Solutionbank M4
Edexcel AS and A Level Modular Mathematics

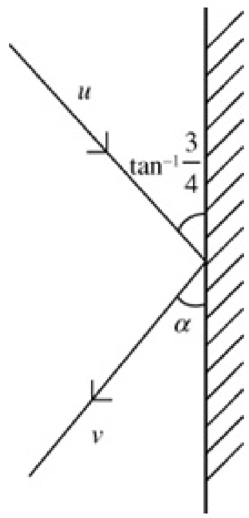
Elastic collisions in two dimensions
Exercise A, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1}\frac{3}{4}$ with the wall. The coefficient of restitution between S and the wall is $\frac{1}{3}$.

Find the speed of S immediately after the collision.

Solution:



R $\uparrow$: $v\cos\alpha = u \times \frac{4}{5}$

law of restitution $\leftrightarrow$: $v\sin\alpha = e \times u \times \frac{3}{5} = \frac{1}{3} \times u \times \frac{3}{5} = u \times \frac{1}{5}$

squaring and adding,

$v^2 = u^2\left(\frac{16}{25} + \frac{1}{25}\right) = u^2 \times \frac{17}{25}$

$v = \frac{u\sqrt{17}}{5}$

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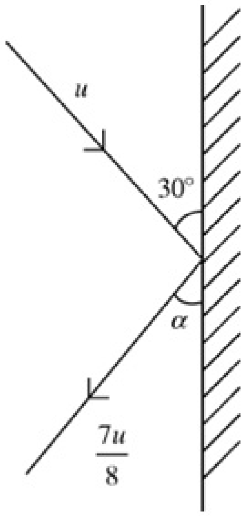
Elastic collisions in two dimensions
Exercise A, Question 2

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $30°$ with the wall. Immediately after the collision the speed of S is $\frac{7}{8}u$.

Find the coefficient of restitution between S and the wall.

Solution:



R $\uparrow$: $\frac{7u}{8}\cos\alpha = u\cos 30°$

law of restitution $\leftrightarrow$: $\frac{7u}{8}\sin\alpha = eu\sin 30°$

squaring and adding:

$$\frac{49u^2}{64} = u^2\left(\frac{3}{4} + \frac{e^2}{4}\right)$$

$$\frac{49}{16} = 3 + e^2$$

$$\frac{1}{16} = e^2, e = \frac{1}{4}$$

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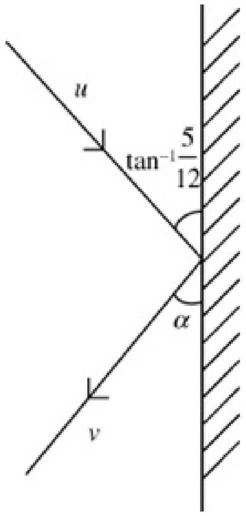
Elastic collisions in two dimensions
Exercise A, Question 3

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1}\frac{5}{12}$ with the wall. The coefficient of restitution between S and the wall is $\frac{3}{5}$.

Find the speed of S immediately after the collision.

Solution:



R ↑: $v\cos\alpha = u \times \frac{12}{13}$

law of restitution ↔: $v\sin\alpha = e \times u \times \frac{5}{13} = \frac{3}{5} \times u \times \frac{5}{13} = u \times \frac{3}{13}$

squaring and adding,

$$v^2 = u^2\left(\frac{144}{169} + \frac{9}{169}\right) = u^2 \times \frac{153}{169}$$

$$v = \frac{3\sqrt{17}u}{13}$$

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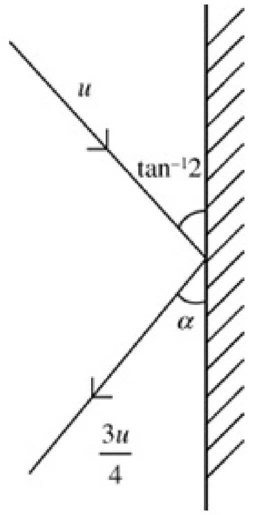
Elastic collisions in two dimensions
Exercise A, Question 4

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $\tan^{-1} 2$ with the wall. Immediately after the collision the speed of S is $\frac{3}{4}u$.

Find the coefficient of restitution between S and the wall.

Solution:



R $\uparrow$: $\frac{3u}{4}\cos\alpha = u \times \frac{1}{\sqrt{5}}$

law of restitution $\leftrightarrow$: $\frac{3u}{4}\sin\alpha = eu \times \frac{2}{\sqrt{5}}$

squaring and adding:

$$\frac{9u^2}{16} = u^2\left(\frac{1}{5} + \frac{4e^2}{5}\right)$$

$$\frac{45}{16} = 1 + 4e^2$$

$$\frac{29}{16} = 4e^2, e = \frac{\sqrt{29}}{8}$$

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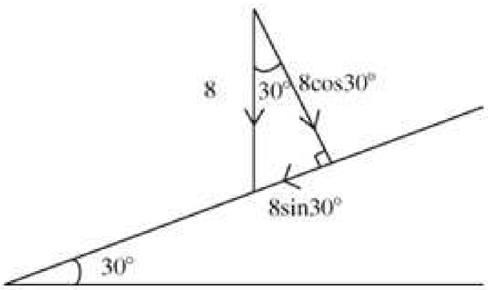
Elastic collisions in two dimensions
Exercise A, Question 5

Question:

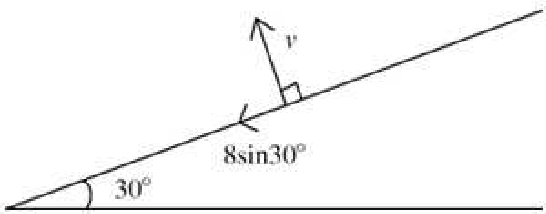
A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle $30°$ to the horizontal. Immediately before striking the plane the ball has speed 8 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$. Find the exact value of the speed of the ball immediately after the impact.

Solution:

Before the impact



After the impact



The component of velocity parallel to the slope $= 8\sin 30° = 8 \times \frac{1}{2} = 4$

Perpendicular to the slope:

$v = e \times 8\cos 30° = \frac{1}{4} \times 8 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Therefore the speed immediately after impact $= \sqrt{4^2 + \sqrt{3}^2} = \sqrt{19} \text{ m s}^{-1}$

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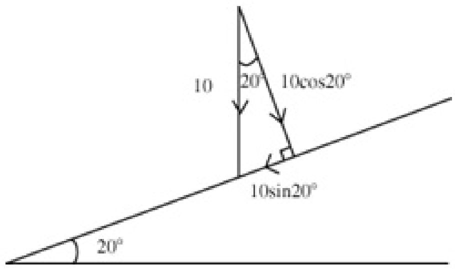
Elastic collisions in two dimensions
Exercise A, Question 6

Question:

A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle $20°$ to the horizontal. Immediately before striking the plane the ball has speed 10 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{2}{5}$. Find the speed, to 3 significant figures, of the ball immediately after the impact.

Solution:

Before the impact



The component of velocity parallel to the slope $= 10\sin 20°$

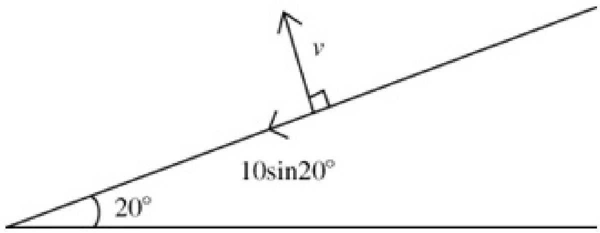
Perpendicular to the slope:

$$v = e \times 10\cos 20° = \frac{2}{5} \times 10\cos 20° = 4\cos 20°$$

Therefore the speed immediately after impact $= \sqrt{(10\sin 20°)^2 + (4\cos 20°)^2}$

$$= \sqrt{25.826\ldots} = 5.08\text{ m s}^{-1}$$

After the impact



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Elastic collisions in two dimensions
Exercise A, Question 7

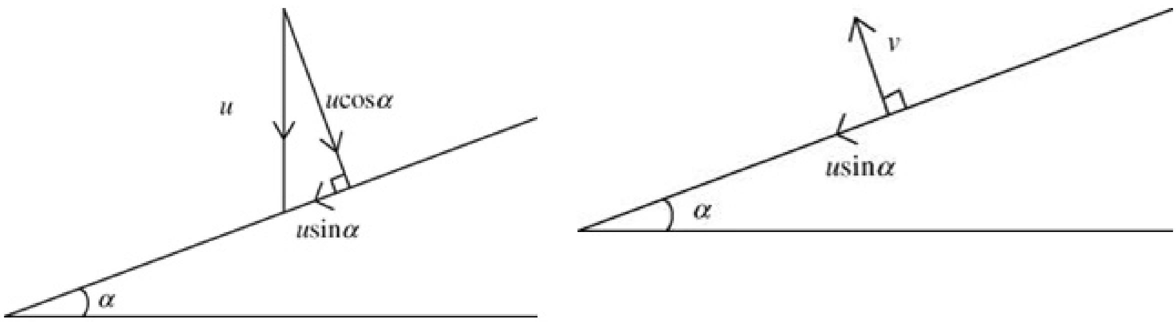
Question:

