Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres and its velocity is $\mathbf{v}$ m s^{-1}. The motion of P is modelled by the differential equation

$$\frac{d\mathbf{v}}{dt} = 3\mathbf{v}.$$

Given that when $t = 0, \mathbf{r} = 3\mathbf{i}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$\dfrac{d\mathbf{v}}{dt} - 3\mathbf{v} = \mathbf{0}$ Auxiliary equation is $\lambda - 3 = 0$

$$\Rightarrow \lambda = 3$$

General solution is $\mathbf{v} = \mathbf{A}e^{3t}$

When $t = 0, \mathbf{v} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = \mathbf{A}$

So, $\mathbf{v} = (\mathbf{i} - \mathbf{j})e^{3t}$

i.e. $\dfrac{d\mathbf{r}}{dt} = (\mathbf{i} - \mathbf{j})e^{3t}$

$\Rightarrow \mathbf{r} = \dfrac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t} + \mathbf{B}$

When $t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \dfrac{1}{3}(\mathbf{i} - \mathbf{j}) + \mathbf{B}$

$$\Rightarrow \dfrac{8}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j} = \mathbf{B}$$

Hence, $\mathbf{r} = \dfrac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t} + \dfrac{8}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j}$

$$= \dfrac{1}{3}\left\{(e^{3t} + 8)\mathbf{i} + (1 - e^{3t})\mathbf{j}\right\}$$

Alternative method

$$\dfrac{d\mathbf{v}}{dt} - 3\mathbf{v} = \mathbf{0}$$

$$\Rightarrow \dfrac{d^2\mathbf{r}}{dt^2} - 3\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$$

Auxiliary equation:

$\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda - 3) = 0$

$$\Rightarrow \lambda = 0 \text{ or } 3$$

$\Rightarrow$ General Solution is $\mathbf{v} = \mathbf{A} + \mathbf{B}e^{3t}$

$t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \mathbf{A} + \mathbf{B}$ ①

$\mathbf{r} = \mathbf{A} + \mathbf{B}e^{3t} \Rightarrow \mathbf{v} = 3\mathbf{B}e^{3t}$

$t = 0, \mathbf{v} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = 3\mathbf{B}$

$$\Rightarrow \dfrac{1}{3}(\mathbf{i} - \mathbf{j}) = \mathbf{B}$$ | Substitute into ① |

$$\mathbf{A} = \dfrac{8}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j}$$

Hence, $\mathbf{r} = \dfrac{8}{3}\mathbf{i} + \dfrac{1}{3}\mathbf{j} + \dfrac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t}$ as before

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Exercise A, Question 2

Question:

The velocity $\mathbf{v}$ m s^{-1} of a particle P at time t seconds satisfies the differential equation

$\dfrac{d\mathbf{v}}{dt} + \mathbf{v} = \mathbf{0}$.

Given that the initial velocity of P is $(12\mathbf{i} + 6\mathbf{j})$, find the velocity of P at $t = \ln 3$.

Solution:

$\dfrac{d\mathbf{v}}{dt} + \mathbf{v} = \mathbf{0} \Rightarrow \lambda + 1 = 0 \Rightarrow \lambda = -1$

So, general solution is $\mathbf{v} = \mathbf{A}e^{-t}$

$t = 0, \mathbf{v} = 12\mathbf{i} + 6\mathbf{j} \Rightarrow \mathbf{v} = (12\mathbf{i} + 6\mathbf{j})e^{-t}$

When $t = \ln 3, \quad \mathbf{v} = (12\mathbf{i} + 6\mathbf{j})e^{-\ln 3}$

$\qquad\qquad\qquad\qquad = 4\mathbf{i} + 2\mathbf{j}$

Velocity is $(4\mathbf{i} + 2\mathbf{j})$ m s^{-1}

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Exercise A, Question 3

Question:

The velocity $\mathbf{v}$ m s^{-1} at time t seconds of a particle moving in a plane satisfies the differential equation

$\dfrac{d\mathbf{v}}{dt} = 6\mathbf{v}$, where $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$ when $t = 0$.

Given that the particle starts, at $t = 0$, at the point with position vector $(\mathbf{i} + \mathbf{j})$m find

a the position vector of the particle P at time t seconds,

b the time when the magnitude of the acceleration of the particle P first equals 100 m s^{-2}.

Solution:

$\dfrac{d\mathbf{v}}{dt} - 6\mathbf{v} = \mathbf{0} \Rightarrow \lambda - 6 = 0 \Rightarrow \lambda = 6$

So, general solution is $\mathbf{v} = \mathbf{A}e^{6t}$

$t = 0, \mathbf{v} = 4\mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{v} = (4\mathbf{i} + 2\mathbf{j})e^{6t}$

$$\Rightarrow \mathbf{r} = \frac{1}{6}(4\mathbf{i} + 2\mathbf{j})e^{6t} + \mathbf{B}$$

$t = 0, \mathbf{r} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = \dfrac{1}{6}(4\mathbf{i} + 2\mathbf{j}) + \mathbf{B}$

$$\Rightarrow \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} = \mathbf{B}$$

a Hence, $\mathbf{r} = \dfrac{1}{3}(2\mathbf{i} + \mathbf{j})e^{6t} + \dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j}$

b $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 6\mathbf{v} = (24\mathbf{i} + 12\mathbf{j})e^{6t}$

$$= 12e^{6t}(2\mathbf{i} + \mathbf{j})$$

$|\mathbf{a}| = 12e^{6t}\sqrt{2^2 + 1^2}$

$\qquad = 12\sqrt{5}e^{6t}$

So, $12\sqrt{5}e^{6t} = 100$

$$\Rightarrow t = \frac{1}{6}\ln\left(\frac{100}{12\sqrt{5}}\right) = 0.2195 \quad (3\,\text{s.f.})$$

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Exercise A, Question 4

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres and its velocity is $\mathbf{v}$ m s^{-1}. The motion of P is described by the differential equation

$$\frac{d\mathbf{v}}{dt} = 4\mathbf{v}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

a find $\mathbf{r}$ in terms of t,

b find the speed of P when $t = 2$,

c find the magnitude of the acceleration of P when $t = 2$.

Solution:

$$\frac{d\mathbf{v}}{dt} - 4\mathbf{v} = \mathbf{0} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = 0 \Rightarrow \lambda^2 - 4\lambda = 0$$

$$\Rightarrow \lambda = 0 \text{ or } 4$$

So, general solution is $\mathbf{r} = \mathbf{A} + \mathbf{B}e^{4t}$

a $t = 0, \mathbf{r} = \mathbf{i} - \mathbf{k} \Rightarrow \mathbf{i} - \mathbf{k} = \mathbf{A} + \mathbf{B}$ ①

$\mathbf{v} = 4\mathbf{B}e^{4t}$; $t = 0, \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{B} = \frac{1}{4}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\Rightarrow \mathbf{A} = \mathbf{i} - \mathbf{k} - \frac{1}{4}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= \frac{3}{4}\mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{5}{4}\mathbf{k} = \frac{1}{4}(3\mathbf{i} - \mathbf{j} - 5\mathbf{k})$

So, $\mathbf{r} = \frac{1}{4}\left\{ 3\mathbf{i} - \mathbf{j} - 5\mathbf{k} + (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{4t} \right\}$

b $\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{4t}$

When $t = 2, \mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{8}$

$|\mathbf{v}| = e^{8}\sqrt{3}$ m s^{-1}

c As $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{v}$

At $t = 2, |\mathbf{a}| = 4e^{8}\sqrt{3}$ m s^{-2}

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Exercise A, Question 5

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres. The motion of P is described by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}.$$

Given that the initial velocity of P is $(2\mathbf{i} - \mathbf{j})\text{m s}^{-1}$, find the speed of P at $t = \ln 3$.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} = 0 \Rightarrow \frac{d\mathbf{v}}{dt} - 2\mathbf{v} = 0$$

$$\Rightarrow \lambda - 2 = 0 \Rightarrow \lambda = 2$$

$$\Rightarrow \mathbf{v} = \mathbf{A}e^{2t}$$

$$t = 0, \mathbf{v} = 2\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{v} = (2\mathbf{i} - \mathbf{j})e^{2t}$$

$$t = \ln 3, \quad \mathbf{v} = (2\mathbf{i} - \mathbf{j})e^{2\ln 3}$$

$$= (18\mathbf{i} - 9\mathbf{j})$$

$$\therefore |\mathbf{v}| = 9\sqrt{2^2 + (-1)^2} = 9\sqrt{5}\text{ m s}^{-1}$$

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Exercise A, Question 6

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = (15\mathbf{i} + 10\mathbf{j})e^{3t}.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} + \mathbf{j}$, find

a $\mathbf{r}$ in terms of t,

b the velocity of P when $t = \ln 4$.

