a constant rate λ kg s^{-1}.

$$\frac{dm}{dt} = -\lambda$$

$$\therefore (M-\lambda t)\frac{dv}{dt}-\lambda U = -kv$$

$$(M-\lambda t)\frac{dv}{dt} = \lambda U - kv$$

$$\frac{dv}{dt} = \frac{\lambda U - kv}{(M-\lambda t)}$$

b

$$\frac{dv}{dt} = \frac{10(U-v)}{M-10t}$$

$\lambda = k = 10$ in **b**.

$$\int\frac{dv}{U-v} = 10\int\frac{dt}{M-10t}$$

$$-\ln(U-v) = -\ln(M-10t)+c$$

$$t=0,\ v=0\quad c=\ln M-\ln U$$

$$\therefore \ln(U-v) = \ln(M-10t)-\ln M+\ln U$$

$$\ln(U-v) = \ln\left[\frac{U(M-10t)}{M}\right]$$

$$\therefore U-v = \frac{U(M-10t)}{M}$$

$$UM-Mv = UM-10tU$$

$$v = \frac{10Ut}{M}$$

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Review Exercise 1
Exercise A, Question 41

Question:

A rocket-driven car moves along a straight horizontal road. The car has total initial mass M. It propels itself forwards by ejecting mass backwards at a constant rate λ per unit time at a constant speed U relative to the car. The car starts from rest at time $t = 0$. At time t the speed of the car is v. The total resistance to motion is modelled as having magnitude kv, where k is a constant.

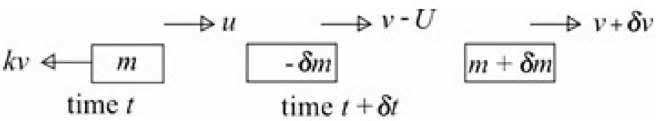
Given that $t < \dfrac{M}{\lambda}$, show that

a $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\lambda U - kv}{M - \lambda t}$,

b $v = \dfrac{\lambda U}{k}\left\{1 - \left(1 - \dfrac{\lambda t}{M}\right)^{\frac{k}{\lambda}}\right\}$. ***E***

Solution:

a



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-U)-mv = -kv\delta t$$

Change of momentum = impulse

$$mv + v\delta m + m\delta v + \delta m\delta v - v\delta m + U\delta m - mv = -kv\delta t$$

Let $\delta t \to 0$

$\dfrac{\delta m}{\delta t}\delta v \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + U\frac{\mathrm{d}m}{\mathrm{d}t} = -kv$$

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\lambda \Rightarrow m = -\lambda t + c$$

Mass is ejected at a constant rate λ per unit time.

$$t = 0,\ m = M \ \therefore\ c = M$$

$$\therefore\ m = M - \lambda t$$

$$\therefore\ (M - \lambda t)\frac{\mathrm{d}v}{\mathrm{d}t} - \lambda U = -kv$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda U - kv}{M - \lambda t}$$

b $\int \frac{dv}{\lambda U - kv} = \int \frac{dt}{M - \lambda t}$

Separate the variables, and integrate.

$$-\frac{1}{k}\ln(\lambda U - kv) = -\frac{1}{\lambda}\ln(M - \lambda t) + \ln A$$

$$t = 0, v = 0 \Rightarrow -\frac{1}{k}\ln \lambda U = -\frac{1}{\lambda}\ln M + \ln A$$

$$\ln A = \ln \frac{M^{\frac{1}{\lambda}} - \ln(\lambda U)^{\frac{1}{k}}}{\frac{1}{\lambda}}$$

$$A = \frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}}$$

$$\therefore \ln\left(\frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}}\right) = \ln\left[\frac{(M - \lambda t)^{\frac{1}{\lambda}}}{(\lambda U - kv)^{\frac{1}{k}}}\right]$$

$$\frac{M^{\frac{1}{\lambda}}}{(\lambda U)^{\frac{1}{k}}} = \frac{(M - \lambda t)^{\frac{1}{\lambda}}}{(\lambda U - kv)^{\frac{1}{k}}}$$

$$\frac{M^{\frac{k}{\lambda}}}{\lambda U} = \frac{(M - \lambda t)^{\frac{k}{\lambda}}}{\lambda U - kv}$$

Raise each side to the power k, so the power is the same as in the required result.

$$\lambda U - kv = \lambda U\left[\frac{M - \lambda t}{M}\right]^{\frac{k}{\lambda}}$$

$$kv = \lambda U\left[1 - \left(\frac{M - \lambda t}{M}\right)^{\frac{k}{\lambda}}\right]$$

$$v = \frac{\lambda U}{k}\left[1 - \left(1 - \frac{\lambda t}{M}\right)^{\frac{k}{\lambda}}\right]$$