A small smooth ball of mass 750 g is falling vertically. The ball strikes a smooth plane which is inclined at an angle $45°$ to the horizontal. Immediately before striking the plane the ball has speed $5\sqrt{2}$ m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$. Find

a the speed, to 3 significant figures, of the ball immediately after the impact,
b the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the impact After the impact



a The component of velocity parallel to the slope $= u\sin\alpha = 5\sqrt{2}\sin 45° = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5$

Perpendicular to the slope:

$$v = e \times u\cos\alpha = \frac{1}{2} \times 5\sqrt{2}\cos 45°$$
$$= \frac{1}{2} \times 5\sqrt{2} \times \frac{1}{\sqrt{2}} = \frac{5}{2}$$

Therefore the speed immediately after impact $= \sqrt{5^2 + 2.5^2} = \sqrt{31.25} = 5.59$ m s^{-1}

b The impulse is perpendicular to the surface:

$$I = \frac{3}{4}\left(\frac{5}{2} - (-5)\right) = \frac{3}{4} \times \frac{15}{2} = 5.625 \text{ Ns}$$

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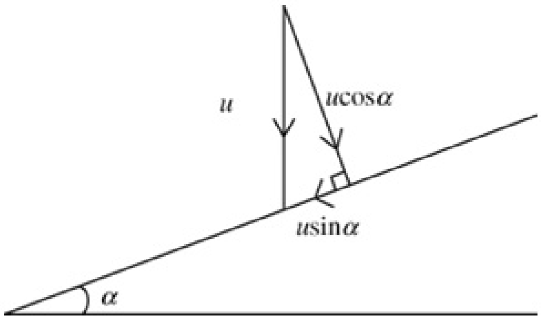
Elastic collisions in two dimensions
Exercise A, Question 8

Question:

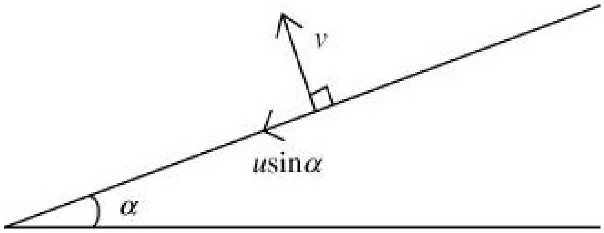
A small smooth ball is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. Immediately before striking the plane the ball has speed 7.5 m s^{-1}. Immediately after the impact the ball has speed 5 m s^{-1}.
Find the coefficient of restitution to 2 significant figures, between the ball and the plane.

Solution:

Before the impact



After the impact



The component of velocity parallel to the slope $= u\sin\alpha = 7.5 \times \frac{3}{5} = 4.5$

Perpendicular to the slope:

$v = eu\cos\alpha = e \times 7.5 \times \frac{4}{5} = 6e$

Combining the two components:

$5^2 = 4.5^2 + 36e^2$

$$\therefore e^2 = \frac{25 - 20.25}{36} = 0.1319\ldots$$

$e = 0.36$

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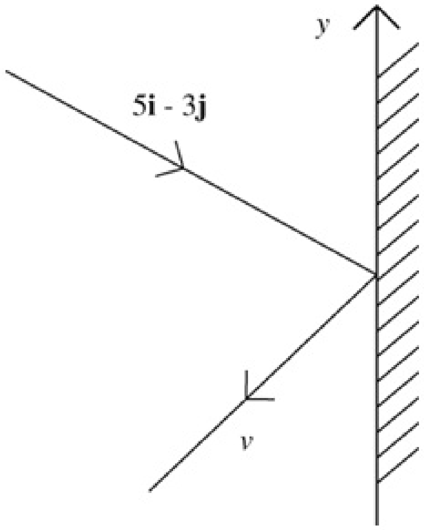
Elastic collisions in two dimensions
Exercise A, Question 9

Question:

A small smooth ball of mass 800 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the y-axis. The velocity of the ball just before impact is $(5\mathbf{i}-3\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

a the velocity of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i}+q\mathbf{j}$

$\uparrow -3=q$

$\leftrightarrow -p=e\times 5=\frac{5}{2}$

so $v=-2.5\mathbf{i}-3\mathbf{j}$

b $\text{K.E. before}=\frac{1}{2}\times\frac{4}{5}\times(5^2+3^2)=13.6$

$\text{K.E. after}=\frac{1}{2}\times\frac{4}{5}\times\left(2.5^2+3^2\right)=6.1$

$\text{K.E. lost}=13.6-6.1=7.5\text{ J}$

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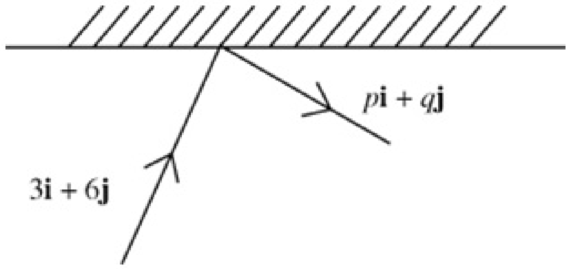
Elastic collisions in two dimensions
Exercise A, Question 10

Question:

A small smooth ball of mass 1 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the x-axis. The velocity of the ball just before impact is $(3\mathbf{i}+6\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

a the speed of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that the velocity of the ball immediately after the impact is $p\mathbf{i}+q\mathbf{j}$

$\leftrightarrow 3 = p$ (parallel to the wall)

$\uparrow -q = \frac{1}{3}\times 6 = 2$ (perpendicular to the wall)

Speed $= \sqrt{3^2+2^2} = \sqrt{13}\text{ m s}^{-1}$.

b K.E. before impact $= \frac{1}{2}\times 1\times(3^2+6^2) = 22.5$, K.E. after impact $= \frac{1}{2}\times 1\times 13 = 6.5$

K.E. lost $= 22.5 - 6.5 = 16\text{ J}$

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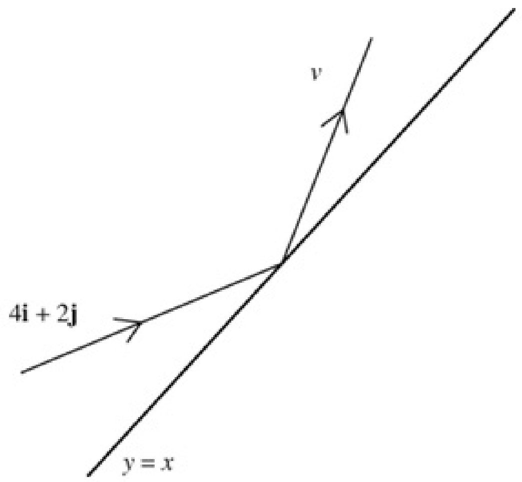
Elastic collisions in two dimensions
Exercise A, Question 11

Question:

A small smooth ball of mass 2 kg is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $y = x$. The velocity of the ball just before impact is $(4\mathbf{i} + 2\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find

a the velocity of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



Suppose that $v = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}$ is parallel to the wall and $\mathbf{b}$ is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector in the direction of the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

$$\nearrow \left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \mathbf{a}$$

$$= \frac{1}{\sqrt{2}} \times 6 \times \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = 3\mathbf{i}+3\mathbf{j}$$

$$\nwarrow \frac{1}{3}\left[(4\mathbf{i}+2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = \mathbf{b}$$

$$= \frac{1}{3} \times \frac{1}{\sqrt{2}} \times (4-2) \times \frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$

$$= \frac{1}{3}(-\mathbf{i}+\mathbf{j})$$

So $\mathbf{v} = 3\mathbf{i}+3\mathbf{j} - \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} = \frac{8}{3}\mathbf{i} + \frac{10}{3}\mathbf{j}$

b K.E. before $= \frac{1}{2} \times 2 \times \left(4^2 + 2^2\right) = 20$, K.E. after $= \frac{1}{2} \times 2 \times \left(\frac{64}{9} + \frac{100}{9}\right) = \frac{164}{9}$

K.E. lost $= 20 - \frac{164}{9} = \frac{16}{9}$ J

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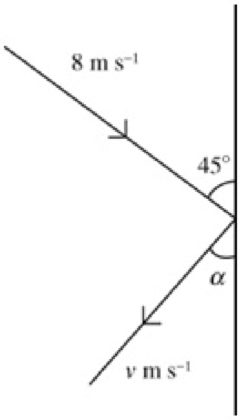
Elastic collisions in two dimensions
Exercise A, Question 12

Question:

A smooth snooker ball strikes a smooth cushion with speed 8 m s^{-1} at an angle of $45°$ to the cushion. Given that the coefficient of restitution between the ball and the cushion is $\frac{2}{5}$, find the magnitude and direction of the velocity of the ball after the impact.