Solution:

a $\quad \dfrac{d\mathbf{r}}{dt} + 2\mathbf{r} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} e^{3t}$

Integrating factor $= e^{\int 2\,dt} = e^{2t}$

$$\frac{d}{dt}\left(e^{2t}\mathbf{r}\right) = \begin{pmatrix} 15 \\ 10 \end{pmatrix} e^{3t}$$

$$e^{2t}\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{5t} + \mathbf{C}$$

$$t = 0, \mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mathbf{c} \Rightarrow \mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{-2t}$$

b $\quad \mathbf{v} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} e^{3t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-2t}$

$$t = \ln 4, \mathbf{v} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} \times 64 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \times \frac{1}{16} = \begin{pmatrix} 576\frac{1}{8} \\ 384\frac{1}{8} \end{pmatrix}$$

velocity is $(576\frac{1}{8}\mathbf{i} + 384\frac{1}{8}\mathbf{j})$ m s^{-1}

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Exercise A, Question 7

Question:

The position vector $\mathbf{r}$ metres of a particle P at time t seconds satisfies the vector differential equation

$$\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{k}e^t.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{k}e^t$$

Integrating factor $= e^{\int 1 dt} = e^t$

$$\Rightarrow \frac{d}{dt}(\mathbf{r}e^t) = \mathbf{k}e^{2t}$$

$$\Rightarrow \mathbf{r}e^t = \int \mathbf{k}e^{2t} dt = \frac{1}{2}\mathbf{k}e^{2t} + \mathbf{c}$$

When $t = 0, \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$$\Rightarrow (2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = \frac{1}{2}\mathbf{k} + \mathbf{c}$$

$$\Rightarrow 2\mathbf{i} + 3\mathbf{j} + \frac{1}{2}\mathbf{k} = \mathbf{c}$$

$$\Rightarrow \mathbf{r} = \frac{1}{2}\mathbf{k}e^t + (2\mathbf{i} + 3\mathbf{j} + \frac{1}{2}\mathbf{k})e^{-t}$$

$$\mathbf{r} = 2e^{-t}\mathbf{i} + 3e^{-t}\mathbf{j} + \frac{1}{2}(e^t + e^{-t})\mathbf{k}$$

Alternative method

$$\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda + 1 = 0 \Rightarrow \lambda = -1$$

C.F. is $\mathbf{r} = \mathbf{A}e^{-t}$

P.I. Try $\mathbf{r} = \mathbf{B}e^t \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{B}e^t$

So, $\mathbf{B}e^t + \mathbf{B}e^t \equiv \mathbf{k}e^t \Rightarrow \mathbf{B} = \frac{1}{2}\mathbf{k}$

So P.I. is $\mathbf{r} = \frac{1}{2}\mathbf{k}e^t$

General solution is C.F. + P.I.

i.e. $\mathbf{r} = \mathbf{A}e^{-t} + \frac{1}{2}\mathbf{k}\,e^t$

$t = 0, \mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} = \mathbf{A} + \frac{1}{2}\mathbf{k}$

$$\Rightarrow \quad 2\mathbf{i} + 3\mathbf{j} + \frac{1}{2}\mathbf{k} = \mathbf{A}$$

So, $\mathbf{r} = (2\mathbf{i} + 3\mathbf{j} + \frac{1}{2}\mathbf{k})e^{-t} + \frac{1}{2}\mathbf{k}e^t$ as before

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Exercise A, Question 8

Question:

At time t the velocity $\mathbf{v}$ of a particle P satisfies the vector differential equation $\frac{d\mathbf{v}}{dt} + \frac{3\mathbf{v}}{T} = \mathbf{0}$, where T is a constant.

At time $t = 0$ the position vector of P is $a(2\mathbf{i} + \mathbf{j})$ and its velocity is $\frac{3a(\mathbf{i} - \mathbf{j})}{T}$.

Find the position vector of P at time t.

Solution:

$$\frac{d\mathbf{v}}{dt} + \frac{3}{T}\mathbf{v} = \mathbf{0} \Rightarrow \lambda + \frac{3}{T} = 0$$

$$\Rightarrow \lambda = \frac{-3}{T}$$

$$\therefore \mathbf{v} = \mathbf{A}e^{\frac{-3}{T}t}$$

$$\Rightarrow \mathbf{r} = -\frac{T}{3}\mathbf{A}e^{\frac{-3t}{T}} + \mathbf{B}$$

When $t = 0, \mathbf{v} = \frac{3a}{T}(\mathbf{i} - \mathbf{j}) \Rightarrow \frac{3a}{T}(\mathbf{i} - \mathbf{j}) = \mathbf{A}$

So, $\mathbf{r} = -a(\mathbf{i} - \mathbf{j})e^{\frac{-3t}{T}} + \mathbf{B}$

When $t = 0, \mathbf{r} = a(2\mathbf{i} + \mathbf{j})$

$$\Rightarrow a(2\mathbf{i} + \mathbf{j}) = -a(\mathbf{i} - \mathbf{j}) + \mathbf{B}$$

$$\Rightarrow a(3\mathbf{i}) = \mathbf{B}$$

So, $\mathbf{r} = 3a\mathbf{i} - a(\mathbf{i} - \mathbf{j})e^{\frac{-3t}{T}}$

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Exercise A, Question 9

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the vector differential equation

$$\frac{d\mathbf{r}}{dt} + 3\mathbf{r} = 4e^{-t}\mathbf{j}.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt} + 3\mathbf{r} = 4e^{-t}\mathbf{j} \quad \text{I.F.} = e^{\int 3\,dt} = e^{3t}$$

$$\Rightarrow \frac{d}{dt}(\mathbf{r}e^{3t}) = 4e^{2t}\mathbf{j}$$

$$\mathbf{r}e^{3t} = 2e^{2t}\mathbf{j} + \mathbf{A}$$

$$t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j} \Rightarrow 2\mathbf{i} - \mathbf{j} = 2\mathbf{j} + \mathbf{A}$$

$$\Rightarrow 2\mathbf{i} - 3\mathbf{j} = \mathbf{A}$$

$$\Rightarrow \mathbf{r} = 2e^{-t}\mathbf{j} + (2\mathbf{i} - 3\mathbf{j})e^{-3t}$$

$$= 2e^{-3t}\mathbf{i} + (2e^{-t} - 3e^{-3t})\mathbf{j}$$

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Exercise A, Question 10

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the vector differential equation

$$\frac{d\mathbf{r}}{dt} - \frac{2}{t}\mathbf{r} = 4\mathbf{j}, t > 0.$$

Given that when $t = 1, \mathbf{r} = \mathbf{i} - \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt} - \frac{2}{t}\mathbf{r} = 4\mathbf{j}$$

I.F. $e^{\int \frac{2}{t}dt} = e^{-2\ln t} = e\ln t^{-2} = \frac{1}{t^2}$

$$\frac{d}{dt}\left(\mathbf{r} \times \frac{1}{t^2}\right) = \frac{4}{t^2}\mathbf{j}$$

$$\frac{1}{t^2}\mathbf{r} = \int 4t^{-2}\mathbf{j}\, dt$$

$$= \frac{-4}{t}\mathbf{j} + \mathbf{c}$$

$t = 1, \mathbf{r} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = -4\mathbf{j} + \mathbf{c}$

$$\mathbf{c} = \mathbf{i} + 3\mathbf{j}$$

So $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})t^2 - 4t\mathbf{j}$

i.e. $\Rightarrow \mathbf{r} = t^2\mathbf{i} + (3t^2 - 4t)\mathbf{j}$

$Check \ \dfrac{d\mathbf{r}}{dt} = 2t\mathbf{i} + (6t - 4)\mathbf{j}$

$LHS = 2t\mathbf{i} + (6t - 4)\mathbf{j} - \dfrac{2}{t}\left\{t^2\mathbf{i} + (3t^2 - 4t)\mathbf{j}\right\}$

$= 4\mathbf{j} = RHS$

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Exercise B, Question 1

Question:

At time t seconds the position vector $\mathbf{r}$ metres of a particle P satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 4\frac{d\mathbf{r}}{dt} + 8\mathbf{r} = \mathbf{0}.$$

At $t = 0, \mathbf{r} = \mathbf{i} + \mathbf{j}$ and the velocity of P is $(2\mathbf{i} - 4\mathbf{j})\mathrm{m\ s}^{-1}$.

Find an expression for $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + 4\frac{d\mathbf{r}}{dt} + 8\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 + 4\lambda + 8 = 0$$

$$\Rightarrow (\lambda + 2)^2 = -4$$

$$\Rightarrow \lambda = -2 \pm 2i$$

General solution is $\mathbf{r} = e^{-2t}(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t)$

$$t = 0, \mathbf{r} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = \mathbf{A}$$

$$\frac{d\mathbf{r}}{dt} = e^{-2t}(-2\mathbf{A}\sin 2t + 2\mathbf{B}\cos 2t) - 2e^{-2t}(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t)$$

$$t = 0, \mathbf{v} = 2\mathbf{i} - 4\mathbf{j} \Rightarrow 2\mathbf{i} - 4\mathbf{j} = 2\mathbf{B} - 2\mathbf{A}$$

$$\Rightarrow \mathbf{B} = \mathbf{i} - 2\mathbf{j} + \mathbf{i} + \mathbf{j} = 2\mathbf{i} - \mathbf{j}$$

So, $\mathbf{r} = e^{-2t}[(\mathbf{i} + \mathbf{j})\cos 2t + (2\mathbf{i} - \mathbf{j})\sin 2t]$

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Exercise B, Question 2

Question:

The position vector **r** metres of a particle P at time t seconds satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 3\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0}.$$

At $t = 0$, the particle is at the point with position vector $2\mathbf{j}$ m moving with velocity $(\mathbf{i} + \mathbf{j})$m s^{-1}.