Solution:



↑: no change parallel to the cushion

$8\cos 45° = v\cos\alpha = 4\sqrt{2}$

↔: Using the law of restitution

$\frac{2}{5} \times 8\sin 45° = v\sin\alpha = \frac{8\sqrt{2}}{5}$

Squaring and adding:

$v^2 = (4\sqrt{2})^2 + \left(\frac{8\sqrt{2}}{5}\right)^2 = 37.12, v \approx 6.09$

Dividing: $\tan\alpha = \frac{\frac{8\sqrt{2}}{5}}{4\sqrt{2}} = \frac{2}{5}, \alpha = \tan^{-1} 0.4$

Velocity is 6.09 m s^{-1} at $21.8°$ to the cushion.

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Elastic collisions in two dimensions
Exercise A, Question 13

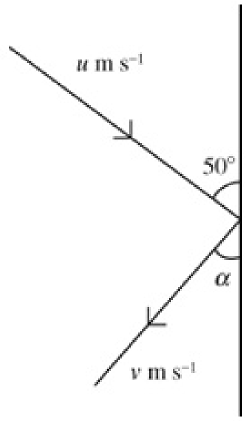
Question:

A smooth snooker ball strikes a smooth cushion with speed u m s^{-1} at an angle of $50°$ to the cushion. The coefficient of restitution between the ball and the cushion is e.

a Show that the angle between the cushion and the rebound direction is independent of u.

b Find the value of e given that the ball rebounds at right angles to its original direction.

Solution:



a $\uparrow$: no change parallel to the cushion

$u\cos 50° = v\cos\alpha$

$\leftrightarrow$: Using the law of restitution, $e \times u\sin 50° = v\sin\alpha$

Dividing: $\dfrac{v\sin\alpha}{v\cos\alpha} = \dfrac{eu\sin 50°}{u\cos 50°}$

$\Rightarrow \tan\alpha = e\tan 50°$, which is independent of the value of u

b If $\alpha = 40°$ then $\tan 40° = e\tan 50°$

$$e = \frac{\tan 40°}{\tan 50°} \approx 0.7$$

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Elastic collisions in two dimensions
Exercise A, Question 14

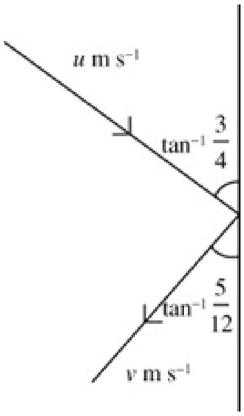
Question:

A smooth billiard ball strikes a smooth cushion at an angle of $\tan^{-1}\frac{3}{4}$ to the cushion.

The ball rebounds at an angle of $\tan^{-1}\frac{5}{12}$ to the cushion. Find

a the fraction of the kinetic energy of the ball lost in the collision,
b the coefficient of restitution between the ball and the cushion.

Solution:



a ↑: no change parallel to the cushion

$$u\times\frac{4}{5}=v\times\frac{12}{13}$$

$$v=u\times\frac{4}{5}\times\frac{13}{12}=\frac{13}{15}u$$

$$\text{K.E. lost}=\frac{1}{2}\times m\times u^2-\frac{1}{2}\times m\times\frac{169}{225}u^2=\frac{1}{2}\times m\times\frac{56}{225}u^2\text{J}$$

$$\text{Fraction of K.E. lost}=\frac{\frac{1}{2}\times m\times\frac{56}{225}u^2}{\frac{1}{2}\times m\times u^2}=\frac{56}{225}$$

b ↔: Using the law of restitution, $e\times u\times\frac{3}{5}=v\times\frac{5}{13}$

$$e\times u\times\frac{3}{5}=\frac{13}{15}u\times\frac{5}{13}$$

$$e=\frac{5}{15}\times\frac{5}{3}=\frac{5}{9}$$

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Elastic collisions in two dimensions
Exercise A, Question 15

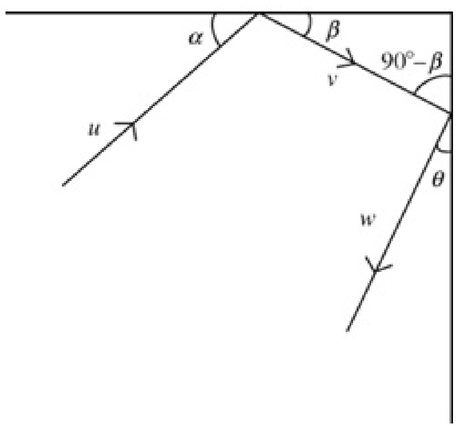
Question:

Two vertical walls meet at right angles at the corner of a room. A small smooth disc slides across the floor and bounces off each wall in turn. Just before the first impact the disc is moving with speed u m s^{-1} at an acute angle α to the wall. The coefficient of restitution between the disc and the wall is e. Find

a the direction of the motion of the disc after the second collision,

b the speed of the disc after the second collision.

Solution:



a First collision:

$\uparrow$: $e \times u \sin\alpha = v \sin\beta$

$\leftrightarrow$: $u\cos\alpha = v\cos\beta$

Second collision:

$\uparrow$: $v\cos(90-\beta) = v\sin\beta = w\cos\theta$

$\leftrightarrow$: $e \times v\sin(90-\beta) = ev\cos\beta = w\sin\theta$

$$\Rightarrow \tan\theta = \frac{e\cos\beta}{\sin\beta} = \frac{e}{\tan\beta}$$

$$= \frac{e}{e\tan\alpha} = \frac{1}{\tan\alpha}$$

$\Rightarrow \theta = 90° - \alpha$, so the path is parallel to the original path but in the opposite direction

b $eu\sin\alpha = v\sin\beta = w\cos\theta = w\cos(90° - \alpha) = w\sin\alpha$

speed after second collision $= w = eu$

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Elastic collisions in two dimensions
Exercise A, Question 16

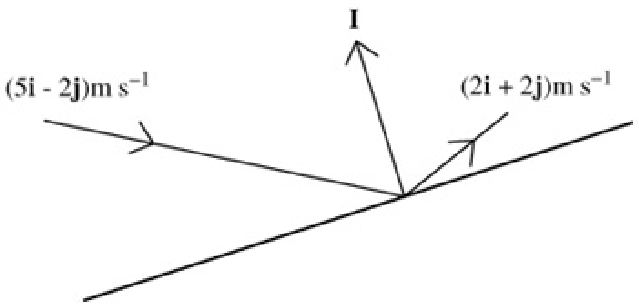
Question:

A small smooth sphere of mass m is moving with velocity $(5\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,

b the coefficient of restitution between the sphere and the wall.

Solution:



a $\mathbf{I} = m\mathbf{v} - m\mathbf{u} = m((2\mathbf{i}+2\mathbf{j})-(5\mathbf{i}-2\mathbf{j}))$

$= m(-3\mathbf{i}+4\mathbf{j})$

The impulse has magnitude $5m$ Ns in the direction parallel to the unit vector

$\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$.

b Component of $(5\mathbf{i}-2\mathbf{j})$ in the direction of the impulse

$= [(5\mathbf{i}-2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-15-8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

$= \frac{-23}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

Component of $(2\mathbf{i}+2\mathbf{j})$ in the direction of the impulse

$= [(2\mathbf{i}+2\mathbf{j}).\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})]\frac{1}{5}(-3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}\times(-6+8)\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

$= \frac{2}{5}\times\frac{1}{5}(-3\mathbf{i}+4\mathbf{j})$

law of restitution

$\Rightarrow \frac{2}{5} = e\times\frac{23}{5}$

$e = \frac{2}{23}$

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Edexcel AS and A Level Modular Mathematics

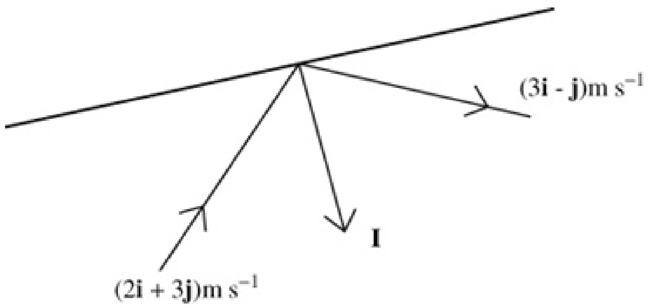
Elastic collisions in two dimensions
Exercise A, Question 17

Question:

A small smooth sphere of mass 2 kg is moving with velocity $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,
b the coefficient of restitution between the sphere and the wall,
c the kinetic energy lost by the sphere in the collision.

Solution:



a $\mathbf{I}=m\mathbf{v}-m\mathbf{u}=m\{(3\mathbf{i}-\mathbf{j})-(2\mathbf{i}+3\mathbf{j})\}$
$=2(\mathbf{i}-4\mathbf{j})$
The impulse has magnitude
$2\sqrt{17}$ Ns in
the direction parallel to the unit vector
$\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$

b Component of $(2\mathbf{i}+3\mathbf{j})$ in the direction of the impulse =
$[(2\mathbf{i}+3\mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})]\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})=\frac{1}{\sqrt{17}}(2-12)\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})=\frac{-10}{\sqrt{17}}\times\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$
Component of $(3\mathbf{i}-\mathbf{j})$ in the direction of the impulse =
$[(3\mathbf{i}-\mathbf{j}).\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})]\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})=\frac{1}{\sqrt{17}}(3+4)\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})=\frac{7}{\sqrt{17}}\times\frac{1}{\sqrt{17}}(\mathbf{i}-4\mathbf{j})$
law of restitution $\Rightarrow\frac{7}{\sqrt{17}}=e\times\frac{10}{\sqrt{17}},\ e=\frac{7}{10}$

c K.E. just before the impact $=\frac{1}{2}\times2\times(2^2+3^2)=13$
K.E. just after the impact $=\frac{1}{2}\times2\times(3^2+1^2)=10$
K.E. lost $=13-10=3$ J

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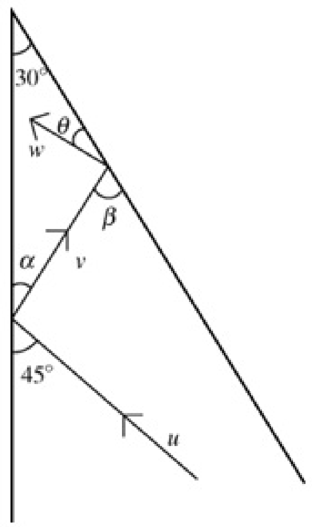
Elastic collisions in two dimensions
Exercise A, Question 18

Question:

Two smooth vertical walls stand on a smooth horizontal floor and intersect at an angle of 30°. A particle is projected along the floor with speed u m s^{-1} at 45° to one of the walls and towards the intersection of the walls. The coefficient of restitution between the particle and each wall is $\frac{1}{\sqrt{3}}$. Find the speed of the particle after one impact with each wall.

Solution:



For the first impact:

$u\cos 45° = \frac{u}{\sqrt{2}} = v\cos\alpha$

$eu\sin 45 = \frac{1}{\sqrt{3}} \times \frac{u}{\sqrt{2}} = v\sin\alpha$

By dividing, $\tan\alpha = \frac{1}{\sqrt{3}}, \alpha = 30°$, and $\beta = 60°$

Squaring and adding, $v^2 = \frac{u^2}{2} + \frac{u^2}{6} = \frac{4u^2}{6}$

For the second impact:

$v\cos 60° = \frac{v}{2} = w\cos\theta$

$ev\sin 60 = v \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{v}{2} = w\sin\theta$

Squaring and adding,

$w^2 = \left(\frac{v}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = \frac{v^2}{2} = \frac{u^2}{3}, \quad w = \frac{\sqrt{3}u}{3}$ m s^{-1}

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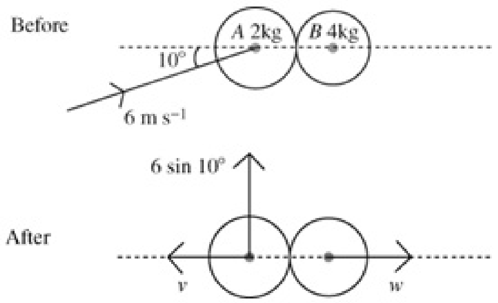
Elastic collisions in two dimensions
Exercise B, Question 1

Question:

A smooth sphere A, of mass 2 kg and moving with speed 6 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of $10°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Parallel to the line of centres
conservation of momentum:
$2 \times 6\cos 10° = 4w - 2v, 2w - v = 6\cos 10°$
law of restitution:
$w + v = \frac{1}{2} \times 6\cos 10° = 3\cos 10°$
$\Rightarrow 3w = 9\cos 10°, w = 3\cos 10°, v = 0$

So, after the impact, A has velocity $6\sin 10° \approx 1.04 \text{ m s}^{-1}$ perpendicular to the line of centres, and B has velocity $3\cos 10° \approx 2.95 \text{ m s}^{-1}$ parallel to the line of centres

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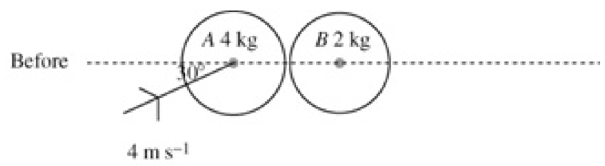
Elastic collisions in two dimensions
Exercise B, Question 2

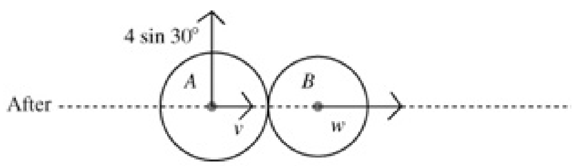
Question:

A smooth sphere A, of mass 4 kg and moving with speed 4 m s^{-1} collides obliquely with a smooth sphere B of mass 2 kg. Just before the impact B is stationary and the velocity of A makes an angle of $30°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{3}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:



Perpendicular to the line of centres, component of the velocity of A is
$4\sin 30° = 2 \text{ m s}^{-1}$



Parallel to the line of centres:

$$4 \times 4\cos 30 = 4v + 2w, 4\sqrt{3} = 2v + w$$

$$w - v = \frac{1}{3} \times 4\cos 30°, 2w - 2v = \frac{4\sqrt{3}}{3}$$

$$\Rightarrow 3w = \frac{16}{3}\sqrt{3}, w = \frac{16\sqrt{3}}{9}$$

and $v = \frac{10\sqrt{3}}{9}$

B has speed $\frac{16\sqrt{3}}{9} \text{ m s}^{-1}$ along the line of centres.

A has speed $\sqrt{(2)^2 + \left(\frac{10\sqrt{3}}{9}\right)^2} = \sqrt{4 + \frac{100}{27}} = \sqrt{\frac{208}{27}} = \frac{4\sqrt{13}}{3\sqrt{3}} = \frac{4\sqrt{39}}{9} \text{ m s}^{-1}$ at an angle of $\tan^{-1}\left(\frac{2}{10\frac{\sqrt{3}}{9}}\right) = \tan^{-1}\left(\frac{18}{10\sqrt{3}}\right) = \tan^{-1}\left(\frac{3\sqrt{3}}{5}\right) = 46.1°$ to the line of centres

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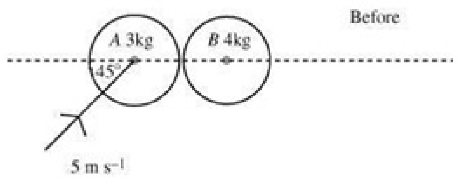
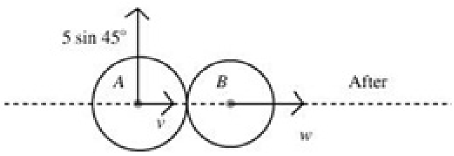
Elastic collisions in two dimensions
Exercise B, Question 3

Question:

A smooth sphere A, of mass 3 kg and moving with speed 5 m s^{-1} collides obliquely with a smooth sphere B of mass 4 kg. Just before the impact B is stationary and the velocity of A makes an angle of $45°$ with the lines of centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the magnitudes and directions of the velocities of A and B immediately after the impact.