Find **r** in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 3\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda + 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -2 \text{ or } -1$$

General solution is

$$\mathbf{r} = \mathbf{A}e^{-2t} + \mathbf{B}e^{-t}$$

$$t = 0, \mathbf{r} = 2\mathbf{j} \Rightarrow 2\mathbf{j} = \mathbf{A} + \mathbf{B} \qquad ①$$

$$\frac{d\mathbf{r}}{dt} = -2\mathbf{A}e^{-2t} - \mathbf{B}e^{-t}$$

$$t = 0, \mathbf{v} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = -2\mathbf{A} - \mathbf{B} \qquad ②$$

$$① + ②: \quad \mathbf{i} + 3\mathbf{j} = -\mathbf{A} \Rightarrow \mathbf{A} = -\mathbf{i} - 3\mathbf{j}$$

$$\Rightarrow \mathbf{B} = \mathbf{i} + 5\mathbf{j}$$

So, $\mathbf{r} = (-\mathbf{i} - 3\mathbf{j})e^{-2t} + (\mathbf{i} + 5\mathbf{j})e^{-t}$

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Exercise B, Question 3

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{j}$, find the distance of P from the origin O when $t = 2$.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda + 1 = 0$$
$$\Rightarrow (\lambda - 1)^2 = 0$$
$$\Rightarrow \lambda = 1 \text{(twice)}$$

General solution is $\mathbf{r} = e^t(\mathbf{A} + \mathbf{B}t)$

$t = 0, \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{i} = \mathbf{A}$

$$\frac{d\mathbf{r}}{dt} = e^t\mathbf{B} + e^t(\mathbf{A} + \mathbf{B}t)$$

$t = 0, \mathbf{v} = \mathbf{j} \Rightarrow \mathbf{j} = \mathbf{B} + \mathbf{A} \Rightarrow \mathbf{B} = -\mathbf{i} + \mathbf{j}$

$\mathbf{r} = e^t[\mathbf{i} + t(-\mathbf{i} + \mathbf{j})]$

When $t = 2, \mathbf{r} = e^2(\mathbf{i} - 2\mathbf{i} + 2\mathbf{j})$
$$= e^2(-\mathbf{i} + 2\mathbf{j})$$
$$|\mathbf{r}| = e^2\sqrt{(-1)^2 + 2^2} = e^2\sqrt{5} \text{ m}$$

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Exercise B, Question 4

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0}.$$

Given that when $t = 0, \mathbf{r} = -2\mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \sqrt{3}\mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \left(\lambda + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

$$\Rightarrow \lambda = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

General solution is $\mathbf{r} = e^{-\frac{1}{2}t}\left(\mathbf{A}\cos\frac{\sqrt{3}}{2}t + \mathbf{B}\sin\frac{\sqrt{3}}{2}t\right)$

$t = 0, \mathbf{r} = -2\mathbf{i} \Rightarrow -2\mathbf{i} = \mathbf{A}$

$$\frac{d\mathbf{r}}{dt} = e^{-\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\mathbf{A}\sin\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\mathbf{B}\cos\frac{\sqrt{3}}{2}t\right) - \frac{1}{2}e^{-\frac{1}{2}t}\left(\mathbf{A}\cos\frac{\sqrt{3}}{2}t + \mathbf{B}\sin\frac{\sqrt{3}}{2}t\right)$$

$t = 0, \mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j} \Rightarrow \mathbf{i} + \sqrt{3}\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{B} - \frac{1}{2}\mathbf{A}$

$$\Rightarrow \mathbf{i} + \sqrt{3}\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{B} + \mathbf{i} \Rightarrow \mathbf{B} = 2\mathbf{j}$$

So, $\mathbf{r} = e^{-\frac{1}{2}t}\left(-2\cos\frac{\sqrt{3}}{2}t\,\mathbf{i} + 2\sin\frac{\sqrt{3}}{2}t\,\mathbf{j}\right)$

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Exercise B, Question 5

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{0}$ and $\dfrac{d\mathbf{r}}{dt} = 4\mathbf{i}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} = \mathbf{0} \Rightarrow \lambda^2 + 2\lambda = 0$$

$$\Rightarrow \lambda(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } -2$$

General Solution is $\mathbf{r} = \mathbf{A} + \mathbf{B}e^{-2t}$

$$t = 0, \mathbf{r} = \mathbf{0} \Rightarrow \mathbf{0} = \mathbf{A} + \mathbf{B} \qquad \textcircled{1}$$

$$\frac{d\mathbf{r}}{dt} = -2\mathbf{B}e^{-2t}$$

$$t = 0, \frac{d\mathbf{r}}{dt} = 4\mathbf{i} \Rightarrow 4\mathbf{i} = -2\mathbf{B} \Rightarrow \mathbf{B} = -2\mathbf{i}$$

$$\Rightarrow \mathbf{A} = 2\mathbf{i}$$

So, $\mathbf{r} = 2\mathbf{i} - 2\mathbf{i}e^{-2t}$

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Exercise B, Question 6

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + \mathbf{r} = 10e^{2t}\mathbf{i}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = 2\mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = \mathbf{A}\cos t + \mathbf{B}\sin t$$

For P.I. try $\mathbf{r} = \mathbf{C}e^{2t}$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{C}e^{2t} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} = 4\mathbf{C}e^{2t}$$

So $4\mathbf{C}e^{2t} + \mathbf{C}e^{2t} \equiv 10e^{2t}\mathbf{i}$

$$\Rightarrow \mathbf{C} = 2\mathbf{i}$$

$$\text{P.I. is } \mathbf{r} = 2\mathbf{i}e^{2t}$$

$\therefore$ general solution is $\mathbf{r} = \mathbf{A}\cos t + \mathbf{B}\sin t + 2e^{2t}\mathbf{i}$

$t = 0, \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{i} = \mathbf{A} + \mathbf{0} + 2\mathbf{i} \Rightarrow \mathbf{A} = -\mathbf{i}$

$$\frac{d\mathbf{r}}{dt} = -\mathbf{A}\sin t + \mathbf{B}\cos t + 4e^{2t}\mathbf{i}$$

$t = 0, \frac{d\mathbf{r}}{dt} = 2\mathbf{j} \Rightarrow 2\mathbf{j} = \mathbf{B} + 4\mathbf{i} \Rightarrow \mathbf{B} = -4\mathbf{i} + 2\mathbf{j}$

So $\mathbf{r} = -\cos t\mathbf{i} + \sin t(-4\mathbf{i} + 2\mathbf{j}) + 2e^{2t}\mathbf{i}$

i.e. $\mathbf{r} = (2e^{2t} - \cos t - 4\sin t)\mathbf{i} + 2\sin t\mathbf{j}$

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Exercise B, Question 7

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + 5\mathbf{r} = 10 \sin t\,\mathbf{i}.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + 5\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda + 5 = 0$$

$$\Rightarrow (\lambda - 1)^2 = -4$$

$$\Rightarrow \lambda = 1 \pm 2i$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t)$$

For P.I. try

$$\mathbf{r} = \mathbf{p}\sin t + \mathbf{q}\cos t$$

$$\Rightarrow \dot{\mathbf{r}} = \mathbf{p}\cos t - \mathbf{q}\sin t$$

$$\Rightarrow \ddot{\mathbf{r}} = -\mathbf{p}\sin t - \mathbf{q}\cos t$$

So, $-\mathbf{p}\sin t - \mathbf{q}\cos t - 2(\mathbf{p}\cos t - \mathbf{q}\sin t) + 5(\mathbf{p}\sin t + \mathbf{q}\cos t) \equiv 10\sin t\,\mathbf{i}$

$$4\mathbf{p}\sin t + 2\mathbf{q}\sin t + 4\mathbf{q}\cos t - 2\mathbf{p}\cos t \equiv 10\sin t\,\mathbf{i}$$

$$4\mathbf{p} + 2\mathbf{q} = 10\mathbf{i} \quad \text{and} \quad 4\mathbf{q} - 2\mathbf{p} = \mathbf{0} \Rightarrow \mathbf{p} = 2\mathbf{q}$$

$$10\mathbf{q} = 10i$$

$$\mathbf{q} = \mathbf{i} \Rightarrow \mathbf{p} = 2\mathbf{i}$$

So P.I. is $\mathbf{r} = 2\mathbf{i}\sin t + \mathbf{i}\cos t$

General solution is

$$\mathbf{r} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t) + 2\mathbf{i}\sin t + \mathbf{i}\cos t$$

$$t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j} \Rightarrow 2\mathbf{i} - \mathbf{j} = \mathbf{A} + \mathbf{i} \Rightarrow \mathbf{A} = \mathbf{i} - \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t) + e^t(-2\mathbf{A}\sin 2t + 2\mathbf{B}\cos 2t) + 2\mathbf{i}\cos t - \mathbf{i}\sin t$$

$$t = 0, \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = \mathbf{A} + 2\mathbf{B} + 2\mathbf{i}$$

So $\mathbf{i} + \mathbf{j} = \mathbf{i} - \mathbf{j} + 2\mathbf{B} + 2\mathbf{i}$

$$\mathbf{j} - \mathbf{i} = \mathbf{B}$$

So $\mathbf{r} = e^t[(\mathbf{i} - \mathbf{j})\cos 2t + (-\mathbf{i} + \mathbf{j})\sin 2t] + 2\mathbf{i}\sin t + \mathbf{i}\cos t$

i.e. $\mathbf{r} = \mathbf{i}(e^t\cos 2t - e^t\sin 2t + 2\sin t + \cos t) + \mathbf{j}(e^t\sin 2t - e^t\cos 2t)$

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Exercise B, Question 8

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} + 4\mathbf{r} = 8\mathbf{i}.$$

At $t = 0$, the particle is at the point with position vector $(2\mathbf{i} - \mathbf{k})$m moving with velocity $(\mathbf{i} + 2\mathbf{j})$ m s^{-1}.

Find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} + 4\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2 \text{ (twice)}$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = e^{2t}(\mathbf{A} + \mathbf{B}t)$$

For P.I. try $\mathbf{r} = \mathbf{C}$

$$\dot{\mathbf{r}} = \mathbf{0}$$

$$\ddot{\mathbf{r}} = \mathbf{0}$$

So $4\mathbf{C} = 8\mathbf{i} \Rightarrow \mathbf{C} = 2\mathbf{i}$

G.S. is $\mathbf{r} = e^{2t}(\mathbf{A} + \mathbf{B}t) + 2\mathbf{i}$

$t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{k} \Rightarrow 2\mathbf{i} - \mathbf{k} = \mathbf{A} + 2\mathbf{i} \Rightarrow \mathbf{A} = -\mathbf{k}$

$$\frac{d\mathbf{r}}{dt} = e^{2t}\mathbf{B} + 2e^{2t}(\mathbf{A} + \mathbf{B}t)$$

$t = 0, \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{i} + 2\mathbf{j} = \mathbf{B} + 2\mathbf{A} \Rightarrow \mathbf{B} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

So $\mathbf{r} = e^{2t}[-\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})] + 2\mathbf{i}$

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Exercise B, Question 9

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 4\mathbf{r} = 12t\mathbf{i} - 2\mathbf{j}.$$

At $t = 0$, the particle is at the point with position vector $(\mathbf{i} + \mathbf{k})$m moving with velocity $2\mathbf{j}$ m s^{-1}.

Find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 4\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$$

$\Rightarrow$ C.F. is $\mathbf{r} = \mathbf{A}e^{2t} + \mathbf{B}e^{-2t}$

For P.I. try $\mathbf{r} = \mathbf{C}t + \mathbf{D}$

$$\dot{\mathbf{r}} = \mathbf{C}$$

$$\ddot{\mathbf{r}} = \mathbf{0}$$

So $\mathbf{0} - 4(\mathbf{C}t + \mathbf{D}) \equiv 12t\mathbf{i} - 2\mathbf{j}$

$\Rightarrow -4\mathbf{C} = 12\mathbf{i} \Rightarrow \mathbf{c} = -3\mathbf{i}$

$-4\mathbf{D} = -2\mathbf{j} \Rightarrow \mathbf{D} = \dfrac{1}{2}\mathbf{j}$

So P.I. is $\mathbf{r} = -3t\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$\therefore$ G.S. is $\mathbf{r} = \mathbf{A}e^{2t} + \mathbf{B}e^{-2t} - 3t\mathbf{i} + \dfrac{1}{2}\mathbf{j}$

$t = 0, \mathbf{r} = \mathbf{i} + \mathbf{k} \Rightarrow \mathbf{i} + \mathbf{k} = \mathbf{A} + \mathbf{B} + \dfrac{1}{2}\mathbf{j}$

$\Rightarrow \mathbf{i} - \dfrac{1}{2}\mathbf{j} + \mathbf{k} = \mathbf{A} + \mathbf{B}$ ①

$$\frac{d\mathbf{r}}{dt} = 2\mathbf{A}e^{2t} - 2\mathbf{B}e^{-2t} - 3\mathbf{i}$$

$t = 0, \dfrac{d\mathbf{r}}{dt} = 2\mathbf{j} \Rightarrow 2\mathbf{j} = 2\mathbf{A} - 2\mathbf{B} - 3\mathbf{i}$

$\Rightarrow 3\mathbf{i} + 2\mathbf{j} = 2\mathbf{A} - 2\mathbf{B}$ ②

$2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = 2\mathbf{A} + 2\mathbf{B}$ ①

Add,

$5\mathbf{i} + \mathbf{j} + 2\mathbf{k} = 4\mathbf{A}$

$\dfrac{5}{4}\mathbf{i} + \dfrac{1}{4}\mathbf{j} + \dfrac{1}{2}\mathbf{k} = \mathbf{A}$

$\Rightarrow \dfrac{-1}{4}\mathbf{i} - \dfrac{3}{4}\mathbf{j} + \dfrac{1}{2}\mathbf{k} = \mathbf{B}$

So, $\mathbf{r} = \begin{pmatrix} \dfrac{5}{4} \\ \dfrac{1}{4} \\ \dfrac{1}{2} \end{pmatrix} e^{2t} + \begin{pmatrix} -\dfrac{1}{4} \\ -\dfrac{3}{4} \\ \dfrac{1}{2} \end{pmatrix} e^{-2t} + \begin{pmatrix} -3t \\ \dfrac{1}{2} \\ 0 \end{pmatrix}$

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Exercise B, Question 10

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} - 8\mathbf{r} = (9\mathbf{i} + 18\mathbf{j})e^t.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i} + 2\mathbf{j}$ and $\dfrac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} - 8\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda - 8 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 4 \text{ or } -2$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = \mathbf{A}e^{4t} + e^{-2t}\mathbf{B}$$

For P.I. try
$$\mathbf{r} = \mathbf{C}e^t$$
$$\dot{\mathbf{r}} = \mathbf{C}e^t$$
$$\ddot{\mathbf{r}} = \mathbf{C}e^t$$

$$\mathbf{C}e^t - 2\mathbf{C}e^t - 8\mathbf{C}e^t \equiv (9\mathbf{i} + 18\mathbf{j})e^t$$
$$-9\mathbf{C} = 9\mathbf{i} + 18\mathbf{j} \Rightarrow \mathbf{C} = -\mathbf{i} - 2\mathbf{j}$$

So P.I. is $\mathbf{r} = (-\mathbf{i} - 2\mathbf{j})e^t$

$\therefore$ G.S. is $\mathbf{r} = \mathbf{A}e^{4t} + \mathbf{B}e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$

$$t = 0, \mathbf{r} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{i} + 2\mathbf{j} = \mathbf{A} + \mathbf{B} + (-\mathbf{i} - 2\mathbf{j})$$
$$\Rightarrow 2\mathbf{i} + 4\mathbf{j} = \mathbf{A} + \mathbf{B} \qquad \text{①}$$

$$\frac{d\mathbf{r}}{dt} = 4\mathbf{A}e^{4t} - 2\mathbf{B}e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$$

$$t = 0, \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} \Rightarrow 2\mathbf{i} + \mathbf{j} = 4\mathbf{A} - 2\mathbf{B} - \mathbf{i} - 2\mathbf{j}$$
$$\Rightarrow 3\mathbf{i} + 3\mathbf{j} = 4\mathbf{A} - 2\mathbf{B} \qquad \text{②}$$

$$\begin{aligned}
2\mathbf{i} + 4\mathbf{j} &= \mathbf{A} + \mathbf{B} & \text{①} \\
3\mathbf{i} + 3\mathbf{j} &= 4\mathbf{A} - 2\mathbf{B} & \text{②} \\
4\mathbf{i} + 8\mathbf{j} &= 2\mathbf{A} + 2\mathbf{B} & \text{② } \times \text{ ①} \\
7\mathbf{i} + 11\mathbf{j} &= 6\mathbf{A}
\end{aligned}$$

$$\frac{7}{6}\mathbf{i} + \frac{11}{6}\mathbf{j} = \mathbf{A} \Rightarrow \mathbf{B} = \frac{5}{6}\mathbf{i} + \frac{13}{6}\mathbf{j}$$

$$\mathbf{r} = \left(\frac{7}{6}\mathbf{i} + \frac{11}{6}\mathbf{j}\right)e^{4t} + \left(\frac{5}{6}\mathbf{i} + \frac{13}{6}\mathbf{j}\right)e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$$

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Exercise C, Question 1

Question:

In each of the following cases calculate the work done by the force **F** as it moves its point of application through the displacement **d**:

a $\mathbf{F} = (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\,\mathrm{N},\, \mathbf{d} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\,\mathrm{m}$.

b $\mathbf{F} = (-4\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,\mathrm{N},\, \mathbf{d} = (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})\,\mathrm{m}$.

c $\mathbf{F} = (\mathbf{i} - 2\mathbf{k})\,\mathrm{N},\, \mathbf{d} = (\mathbf{j} - 3\mathbf{k})\,\mathrm{m}$.

Solution:

a $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 3 + 1 + 4 = 8\,\mathrm{J}$

b $\begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = -12 + 1 + 8 = -3\,\mathrm{J}$

c $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 6\,\mathrm{J}$

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Exercise C, Question 2

Question:

In each of the following cases calculate the work done by the force $\mathbf{F}$ as it moves its point of application from the point A with position vector $\mathbf{r}_A$ to the point B with position vector $\mathbf{r}_B$:

a $\mathbf{F} = (\mathbf{i}+\mathbf{j}-2\mathbf{k})\mathrm{N}, \mathbf{r}_A = (\mathbf{i}+\mathbf{j}-2\mathbf{k})\,\mathrm{m}, \mathbf{r}_B = (2\mathbf{i}-3\mathbf{j}+\mathbf{k})\,\mathrm{m}$.

b $\mathbf{F} = (2\mathbf{i}-\mathbf{j}+3\mathbf{k})\mathrm{N}, \mathbf{r}_A = (2\mathbf{i}-\mathbf{j}+\mathbf{k})\,\mathrm{m}, \mathbf{r}_B = (4\mathbf{i}-3\mathbf{j}-2\mathbf{k})\,\mathrm{m}$.

c $\mathbf{F} = (\mathbf{i}-\mathbf{k})\mathrm{N}, \mathbf{r}_A = (2\mathbf{i}-\mathbf{j})\,\mathrm{m}, \mathbf{r}_B = (3\mathbf{i}-\mathbf{j}+\mathbf{k})\,\mathrm{m}$.