Solution:





Perpendicular to the line of centres, component of the velocity of A is

$5\sin 45° = \frac{5\sqrt{2}}{2}\text{ m s}^{-1}$

Parallel to the line of centres:

$3\times 5\cos 45 = 3v + 4w$

$w - v = \frac{1}{2}\times 5\cos 45°, 3w - 3v = \frac{15\sqrt{2}}{4}$

$\Rightarrow 7w = \frac{45\sqrt{2}}{4}, w = \frac{45\sqrt{2}}{28}$

and $v = \frac{10\sqrt{2}}{28} = \frac{5\sqrt{2}}{14}$

B has speed $\frac{45\sqrt{2}}{28}\text{ m s}^{-1}$ along the line of centres

A has speed $\sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{14}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{1^2 + \left(\frac{1}{7}\right)^2} = \frac{5\sqrt{2}}{2}\sqrt{\frac{50}{49}} = \frac{50}{14} = \frac{25}{7}\text{ m s}^{-1}$ at an

angle of $\tan^{-1}\left(\frac{5\sin 45°}{v}\right) = \tan^{-1}\left(\frac{\frac{5\sqrt{2}}{2}}{\frac{5\sqrt{2}}{14}}\right) = \tan^{-1}7 \approx 81.9°$ to the line of centres

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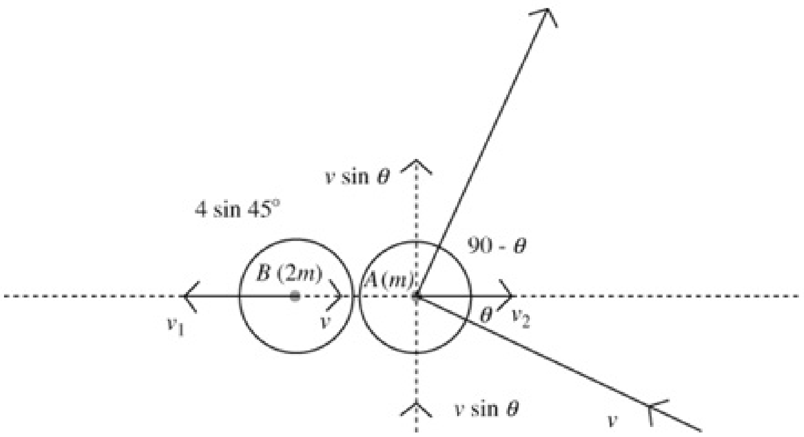
Elastic collisions in two dimensions
Exercise B, Question 4

Question:

A small smooth sphere A of mass m and a small smooth sphere B of the same radius but mass $2m$ collide. At the instant of impact, B is stationary and the velocity of A makes an angle θ with the line of centres. The direction of motion of A is turned through $90°$ by the impact. The coefficient of restitution between the spheres is e. Show that

$$\tan^2\theta = \frac{2e-1}{3}.$$

Solution:



Components perpendicular to the line of centres are unchanged. For A, the component perpendicular to the line of centres is $v\sin\theta$.
Parallel to the line of centres:
conservation of momentum $\Rightarrow mv\cos\theta = 2mv_1 - mv_2$
law of restitution

$$\Rightarrow v_1 + v_2 = ev\cos\theta$$

$$\Rightarrow v\cos\theta = 2(ev\cos\theta - v_2) - v_2 = 2ev\cos\theta - 3v_2$$

$$v_2 = \frac{v\cos\theta(2e-1)}{3}$$

$$\Rightarrow \tan(90-\theta) = \frac{1}{\tan\theta} = \frac{v\sin\theta}{v_2} = \frac{3v\sin\theta}{v\cos\theta(2e-1)},\ \therefore \frac{1}{\tan\theta} = \frac{3\tan\theta}{2e-1} \quad \therefore \tan^2\theta = \frac{2e-1}{3}$$

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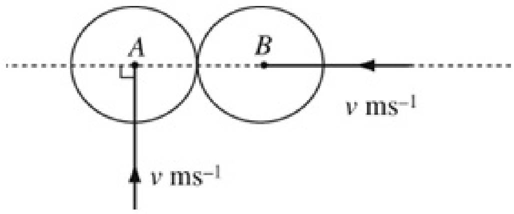
Elastic collisions in two dimensions
Exercise B, Question 5

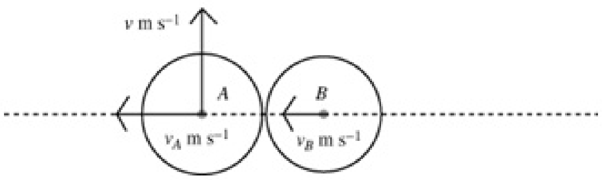
Question:

Two smooth spheres A and B are identical and are moving with equal speeds on a smooth horizontal surface. In the instant before impact, A is moving in a direction perpendicular to the line of centres of the spheres, and B is moving along the line of centres, as



shown in the diagram. The coefficient of restitution between the spheres is $\frac{2}{3}$. Find the speeds and directions of motion of the spheres after the collision.

Solution:



Components perpendicular to the line of centres are unchanged.
Conservation of momentum: $mv = mv_A + mv_B$, $\quad v = v_A + v_B$
law of restitution:

$$v_A - v_B = \frac{2}{3}v$$

$$\Rightarrow 2v_A = \frac{5}{3}v, v_A = \frac{5}{6}v, \quad v_B = \frac{1}{6}v$$

A has speed $\sqrt{1^2 + \left(\frac{5}{6}\right)^2}\,v = \sqrt{\frac{61}{36}}\,v = \frac{\sqrt{61}v}{6}$ m s^{-1} and is moving at

$\tan^{-1}\left(\frac{1}{\left(\frac{5}{6}\right)}\right) = \tan^{-1}\frac{6}{5} = 50.2°$ to the line of centres.

B is moving along the line of centres with speed $\frac{1}{6}v$ m s^{-1}.

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Elastic collisions in two dimensions
Exercise B, Question 6

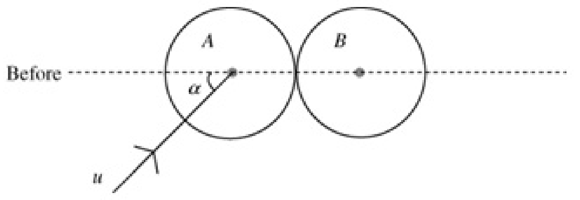
Question:

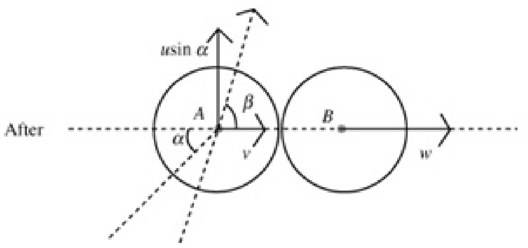
A smooth sphere A collides obliquely with an identical smooth sphere B. Just before the impact B is stationary and the velocity of A makes an angle of α with the lines of centres of the two spheres. The coefficient of restitution between the spheres is e $(e \neq 1)$. Immediately after the collision the velocity of A makes an angle of β with the line of centres.

a Show that $\tan\beta = \dfrac{2\tan\alpha}{1-e}$.

b Hence show that in the collision the direction of motion of A turns through an angle equal to $\tan^{-1}\left(\dfrac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}\right)$.

Solution:





Perpendicular to the line of centres, component of velocity of A is $u\sin\alpha$. Parallel to the line of centres:
conservation of momentum: $mu\cos\alpha = mv + mw$, $u\cos\alpha = v + w$
law of restitution: $w - v = eu\cos\alpha$, so $2v = u\cos\alpha - eu\cos\alpha = u\cos\alpha(1-e)$

$$\Rightarrow \tan\beta = \frac{u\sin\alpha}{v} = \frac{2u\sin\alpha}{u\cos\alpha(1-e)} = \frac{2\tan\alpha}{1-e}$$

b The path of A has been deflected through an angle equal to $\beta - \alpha$.

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\alpha\tan\beta} = \frac{\dfrac{2\tan\alpha}{1-e} - \tan\alpha}{1+\tan\alpha\dfrac{2\tan\alpha}{1-e}} = \frac{2\tan\alpha-(1-e)\tan\alpha}{1-e+2\tan^2\alpha}$$

$$= \frac{(1+e)\tan\alpha}{2\tan^2\alpha+1-e}$$

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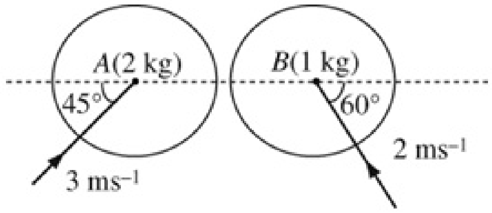
Elastic collisions in two dimensions
Exercise B, Question 7

Question:



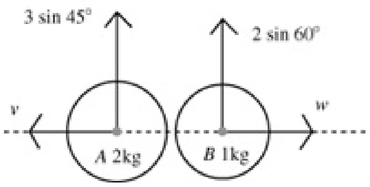
A small smooth sphere A of mass 2 kg collides with a small smooth sphere B of mass 1 kg. Just before the impact A is moving with a speed of 3 m s^{-1} in a direction at $45°$ to the line of centres and B is moving with speed 2 m s^{-1} at $60°$ to the line of centres, as shown in the diagram. The coefficient of restitution between the spheres is $\frac{\sqrt{2}}{3}$.