Solution:

a $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = 1-4-6 = -9\,\mathrm{J}$

b $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = 4+2-9 = -3\,\mathrm{J}$

c $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1+0-1 = 0\,\mathrm{J}$

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Exercise C, Question 3

Question:

In each of the following cases a particle P of mass $0.5 \, \text{kg}$ is moved from the point A with position vector $\mathbf{r}_A$ to the point B with position vector $\mathbf{r}_B$ by a force $\mathbf{F}$. Assuming that in each case there are no other forces, apart from $\mathbf{F}$, doing work on P and that the speed of P at the point A is $4 \, \text{m s}^{-1}$, find in each case the speed of P when it reaches the point B:

a $\mathbf{F} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}, \mathbf{r}_A = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \, \text{m}, \mathbf{r}_B = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \, \text{m}$.

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j} - 3\mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) \, \text{m}, \mathbf{r}_B = (4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \, \text{m}$.

c $\mathbf{F} = (\mathbf{i} - \mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j}) \, \text{m}, \mathbf{r}_B = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \, \text{m}$.

Solution:

a   work done = K.E. gain

$$\begin{pmatrix} 1 \\ -1 \\ +2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$1 + 4 + 6 = \frac{1}{4}(v^2 - 16)$$

$$60 = v^2$$

$$\sqrt{60} \, \text{m s}^{-1} = v$$

b
$$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$4 + 2 + 9 = \frac{1}{4}(v^2 - 16)$$

$$76 = v^2$$

$$\sqrt{76} \, \text{m s}^{-1} = v$$

c
$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$1 + 0 - 1 = \frac{1}{4}(v^2 - 16)$$

$$16 = v^2$$

$$4 \, \text{m s}^{-1} = v$$

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Exercise C, Question 4

Question:

Forces of magnitudes 6 N, 7 N and 9 N act in the directions $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}, 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, and $7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, respectively. The three forces act on a particle causing a displacement of $(34\mathbf{i} + 10\mathbf{j})$m .

a Find the work done by each force.

b Verify that the total work done by all three forces is equal to the work done by the resultant force.

Solution:

$$\mathbf{F}_1 = 6 \times \frac{1}{\sqrt{2^2 + 2^2 + 1^2}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$$\mathbf{F}_2 = 7 \times \frac{1}{\sqrt{6^2 + (-3)^2 + 2^2}} \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$$

$$\mathbf{F}_3 = 9 \times \frac{1}{\sqrt{7^2 + 4^2 + (-4)^2}} \begin{pmatrix} 7 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -4 \end{pmatrix}$$

a
$$\begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 34 \\ 10 \\ 0 \end{pmatrix} = 136 + 40 = 176 \, \text{J}$$

$$\begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 34 \\ 10 \\ 0 \end{pmatrix} = 204 - 30 = 174 \, \text{J}$$

$$\begin{pmatrix} 7 \\ 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 34 \\ 10 \\ 0 \end{pmatrix} = 238 + 40 = 278 \, \text{J}$$

b Total work done = 628 J

$$\rho = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 17 \\ 5 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 17 \\ 5 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 34 \\ 10 \\ 0 \end{pmatrix} = 578 + 50 = 628 \, \text{J}$$

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Exercise C, Question 5

Question:

In each of the following cases the force $\mathbf{F}$ acts through the point with position vector $\mathbf{r}$ relative to the origin O. Find the vector moment of $\mathbf{F}$ about O.

a $\mathbf{F} = (\mathbf{i} + 2\mathbf{j})\text{N}, \mathbf{r} = (\mathbf{i} - \mathbf{j})\text{ m}$.

b $\mathbf{F} = (\mathbf{i} + 2\mathbf{k})\text{N}, \mathbf{r} = (2\mathbf{i} - \mathbf{k})\text{ m}$.

c $\mathbf{F} = (\mathbf{i} - \mathbf{j})\text{N}, \mathbf{r} = 3\mathbf{k}\text{ m}$.

d $\mathbf{F} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{N}, \mathbf{r} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\text{ m}$.

Solution:

a $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ Nm

b $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$ Nm

c $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ Nm

d $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$ Nm

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Exercise C, Question 6

Question:

In each of the following cases the force $\mathbf{F}$ acts through the point with position vector $\mathbf{r}$ relative to the origin O. Find the vector moment of $\mathbf{F}$ about the point A.

a $\mathbf{F} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\,\mathrm{N}, \mathbf{r} = (3\mathbf{i} - 2\mathbf{j})\,\mathrm{m}, A(0,1,0)$,

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j})\,\mathrm{N}, \mathbf{r} = (\mathbf{i} + 2\mathbf{j})\,\mathrm{m}, A(0,1)$,

c $\mathbf{F} = (\mathbf{i} + 2\mathbf{k})\,\mathrm{N}, \mathbf{r} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\,\mathrm{m}, A(2,0,2)$.

Solution:

a $\begin{pmatrix} 3-0 \\ -2-1 \\ 0-0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \mathrm{Nm}$

b $\begin{pmatrix} 1-0 \\ 2-1 \\ 0-0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \mathrm{Nm}$

c $\begin{pmatrix} 1-2 \\ 1-0 \\ -1-2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \mathrm{Nm}$

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Exercise C, Question 7

Question:

A force $\mathbf{F}$ acts through a point with position vector $\mathbf{p}$. Find, in terms of $\mathbf{F}$, $\mathbf{p}$ and $\mathbf{q}$, the vector moment of $\mathbf{F}$ about the point with position vector $\mathbf{q}$.

Solution:

$$(\mathbf{p} - \mathbf{q}) \times \mathbf{F} = \mathbf{p} \times \mathbf{F} - \mathbf{q} \times \mathbf{F}$$

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Exercise C, Question 8

Question:

In each of the following cases the force $\mathbf{F}$ has vector moment $\mathbf{n}$ about the origin O.
Find a vector equation of the line of action of $\mathbf{F}$ if

a $\mathbf{F} = (\mathbf{i} + \mathbf{j})\mathrm{N}, \mathbf{n} = 4\mathbf{k} \ \mathrm{Nm}$,

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j})\mathrm{N}, \mathbf{n} = (\mathbf{i} + 2\mathbf{j}) \ \mathrm{Nm}$,

c $\mathbf{F} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\mathrm{N}, \mathbf{n} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \ \mathrm{Nm}$.

Solution:

a　$\mathbf{r} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -z \\ z \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

$$\Rightarrow z = 0, x - y = 4$$

Let $y = 0, x = 4$

Equation is $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

b　$\mathbf{r} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} z \\ 2z \\ -x-2y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow z = 1, -x - 2y = 0$$

Let $y = 0, x = 0$

equation is $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

c　$\mathbf{r} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} -y-z \\ z+x \\ x-y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow -y - z = 3; z + x = -2; x - y = 1$$

Let $y = 0 \Rightarrow z = -3$ and $x = 1$

Equation is $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

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Exercise C, Question 9

Question:

In each of the following cases the force $\mathbf{F}$ acts through a point P. Find a vector equation of the axis through the point Q about which the moment of $\mathbf{F}$ is calculated.

a $\mathbf{F} = (\mathbf{i} - 2\mathbf{j})\mathrm{N}, P(0,1,0), Q(0,0,0),$

b $\mathbf{F} = (\mathbf{j} + 2\mathbf{k})\mathrm{N}, P(\mathbf{i} + \mathbf{k})\mathrm{m}, Q(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\mathrm{m},$

c $\mathbf{F} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\mathrm{N}, P(\mathbf{i} + \mathbf{j} + 2\mathbf{k})\mathrm{m}, Q(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})\mathrm{m}.$

Solution:

a
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{is equation of axis.}$$

b
$$\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} \quad \text{is equation of axis.}$$

c
$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix} \quad \text{is equation of axis.}$$

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Exercise C, Question 10

Question:

The moment of a non-zero force $\mathbf{F}$ about a point P is equal to its moment about another point Q. Show that the line of action of $\mathbf{F}$ is parallel to the line PQ.

Solution:

Let $\mathbf{T}$ be the position vector of a point on the line of action of $\mathbf{F}$.

Let $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$

Then the moment of $\mathbf{F}$ about $\mathbf{P}$ is $(\mathbf{T} - \mathbf{P}) \times \mathbf{F}$ and above Q is $(\mathbf{r} - \mathbf{q}) \times \mathbf{F}$

Therefore $(\mathbf{r} - \mathbf{p}) \times \mathbf{F} = (\mathbf{r} - \mathbf{q}) \times \mathbf{F} \Rightarrow \mathbf{T} \times \mathbf{F} - \mathbf{p} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} - \mathbf{q} \times \mathbf{F}$

$$\Rightarrow \mathbf{p} \times \mathbf{F} = \mathbf{q} \times \mathbf{F}$$

$$\Rightarrow \mathbf{p} \times \mathbf{F} - \mathbf{q} \times \mathbf{F} = 0$$

$$\Rightarrow (\mathbf{p} - \mathbf{q}) \times \mathbf{F} = 0$$

$$\Rightarrow \overrightarrow{QP} \times \mathbf{F} = 0 \quad \longleftarrow \quad \boxed{\text{Since } \mathbf{F} \neq 0, \overrightarrow{QP} \text{ must be parallel to } \mathbf{F}}$$

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Exercise D, Question 1

Question:

Prove that the following system of forces reduces to a couple:
$\mathbf{F_1} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\,\mathrm{N}$ acting at the point with position vector $\mathbf{r_1} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\mathrm{m}$,
$\mathbf{F_2} = (-3\mathbf{i} + \mathbf{j} - 3\mathbf{k})\,\mathrm{N}$ acting at the point with position vector $\mathbf{r_2} = (3\mathbf{i} + \mathbf{k})\mathrm{m}$,
$\mathbf{F_3} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\mathrm{N}$ acting at the point with position vector $\mathbf{r_3} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\mathrm{m}$,
$\mathbf{F_4} = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\mathrm{N}$ acting at the point with position vector $\mathbf{r_4} = (\mathbf{j} - 2\mathbf{k})\mathrm{m}$.

Solution:

$$\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} \qquad \text{Hence system is a couple of vector moment } (-\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})\mathrm{Nm}.$$

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Exercise D, Question 2

Question:

Prove that the following system of forces is in equilibrium:

$F_1 = (i - j)N$ acting at the point with position vector $r_1 = (i + k)m$,

$F_2 = (2j + k)N$ acting at the point with position vector $r_2 = (i - 2j)m$,

$F_3 = (-2i - j)N$ acting at the point with position vector $r_3 = (3i + j + k)m$,

$F_4 = (i - k)N$ acting at the point with position vector $r_4 = 2i \ m$.

Solution:

$$\sum F_i = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sum r_i \times F_i = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{Hence system is in equilibrium.}$$

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Exercise D, Question 3

Question:

A system consists of three forces $\mathbf{F_1}, \mathbf{F_2}$ and $\mathbf{F_3}$. The forces $\mathbf{F_1}$ and $\mathbf{F_2}$ act at the points $(2, -1, 0)$ and $(2, 0, 1)$ respectively. $\mathbf{F_1} = (\mathbf{i} - 2\mathbf{j})\text{N}; \mathbf{F_2} = (\mathbf{j} + \mathbf{k})\text{N}; \mathbf{F_3}$ has magnitude $\sqrt{11}\,\text{N}$ and acts along the line whose equation is $\mathbf{r} = 5\mathbf{i} - \mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.
Prove that the system reduces to a single force and find a vector equation for its line of action.

Solution:

$$\mathbf{F}_3 = \frac{1}{\sqrt{3^2 + 1^2 + (-1)^2}} \cdot \sqrt{11} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i \cdot \sum \mathbf{F}_i = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = 0$$

Hence the resultant is coplanar with the couple
Hence system reduces to a single force.

$$\mathbf{r} \times \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \text{ for a point } \mathbf{r} \text{ on the line of action}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 4z \\ -4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \Rightarrow z = 0, y = -1 \quad x = \lambda \text{ (anything)}$$

$$\therefore \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ is vector equation.}$$

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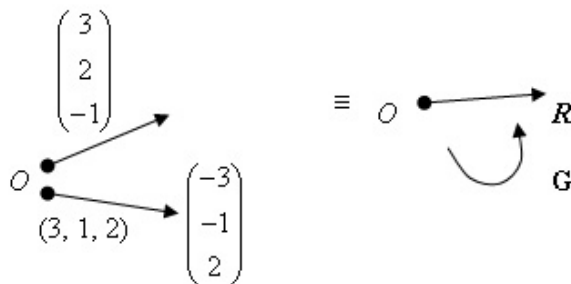
Exercise D, Question 4

Question:

The line of action of a force $(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})\mathrm{N}$ passes through the origin O and the line of action of a force $(-3\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,\mathrm{N}$ passes through the point with position vector $(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})\mathrm{m}$.

  a Reduce the system of two forces to a single force acting through the origin O together with a couple.

  b Find the magnitude of the couple.

Solution:

a

$$\mathbf{R} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Comparing moments about O,

$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix} = \mathbf{G}$$

$$\begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \mathbf{G}$$

Single force is $(\mathbf{j} + \mathbf{k})\mathrm{N}$

Couple has vector moment $(4\mathbf{i} - 12\mathbf{j})\mathrm{Nm}$

b $|\mathbf{G}| = \sqrt{4^2 + (-12)^2} = \sqrt{160} = 4\sqrt{10}\,\mathrm{Nm}$

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Exercise D, Question 5

Question:

Two forces $4\mathbf{j}$ N and $3\mathbf{k}$ N act through the points with position vectors $(\mathbf{i}+\mathbf{j})$ m and $(\mathbf{j}+\mathbf{k})$ m respectively. A third force acts through the point with position vector $(\mathbf{i}+\mathbf{k})$ m and is such that the three forces are equivalent to a couple. Find the vector moment and the magnitude of this couple.

Solution:

$$4\mathbf{j} + 3\mathbf{k} + \mathbf{F}_3 = \mathbf{0} \quad \text{(since system reduces to a couple)}$$
$$\Rightarrow \mathbf{F}_3 = (-4\mathbf{j} - 3\mathbf{k})\text{N}$$

$$\mathbf{G} = \sum \mathbf{r}_1 \times \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} \text{Nm} = (7\mathbf{i} + 3\mathbf{j})\text{Nm}$$

$$|\mathbf{G}| = \sqrt{7^2 + 3^2} = \sqrt{58} \text{ Nm}$$

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Exercise D, Question 6

Question:

In each of the following cases find the simplest system of forces which is equivalent to the given system:

a $F_1 = (-2i + 3j + k)N$ acting at the point with position vector $r_1 = (-i + 2j - k)m$,

    $F_2 = (-2j - k)N$ acting at the point with position vector $r_2 = 3j\,m$,

    $F_3 = (i - j + k)N$ acting at the point with position vector $r_3 = (i - 2k)m$.

b $F_1 = (2i - j)N$ acting at the point with position vector $r_1 = (i + 2j)m$,

    $F_2 = (3i + j)N$ acting at the point with position vector $r_2 = (2i + 3j)m$,

    $F_3 = (-i + j)N$ acting at the point with position vector $r_3 = -2j\,m$.

c   Forces PQ, QR and RP where the points P, Q and R have position vectors p, q and r respectively.

d   Forces AB, BC, CD and DA where the $ABCD$ is a regular tetrahedron.

e   Force $(i + 3j)N$ acting at the origin O and a couple of vector moment $3k\ Nm$.

Solution:

a $\quad \sum \mathbf{F}_i = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} +5 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence system reduces to a single force $(-\mathbf{i} + \mathbf{k})\mathrm{N}$ acting through the origin O.

b $\quad \sum \mathbf{F}_i = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -14 \end{pmatrix}$$

$$\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -14 \end{pmatrix} = 0$$

Hence resultant force is coplanar with the couple.
Point on line of action given by

$$\mathbf{r} \times \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -14 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -14 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -z \\ 4z \\ x - 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -14 \end{pmatrix} \qquad \Rightarrow \begin{cases} z = 0 \\ x - 4y = -14 \end{cases}$$

Let $y = 0 \Rightarrow x = -14$
Single force $(4\mathbf{i} + \mathbf{j})\mathrm{N}$

$\therefore$ Line of action has equation $\mathbf{r} = \begin{pmatrix} -14 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

c $\sum \mathbf{F}_i = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{PR} + \overrightarrow{RP} = \mathbf{0}$ Hence couple or equilibrium

$\sum \mathbf{r}_i \times \mathbf{F}_i = (\mathbf{p} \times \overrightarrow{PQ}) + (\mathbf{q} \times \overrightarrow{QR}) + (\mathbf{r} \times \overrightarrow{RP})$

$\qquad = \mathbf{p} \times (\mathbf{q} - \mathbf{p}) + \mathbf{p} \times (\mathbf{r} - \mathbf{q}) + \mathbf{r} \times (\mathbf{p} - \mathbf{r})$

$\qquad = \mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r} + \mathbf{r} \times \mathbf{p}$

Hence system reduces to a couple of vector moment $(\mathbf{p} \times \mathbf{q}) + (\mathbf{q} + \mathbf{r}) + (\mathbf{r} \times \mathbf{p})$

d $\sum \mathbf{F}_i = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{a} \times \overrightarrow{AB} + \mathbf{b} \times \overrightarrow{BC} + \mathbf{c} \times \overrightarrow{CD} + \mathbf{p} \times \overrightarrow{DA}$

$\qquad = \mathbf{a} \times (\mathbf{b} - \mathbf{a}) + \mathbf{b} \times (\mathbf{c} - \mathbf{b}) + \mathbf{c} \times (\mathbf{d} - \mathbf{c}) + \mathbf{d} \times (\mathbf{a} - \mathbf{d})$

$\qquad = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a}$

Hence system reduces to a couple of vector moment $(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a})$.

e $\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0} + 3\mathbf{k}$

$\qquad = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

Since $\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0$ force is coplanar with the couple.

Point on time of action given by

$\mathbf{r} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

$\qquad\qquad\qquad\qquad \begin{pmatrix} -3z \\ z \\ 3x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

$\Rightarrow z = 0, 3x - y = 3$

Let $x = 0, y = -3$

$\therefore$ Equation of line of action is $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

Hence system reduces to a force $(\mathbf{i} + 3\mathbf{j})N$ acting at the point $(0, -3, 0)$.

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Exercise D, Question 7

Question:

$F_1 = (3i - j + 2k)N$ acting at the point with position vector $r_1 = (3i - k)m$,

$F_2 = (-i - 4j + k)N$ acting at the point with position vector $r_2 = (2i - 4j)m$,

$F_3 = (i + j - 2k)N$ acting at the point with position vector $r_3 = (-3j + 5k)m$.

When a fourth force F_4 is added the system is in equilibrium.

a Find the force F_4.

b Find a vector equation of its line of action.

Solution:

a $\sum \mathbf{F}_i = \mathbf{0} \Rightarrow \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mathbf{F}_4 = 0$

$$\Rightarrow \mathbf{F}_4 = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} \text{N}$$

i.e. $\mathbf{F}_4 = (-3\mathbf{i} + 4\mathbf{j} - \mathbf{k})\text{N}$

b $\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ -9 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -12 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ -6 \\ -12 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -y - 4z \\ -3z + x \\ 4x + 3y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$\Rightarrow -y - 4z = 4$$
$$-3z + x = 6$$
$$4x + 3y = 12$$

Let $x = 0 \Rightarrow y = 4$ and $z = -2$

∴ Equation of line of action is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$$

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Exercise D, Question 8

Question:

A force $(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$N acts through the point $(-1, -1, 1)$. Show that this force is equivalent to an equal force acting through the origin together with a couple. Find the magnitude of this couple.