Find

a the kinetic energy lost in the impact,

b the magnitude of the impulse exerted by A on B.

Solution:



No change in the components of velocity perpendicular to the line of centres. Parallel to the line of centres:

conservation of momentum: $1\times 2\cos 60° - 2\times 3\cos 45° = 2v - w = 1 - 3\sqrt{2}$

law of restitution: $v + w = \frac{\sqrt{2}}{3}(3\cos 45° + 2\cos 60°)$

$$= \frac{\sqrt{2}}{3}\left(\frac{3\sqrt{2}}{2} + 1\right) = 1 + \frac{\sqrt{2}}{3}$$

Solving the simultaneous equations gives $3v = 2 - \frac{8\sqrt{2}}{3}$, $v = \frac{2}{3} - \frac{8\sqrt{2}}{9} \approx -0.590$ and

$w = 1 + \frac{\sqrt{2}}{3} - \frac{2}{3} + \frac{8\sqrt{2}}{9} = \frac{1}{3} + \frac{11\sqrt{2}}{9} \approx 2.06$

a K.E. lost in the impact

$$= \frac{1}{2}\times 2\times((3\cos 45°)^2 - 0.590^2) + \frac{1}{2}\times 1\times((2\cos 60°)^2 - 2.06^2) \approx 2.53\text{ J}$$

b Impulse on $B = 1(w + 2\cos 60°) \approx 3.06$ Ns

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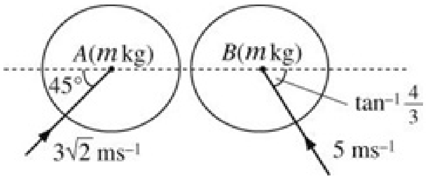
Elastic collisions in two dimensions
Exercise B, Question 8

Question:

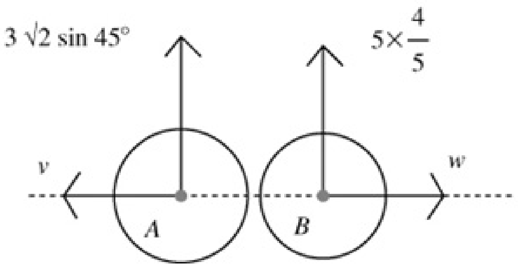
A small smooth sphere A collides with an identical small smooth sphere B. Just before the impact A is moving with a speed of $3\sqrt{2}$ m s^{-1} in a direction at 45° to the line of centres and B is moving with speed 5 m s^{-1} at $\tan^{-1}\frac{4}{3}$ to the line of centres, as shown in the diagram.



The coefficient of restitution between the spheres is $\frac{2}{3}$. Find

a the speeds of both spheres immediately after the impact,
b the fraction of the kinetic energy lost in the impact.

Solution:



After the collision the components of velocity perpendicular to the line of centres are 3 m s^{-1} and 4 m s^{-1}. (No change in this direction.)
Parallel to the line of centres:

conservation of momentum: $m\times 3\sqrt{2}\cos 45° - m\times 5\times\frac{3}{5} = mw - mv = 0$

law of restitution: $v + w = \frac{2}{3}\left(3\sqrt{2}\cos 45° + 5\times\frac{3}{5}\right) = 4$

so $v = w = 2$

a speed of $A = \sqrt{2^2+3^2} = \sqrt{13}$ m s^{-1}

speed of $B = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$ m s^{-1}

b Total K.E. just before impact $= \frac{1}{2}\times m\times(3\sqrt{2})^2 + \frac{1}{2}\times m\times 5^2 = \frac{m\times 43}{2}$ J

Total K.E. just after impact $= \frac{1}{2}\times m\times(\sqrt{13})^2 + \frac{1}{2}\times m\times(2\sqrt{5})^2 = \frac{m\times 33}{2}$ J

Fraction of K.E. lost $= \frac{43-33}{43} = \frac{10}{43}$

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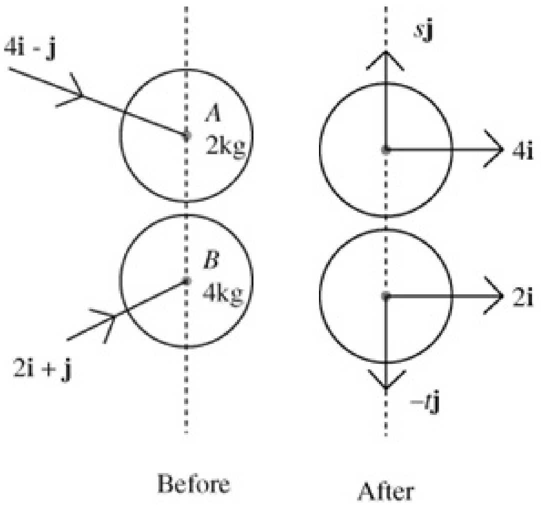
Elastic collisions in two dimensions
Exercise B, Question 9

Question:

A smooth sphere A of mass 2 kg is moving on a smooth horizontal surface with velocity $(4\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Another smooth sphere B of mass 4 kg and the same radius as A is moving on the same surface with velocity $(2\mathbf{i}+\mathbf{j})\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{j}$. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find the velocities of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{j} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{i}$.
Conservation of momentum: $-2\times 1+4\times 1=2\times s-4\times t=2$

law of restitution: $s+t=\frac{1}{2}(1+1), s+t=1$

$$s-2t=1$$
$$3s=3$$
$$s=1, t=0$$

velocity of A is $4\mathbf{i}+\mathbf{j}\text{ m s}^{-1}$
velocity of B is $2\mathbf{i}\text{ m s}^{-1}$

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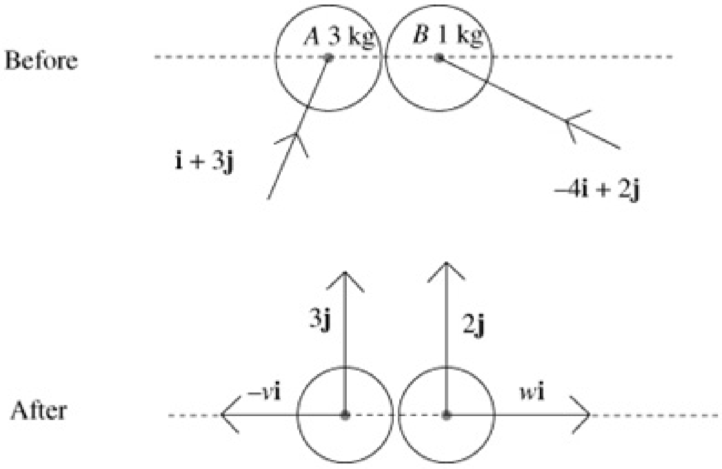
Elastic collisions in two dimensions
Exercise B, Question 10

Question:

A smooth sphere A of mass 3 kg is moving on a smooth horizontal surface with velocity $(\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$. Another smooth sphere B of mass 1 kg and the same radius as A is moving on the same surface with velocity $(-4\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of both spheres after the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{j}$

conservation of momentum: $3\times1-1\times4=1\times w-3\times v=-1$

law of restitution: $v+w=\frac{3}{4}(4+1), 4v+4w=15$

$$4w-12v=-4$$

$$16v=19$$

$$v=\frac{19}{16}, w=\frac{41}{16}$$

After the impact, speed of $A=\sqrt{3^2+\left(\frac{19}{16}\right)^2}\approx3.23\text{ m s}^{-1}$,

speed of $B=\sqrt{2^2+\left(\frac{41}{16}\right)^2}\approx3.25\text{ m s}^{-1}$

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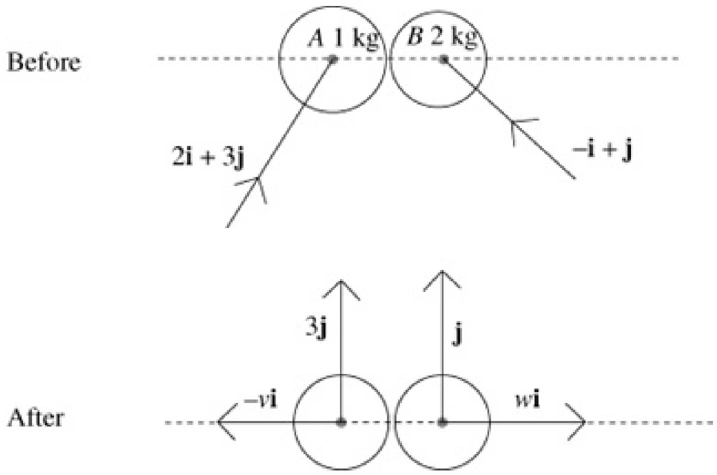
Elastic collisions in two dimensions
Exercise B, Question 11

Question:

A smooth sphere A of mass 1 kg is moving on a smooth horizontal surface with velocity $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$. Another smooth sphere B of mass 2 kg and the same radius as A is moving on the same surface with velocity $(-\mathbf{i}+\mathbf{j})\text{ms}^{-1}$. The spheres collide when their line of centres is parallel to $\mathbf{i}$. The coefficient of restitution between the spheres is $\frac{3}{5}$. Find the kinetic energy lost in the impact.