Solution:

$$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \mathbf{O} + \mathbf{G} \text{, comparing moments about } O.$$

$$\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \mathbf{G} \text{ so } |\mathbf{G}| = \sqrt{(-1)^2 + 3^2 + 2^2}$$

$$= \sqrt{14} \text{ Nm}$$

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Exercise D, Question 9

Question:

Prove that the following system of forces can be reduced to a force together with a non-coplanar couple.

$F_1 = (i + j)N$ acting at the point with position vector $r_1 = (3i + j + k)$ m ,

$F_2 = (i + k)N$ acting at the point with position vector $r_2 = i$ m ,

$F_3 = (2j - k)N$ acting at the point with position vector $r_3 = 2j$ m .

Solution:

$$\sum F_i = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$\sum r_i \times F_i = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

$$\sum F_i \cdot \sum r_i \times F_i = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} = -6 \neq 0$$

Hence, resultant force is *not* coplanar with couple.

So system can be reduced to a force $(2i + 3j)N$ acting through the origin O together with a couple of vector moment $(-3i + 2k)Nm$.

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Exercise D, Question 10

Question:

A force $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$N acts through the point $(2, 0, 0)$ together with a couple of vector moment $(2\mathbf{i}-\mathbf{k})$Nm.

a Show that this system cannot be reduced.
b Find an equivalent system where the force acts through the point $(1,-3,4)$.

Solution:

a　$\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

$\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 3 \neq 0$

Hence, resultant force is non-coplanar with the couple ∴ no further reduction possible.

b　Force $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ at $(2, 0, 0)$ + Couple $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \equiv$ Force $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ at $(1, -3, 4)$ + Couple $\mathbf{G}$

$(2,0,0)$　$(1,-3,4)$

Comparing moments about O,

From **a**,

$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mathbf{G}$

$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \mathbf{G}$

$\begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix} = \mathbf{G}$

Equivalent system is $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\text{N}$ acting at $(1, -3, 4)$ plus a couple of moment $(7\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})\text{Nm}$

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Exercise E, Question 1

Question:

The vertices of a tetrahedron $PQRS$ have position vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ respectively, where

$\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ $\mathbf{q} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\mathbf{r} = 4\mathbf{i} + \mathbf{k}$ $\mathbf{s} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Forces of magnitude 30 and $\sqrt{117}$ act along RQ and RS respectively. A third force acts at P. Given that the system reduces to a couple,

a find the magnitude of this couple,

b find the force acting at P,

c find a unit vector along the axis of the couple.

Solution:

Find the third force $\mathbf{F}_3$ first.

a $\overrightarrow{RQ} = \mathbf{q} - \mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$

$\mathbf{F}_1 = \dfrac{1}{5}\begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} \times 30 = \begin{pmatrix} 0 \\ 24 \\ -18 \end{pmatrix}$

$\overrightarrow{RS} = \mathbf{s} - \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$

$\mathbf{F}_2 = \dfrac{1}{\sqrt{13}}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}\sqrt{117} = \dfrac{1}{\sqrt{13}}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}3\sqrt{13} = \begin{pmatrix} -9 \\ -6 \\ 0 \end{pmatrix}$

$\sum \mathbf{F}_i = \begin{pmatrix} 0 \\ 24 \\ -18 \end{pmatrix} + \begin{pmatrix} -9 \\ -6 \\ 0 \end{pmatrix} + \mathbf{F}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_3 = \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$

Sum of moments about R:

$\overrightarrow{RP} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix} = \left\{ \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \right\} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$

$= \begin{pmatrix} -72 \\ 18 \\ 54 \end{pmatrix} = 18\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

$\therefore$ Magnitude $= 18\sqrt{26}$ Nm

b $\begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$

c   unit vector along axis $\dfrac{1}{\sqrt{26}}\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

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Exercise E, Question 2

Question:

Two forces $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})\text{N}$ and $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})\text{N}$ act at the points $D(4,-1,1)$ and $E(3, 1, 6)$ respectively.
a Find the force through the origin O and the couple which together are equivalent to these two forces.
b Find the magnitude of the couple.
c Show that the lines of action of the forces through D and E meet and find the position vector of the point of intersection.

Solution:

a $\sum \mathbf{F}_i = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 11 \end{pmatrix} + \begin{pmatrix} -9 \\ -3 \\ 5 \end{pmatrix}$

$= \begin{pmatrix} -12 \\ -4 \\ 16 \end{pmatrix} = \mathbf{G}$

Hence force is $(4\mathbf{i}+4\mathbf{j}+4\mathbf{k})\text{N}$ through O plus a couple of vector moment $(-12\mathbf{i}-4\mathbf{j}+16\mathbf{k})\text{Nm}$.

b $|\mathbf{G}| = 4\sqrt{(-3)^2 + (-1)^2 + 4^2} = 4\sqrt{26}\ \text{Nm}$

c $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow 4+3\lambda = 3+\mu \qquad ①$
$-1+2\lambda = 1+2\mu \qquad ②$
$1+\lambda = 6+3\mu \qquad ③$
$②-①:\ 0+3\lambda = 7+5\mu \qquad ④$
$①-④:\ 4 = -4-4\mu$
$-2 = \mu \Rightarrow \lambda = -1$

Point of intersection has position vector $(\mathbf{i}-3\mathbf{j})\text{m}$.

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Exercise E, Question 3

Question:

A system of five forces consists of the forces $\mathbf{F}_1 = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\,\mathrm{N}, \mathbf{F}_2 = (2\mathbf{i} + 6\mathbf{k})\,\mathrm{N}$ and $\mathbf{F}_3 = (\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})\,\mathrm{N}$, all acting through the origin O, together with a force $\mathbf{F}_4 = (\mathbf{i} - 2\mathbf{j} - \mathbf{k})\,\mathrm{N}$ acting through the point $(-2, 4, 2)$ and a force $\mathbf{F}_5 = (-\mathbf{i} - 2\mathbf{j} - 7\mathbf{k})\,\mathrm{N}$ acting through the point $(1, -2, -1)$.

a Reduce the system to a force $\mathbf{R}$ acting at O together with a couple $\mathbf{G}$.

b Hence, or otherwise, verify that the system is equivalent to a single force $(4\mathbf{i} - 8\mathbf{j} - 4\mathbf{k})\,\mathrm{N}$ acting through the point $(1, -1, 1)$.

Solution:

a
$$\mathbf{R} = \sum \mathbf{F}_i = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ -7 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$$

$$\mathbf{G} = \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ -7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 12 \\ 8 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 \\ 8 \\ -4 \end{pmatrix}$$

$$\mathbf{R} = (4\mathbf{i} - 8\mathbf{j} - 4\mathbf{k})\,\mathrm{N}; \mathbf{G} = (12\mathbf{i} + 8\mathbf{j} - 4\mathbf{k})\,\mathrm{Nm}$$

b
$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ -4 \end{pmatrix} = \mathbf{G} \quad \text{QED}$$

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Exercise E, Question 4

Question:

A system consists of three forces:

$\mathbf{F_1} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\mathrm{N}$ acting at the point with position vector $\mathbf{r_1} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\,\mathrm{m}$,

$\mathbf{F_2} = (\mathbf{i} + 3\mathbf{j} - \mathbf{k})\mathrm{N}$ acting at the point with position vector $\mathbf{r_2} = (\mathbf{j} + 2\mathbf{k})\,\mathrm{m}$,

$\mathbf{F_3} = (s\mathbf{i} + t\mathbf{j} + 2\mathbf{k})\mathrm{N}$ acting at the point with position vector $\mathbf{r_3} = \mathbf{k}\,\mathrm{m}$.

a Obtain, in terms of s and t, the equivalent system consisting of a single force $\mathbf{F}$ acting through the origin and a couple of moment $\mathbf{G}$.

b Determine the values of s and t such that $\mathbf{G}$ is parallel to $\mathbf{F}$.

Solution:

a
$$\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} s \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+s \\ 2+t \\ 3 \end{pmatrix} = \mathbf{F}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} s \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -t \\ s \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -8-t \\ -3+s \\ -3 \end{pmatrix} = \mathbf{G}$$

b $\mathbf{G}$ parallel to $\mathbf{F} \Rightarrow -(2+s) = -8-t \Rightarrow 6 = s-t$

$$-(2+t) = -3+s \Rightarrow 1 = s+t$$

$$\Rightarrow s = \frac{7}{2}; t = \frac{-5}{2}$$

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Exercise E, Question 5

Question:

A force $\mathbf{F_1}$ of magnitude 26 N, acts along the direction of the vector $(4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k})$.
Given that the line of action of $\mathbf{F_1}$ passes through the point $(2, 1, -1)$,
a find the moment of $\mathbf{F_1}$ about O.
A bead moves along a smooth straight wire from the point $P(3, -2, 1)$ to the point
$Q(5, -22, 2)$, under the influence of $\mathbf{F_1}$ and the reaction from the wire only.
b Find the work done by $\mathbf{F_1}$ in this motion.