Solution:



Line of centres parallel to $\mathbf{i} \Rightarrow$ no change in the components of velocity parallel to $\mathbf{j}$

conservation of momentum: $1\times2-2\times1=2\times w-1\times v=0$

law of restitution: $v+w=\frac{3}{5}(2+1)$

$2w-v=0, 3w=\frac{9}{5}, w=\frac{3}{5}$

$v=\frac{6}{5}$

$\text{K.E. lost}=\frac{1}{2}\times1\times\left(2^2-\left(\frac{6}{5}\right)^2\right)+\frac{1}{2}\times2\times\left(1^2-\left(\frac{3}{5}\right)^2\right)=\frac{48}{25}=1.92\text{ J}$

Components of velocity unchanged parallel to $\mathbf{j} \Rightarrow$ all K.E. lost parallel to $\mathbf{i}$.

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Elastic collisions in two dimensions
Exercise B, Question 12

Question:

Two small smooth spheres A and B have equal radii. The mass of A is m kg and the mass of B is $2m$ kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A *is* $(3\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

a Conservation of momentum $\Rightarrow m(2\mathbf{i}+5\mathbf{j})+2m(3\mathbf{i}-\mathbf{j})=m(3\mathbf{i}+2\mathbf{j})+2m\mathbf{v}$

$$2\mathbf{v}=\mathbf{i}(2+2\times3-3)+\mathbf{j}(5-2\times1-2)=5\mathbf{i}+\mathbf{j}$$

$$\mathbf{v}=\frac{5}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}$$

b Impulse on $A=m((3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+5\mathbf{j}))=m(\mathbf{i}-3\mathbf{j})$

$\Rightarrow$ line of centres parallel to $\frac{1}{\sqrt{10}}(\mathbf{i}-3\mathbf{j})$

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Elastic collisions in two dimensions
Exercise B, Question 13

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $3m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(3\mathbf{i}-5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(4\mathbf{i}+\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(4\mathbf{i}-4\mathbf{j})\text{m s}^{-1}$. Find

a the speed of B immediately after the collision,

b the kinetic energy lost in the collision.

Solution:

a Conservation of momentum $\Rightarrow 3m(3\mathbf{i}-5\mathbf{j})+m(4\mathbf{i}+\mathbf{j})=3m(4\mathbf{i}-4\mathbf{j})+m\mathbf{v}$

$$\mathbf{v}=\mathbf{i}(3\times 3+4-3\times 4)+\mathbf{j}(-3\times 5+1+3\times 4)=\mathbf{i}-2\mathbf{j}$$

Speed of B is $\sqrt{1^2+2^2}=\sqrt{5}\text{ m s}^{-1}$

b $\text{K.E. lost}=\frac{3m}{2}((3^2+5^2)-(4^2+4^2))+\frac{m}{2}((4^2+1^2)-5)$

$$=\frac{m}{2}(3(34-32)+(17-5))=\frac{m}{2}(6+12)=9m\text{ J}$$

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Elastic collisions in two dimensions
Exercise B, Question 14

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $2m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+5\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i}+4\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b the coefficient of restitution between the two spheres.

Solution:

a Conservation of momentum $\Rightarrow 2m(2\mathbf{i}+5\mathbf{j})+m(2\mathbf{i}-2\mathbf{j}) = 2m(3\mathbf{i}+4\mathbf{j})+m\mathbf{v}$

$$\mathbf{v} = \mathbf{i}(2\times2+2-2\times3)+\mathbf{j}(2\times5-2-2\times4) = 0$$

b B is brought to a halt in the collision $\Rightarrow$ the line of centres must be parallel to the original direction of motion of B, i.e. $\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$

In this direction,

speed of A before $= ((2\mathbf{i}+5\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j}) = \frac{\sqrt{2}}{2}(2-5) = -3\frac{\sqrt{2}}{2}$

speed of A after $= (3\mathbf{i}+4\mathbf{j}).\frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j}) = \frac{\sqrt{2}}{2}(3-4) = -\frac{\sqrt{2}}{2}$

speed of B before $= 2\sqrt{2}$

speed of B after $= 0$

Therefore the impact law gives $\dfrac{\frac{\sqrt{2}}{2}}{3\frac{\sqrt{2}}{2}+2\sqrt{2}} = e = \frac{1}{7}$

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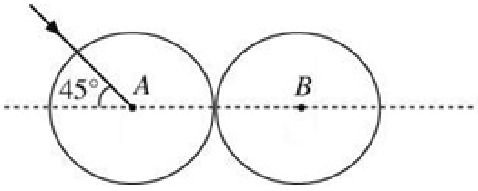
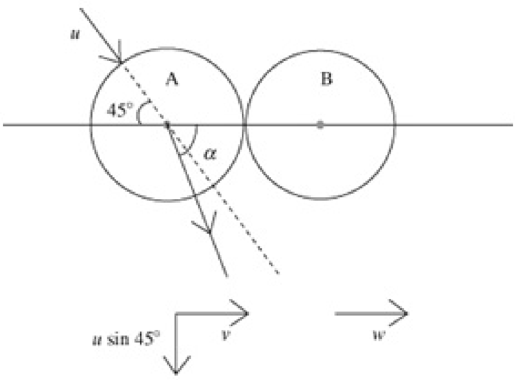
Elastic collisions in two dimensions
Exercise B, Question 15

Question:

A smooth uniform sphere A, moving on a smooth horizontal table, collides with an identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of $45°$ with the direction of motion of A, as shown in the diagram. The coefficient of restitution between the spheres is e. The direction of motion of A is deflected through an angle θ by the collision. Show that $\tan\theta = \frac{1+e}{3-e}$.



Solution:



Parallel to the line of centres, using conservation of momentum and the law of restitution gives $mu\cos 45° = mv + mw$ and $w - v = eu\cos 45°$
By subtracting

$$2v = u\cos 45°(1-e)$$

$$v = \frac{u\sqrt{2}(1-e)}{4}$$

$$\Rightarrow \tan\alpha = \frac{u\sin 45°}{\left(\frac{u\sqrt{2}(1-e)}{4}\right)} = \frac{2}{1-e}$$

$$\theta = \alpha - 45° \Rightarrow \tan\theta = \frac{\tan\alpha - \tan 45°}{1+\tan\alpha\tan 45°} = \frac{\frac{2}{1-e}-1}{1+\frac{2}{1-e}} = \frac{2-1+e}{1-e+2} = \frac{1+e}{3-e}$$

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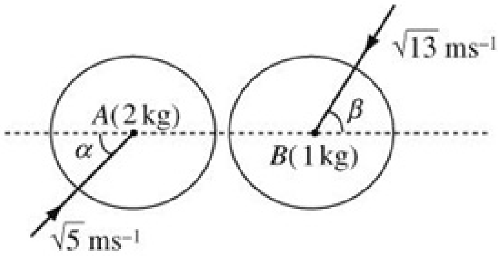
Elastic collisions in two dimensions
Exercise B, Question 16

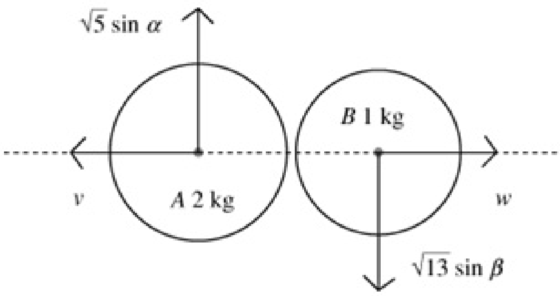
Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{5}$ m s^{-1} and the speed of B is $\sqrt{13}$ m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{3}{2}$, as shown in the diagram above. The coefficient of restitution between A and B is $\frac{1}{2}$.

Find the speed of each sphere after the collision.