Solution:

a $\mathbf{F_1} = \dfrac{1}{13}\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \times 26 = \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 18 \\ -56 \\ -20 \end{pmatrix}$ Nm $= (18\mathbf{i} - 56\mathbf{j} - 20\mathbf{k})$Nm

b $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -20 \\ 1 \end{pmatrix}$

work done $= \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -20 \\ 1 \end{pmatrix} = 16 + 120 + 24 = 160$ J

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Exercise E, Question 6

Question:

A system consists of three forces:
$F_1 = (4i + j + 2k)N$ acting at the point with position vector $r_1 = (6i + 4j + k)$ m ,
$F_2 = (i - 2j + k)N$ acting at the point with position vector $r_2 = (i + 5j - 2k)$ m ,
$F_3 = (-5i + j - 3k)N$ acting at the point with position vector $r_3 = (i + j + k)$ m .

a Show that this system reduces to a couple and find its magnitude.
The force F_3 is now removed from the system and replaced by the force F_4 such
that the forces F_1, F_2 and F_4 are in equilibrium. Find
b the magnitude of F_4,
c a vector equation for the line of action of F_4.

Solution:

a　$\sum \mathbf{F}_i = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix}$

$\quad\quad = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix}$

$\quad\quad = \begin{pmatrix} 7 \\ -8 \\ -10 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -7 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ 6 \end{pmatrix}$

$\quad\quad = \begin{pmatrix} 4 \\ -13 \\ -11 \end{pmatrix} = \mathbf{G}$

$\quad |\mathbf{G}| = \sqrt{4^2 + (-13)^2 + (-11)^2} = \sqrt{16 + 169 + 121} = \sqrt{306} \text{ Nm}$

b　$\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \mathbf{F}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\mathbf{F}_4 = \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix} \Rightarrow |\mathbf{F}_4| = \sqrt{25 + 1 + 9} = \sqrt{35} \text{ N}$

c　From **a**,

$\sum \mathbf{r}_i \times \mathbf{F}_1 = \begin{pmatrix} 7 \\ -8 \\ -10 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ -7 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \\ 17 \end{pmatrix}$

$\begin{pmatrix} -3y - z \\ -5z + 3x \\ x + 5y \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \\ 17 \end{pmatrix}$

Put $y = 0 : x = 17, z = 8$

Equation of line of action of $\mathbf{F}_4$ is $\mathbf{r} = \begin{pmatrix} 17 \\ 0 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 1 \\ -3 \end{pmatrix}$

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Exercise E, Question 7

Question:

Three forces of magnitudes 26, $4\sqrt{41}$ and 15 N act respectively along the sides OP, PQ and QO of the triangle POQ where O is the origin. Relative to O, the coordinates of P and Q are $(5, 12, 0)$ and $(3, 0, 4)$ respectively.

a Show that the resultant is $(-3\mathbf{i} - 4\mathbf{k})$N.

b Find the magnitude of the moment of the resultant about O.

Solution:

a $\quad \mathbf{F}_1 = \dfrac{1}{13}\begin{pmatrix} 5 \\ 12 \\ 0 \end{pmatrix} \times 26 = \begin{pmatrix} 10 \\ 24 \\ 0 \end{pmatrix}$ N

$$\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 12 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \\ 4 \end{pmatrix}$$

$$\mathbf{F}_2 = \frac{1}{\sqrt{(-2)^2 + (-12)^2 + 4^2}}\begin{pmatrix} -2 \\ -12 \\ 4 \end{pmatrix} \cdot 4\sqrt{41}$$

$$= \frac{1}{\sqrt{164}}\begin{pmatrix} -2 \\ -12 \\ 4 \end{pmatrix} 4\sqrt{41} = \begin{pmatrix} -4 \\ -24 \\ 8 \end{pmatrix}$$ N

$$\overrightarrow{QO} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$$

$$\mathbf{F}_3 = \frac{1}{5}\begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix} \cdot 15 = \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix}$$ N

$$\sum \mathbf{F}_i = \begin{pmatrix} 10 \\ 24 \\ 0 \end{pmatrix} + \begin{pmatrix} -4 \\ -24 \\ 8 \end{pmatrix} + \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix} = (-3\mathbf{i} - 4\mathbf{k})$$N

b $\quad \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 5 \\ 12 \\ 0 \end{pmatrix} \times \begin{pmatrix} -4 \\ -24 \\ 8 \end{pmatrix} = \begin{pmatrix} 96 \\ -40 \\ -72 \end{pmatrix} = \mathbf{G}$

$$|\mathbf{G}| = 8\sqrt{12^2 + (-5)^2 + (-9)^2}$$
$$= 8\sqrt{144 + 25 + 81}$$
$$= 8\sqrt{250}$$
$$= 40\sqrt{10} \text{ Nm}$$

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Exercise E, Question 8

Question:

The point A has coordinates $(2,-5,1)$, the point B has coordinates $(-8,-1,4)$, the point C has coordinates $(0,-13,5)$ and the point D has coordinates $(4,3,-3)$.

a Show that the lines AB and CD intersect at right angles.

b Find the coordinates of the point of intersection.

A force of magnitude F acting in the direction AD moves a particle from B to D.

c Find the work done by the force.

Solution:

a $\overrightarrow{AB} = \begin{pmatrix} -8-2 \\ -1+5 \\ 4-1 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix}$

$\overrightarrow{CD} = \begin{pmatrix} 4-0 \\ 3+13 \\ -3-5 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix} = -40 + 64 - 24 = 0$

$\therefore$ perpendicular

b $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -13 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix}$

$\Rightarrow 2 - 10\lambda = 4\mu \qquad$ ①

$-5 + 4\lambda = -13 + 16\mu \qquad$ ②

$1 + 3\lambda = 5 - 8\mu \qquad$ ③

$2 + 6\lambda = 10 - 16\mu \qquad$ ③×3

$-3 + 10\lambda = -3$

$\lambda = 0 \Rightarrow \mu = \dfrac{1}{2}$

Point of intersection is $(2, -5, 1)$

c $\mathbf{F} = F \times$ unit vector along $\overrightarrow{AD}$

$\overrightarrow{AD} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \quad |\overrightarrow{AD}| = \sqrt{2^2 + 8^2 + (-4)^2}$

$= \sqrt{4 + 64 + 16}$

$= \sqrt{84}$

So, $\mathbf{F} = \dfrac{F}{\sqrt{84}} \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}; \quad \overrightarrow{BD} = \begin{pmatrix} 4+8 \\ 3+1 \\ -3-4 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -7 \end{pmatrix}$

work done $= \dfrac{F}{\sqrt{84}} \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 4 \\ -7 \end{pmatrix}$

$= \dfrac{F}{\sqrt{84}} (24 + 32 + 28) = F\sqrt{84} = 2F\sqrt{21}$

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Exercise E, Question 9

Question:

The force $\mathbf{F} = (X\mathbf{i} + Y\mathbf{j})\text{N}$ acts through the point with position vector $\mathbf{r} = (x\mathbf{i} - y\mathbf{j})$ m .

a Prove that this force is equivalent to an equal force at O together with a couple.

Three variable forces $\mathbf{F}_1 = 2\cos t\mathbf{i}$, $\mathbf{F}_2 = \cos t\mathbf{i} + 2\sin t\mathbf{j}$, $\mathbf{F}_3 = 3\sin t\mathbf{i} + \cos t\mathbf{j}$ act at the points with position vectors $\mathbf{O}$, $\mathbf{i} + \mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.

If the system is reduced to a single force $\mathbf{R}$ acting at O with a couple $\mathbf{G}$,

b find the values of $\mathbf{R}$ and $\mathbf{G}$.

c reduce an equation of the line of action of the resultant.

d Show that this line passes through a fixed point which is independent of t.

Solution:

a $\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \times \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xY + yX \end{pmatrix} = \mathbf{G}$

Couple is $(xY + yX)\mathbf{k}$ Nm .

b $\mathbf{R} = \sum \mathbf{F}_i = \begin{pmatrix} 2\cos t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t \\ 2\sin t \\ 0 \end{pmatrix} + \begin{pmatrix} 3\sin t \\ \cos t \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix} \text{N}$

$\mathbf{G} = \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos t \\ 2\sin t \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3\sin t \\ \cos t \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ 2\sin t - \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -3\cos t - 6\sin t \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$

$\begin{pmatrix} -z(2\sin t + \cos t) \\ z(3\cos t + 3\sin t) \\ x(2\sin t + \cos t) - y(3\cos t + 3\sin t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$

$\Rightarrow 2x - 3y = -4; \ x - 3y = -4; \ z = 0$

If $x = 0, y = \dfrac{4}{3}, z = 0 \ (\forall \text{ (for all) } t)$

Hence line of action is $\mathbf{r} = \begin{pmatrix} 0 \\ \dfrac{4}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix}$

d This line passes through the fixed point $\begin{pmatrix} 0 \\ \dfrac{4}{3} \\ 0 \end{pmatrix}$ which is independent of t.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 10

Question:

A force of unit magnitude has equal vector moments about points with position vectors $\mathbf{j}$ m and $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ m. Find the possible forces.

Solution:

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{pmatrix} -z-y \\ x+z \\ -y+x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow y = -z$$
$$x = -z$$
$$\Rightarrow x = y = -z$$

and $x^2 + y^2 + z^2 = 1$
$$3x^2 = 1$$
$$x = \pm\frac{1}{\sqrt{3}}$$

$$x = \frac{1}{\sqrt{3}}; y = \frac{1}{\sqrt{3}}; z = -\frac{1}{\sqrt{3}}$$

$$x = \frac{-1}{\sqrt{3}}; y = \frac{-1}{\sqrt{3}}; z = \frac{1}{\sqrt{3}}$$

Possible forces $\pm\dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})$N