Solution:



Before collision, components of velocity of A are 1 m s^{-1} perpendicular to the lines of centres and 2 m s^{-1} parallel to the line. The components of the velocity of B are 3 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

conservation of momentum: $2\times2-1\times2=1\times w-2\times v, 2=w-2v$

law of restitution: $w+v=e(2+2), w+v=4e=2$

Solving the simultaneous equations $\Rightarrow w=2, v=0$

$\Rightarrow$ speed of A is 1 m s^{-1} and speed of B is $\sqrt{3^2+2^2}=\sqrt{13}$ m s^{-1}

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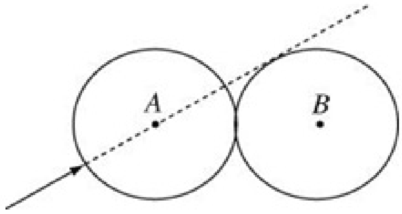
Elastic collisions in two dimensions
Exercise B, Question 17

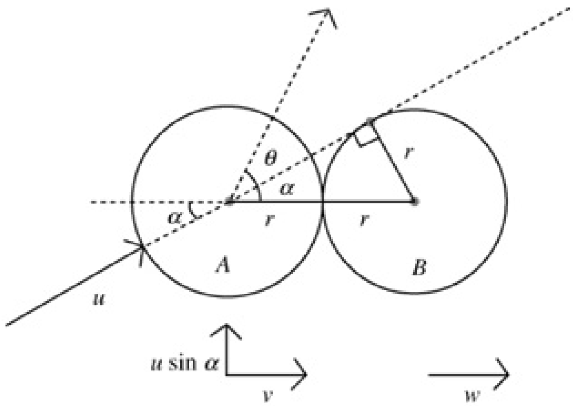
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{1}{2}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\tan\theta = \frac{3\sqrt{3}}{7}$.

Solution:



Tangent perpendicular to radius $\Rightarrow \sin\alpha = \frac{1}{2}$

Initial components of velocity of A are $u\cos\alpha$ parallel to the line of centres, and $u\sin\alpha$ perpendicular to the line of centres.

momentum $\Rightarrow mu\cos\alpha = mv + mw$

$$u\cos\alpha = v + w$$

$$\text{impact} \Rightarrow w - v = eu\cos\alpha$$

Subtracting gives

$$2v = u\cos\alpha - eu\cos\alpha \qquad v = \frac{u\cos\alpha\left(1-\frac{1}{2}\right)}{2} = \frac{u\times\frac{\sqrt{3}}{2}\times\frac{1}{2}}{2} = \frac{u\sqrt{3}}{8}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\frac{u}{2}\right)}{\left(\frac{u\sqrt{3}}{8}\right)} = \frac{4}{\sqrt{3}}$$

$$\tan\theta = \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{\frac{4}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{1+\frac{4}{\sqrt{3}}\times\frac{1}{\sqrt{3}}} = \frac{\left(\frac{3}{\sqrt{3}}\right)}{\left(\frac{3+4}{3}\right)} = \frac{3\sqrt{3}}{7}$$

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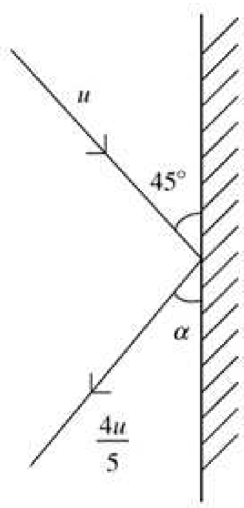
Elastic collisions in two dimensions
Exercise C, Question 1

Question:

A smooth sphere S is moving on a smooth horizontal plane with speed u when it collides with a smooth fixed vertical wall. At the instant of collision the direction of motion of S makes an angle of $45°$ with the wall. Immediately after the collision the speed of S is $\frac{4}{5}u$. Find the coefficient of restitution between S and the wall.

Solution:



$$R \uparrow : \frac{4u}{5}\cos\alpha = u\cos 45°$$

law of restitution $\leftrightarrow$: $\frac{4u}{5}\sin\alpha = eu\sin 45°$

squaring and adding:

$$\frac{16u^2}{25} = u^2\left(\frac{1}{2}+\frac{e^2}{2}\right)$$

$$\frac{32}{25} = 1+e^2$$

$$\frac{7}{25} = e^2, e = \frac{\sqrt{7}}{5}$$

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Elastic collisions in two dimensions
Exercise C, Question 2

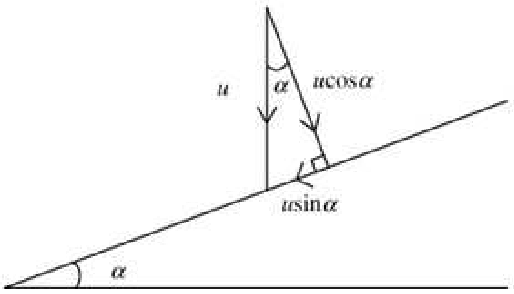
Question:

A small smooth ball of mass $\frac{1}{2}$kg is falling vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. Immediately before striking the plane the ball has speed 5.2 m s^{-1}. The coefficient of restitution between ball and plane is $\frac{1}{4}$. Find

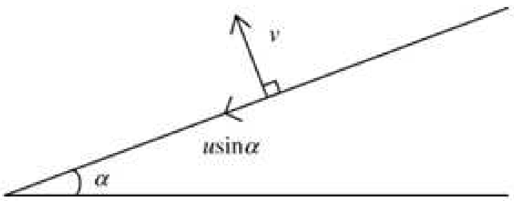
a the speed, to 3 significant figures, of the ball immediately after the impact,
b the magnitude of the impulse received by the ball as it strikes the plane.

Solution:

Before the collision:



After the collision:



a Considering the component of velocity parallel to the plane:

$$u\sin\alpha = 5.2 \times \frac{5}{13} = 2$$

Perpendicular to the plane:

$$v = eu\cos\alpha = \frac{1}{4} \times 5.2 \times \frac{12}{13} = 1.2$$

$$\text{speed} = \sqrt{2^2 + 1.2^2} = \sqrt{5.44} = 2.33\text{ m s}^{-1}$$

b $\text{Impulse} = \frac{1}{2}(1.2 - (-4.8)) = 3\text{ Ns}$

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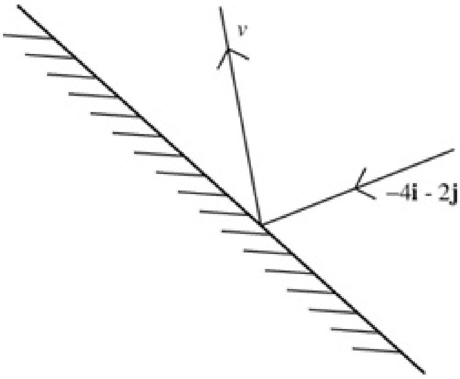
Elastic collisions in two dimensions
Exercise C, Question 3

Question:

A small smooth ball of mass 500 g is moving in the xy-plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$. The velocity of the ball just before impact is $(-4\mathbf{i} - 2\mathbf{j})\text{ms}^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find

a the velocity of the ball immediately after the impact,
b the kinetic energy lost as a result of the impact.

Solution:



a Suppose that $\mathbf{v} = \mathbf{a} + \mathbf{b}$ where $\mathbf{a}$ is parallel to the wall and $\mathbf{b}$ is perpendicular to the wall.

$\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$ is a unit vector parallel to the wall and $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$ is a unit vector perpendicular to the wall.

$$\nwarrow \mathbf{a} = \left[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})\right]\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j})$$

$$= \frac{1}{\sqrt{2}}(4-2)\times\frac{1}{\sqrt{2}}(-\mathbf{i}+\mathbf{j}) = (-\mathbf{i}+\mathbf{j})$$

$$\nearrow \mathbf{b} = -\frac{1}{2}[(-4\mathbf{i}-2\mathbf{j}).\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})]\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

$$= -\frac{1}{2}\times\frac{1}{\sqrt{2}}(-4-2)\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$

$$= \frac{3}{2}(\mathbf{i}+\mathbf{j})$$

So $\mathbf{v} = (-\mathbf{i}+\mathbf{j}) + \frac{3}{2}(\mathbf{i}+\mathbf{j}) = \frac{1}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$

b K.E. before impact $= \frac{1}{2}\times\frac{1}{2}\times(4^2+2^2) = 5$

K.E. after impact $= \frac{1}{2}\times\frac{1}{2}\times\left(\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2\right) = \frac{1}{4}\times\frac{26}{4} = \frac{13}{8}$

K.E. lost $= 5 - \frac{13}{8} = 3.375\text{ J}$

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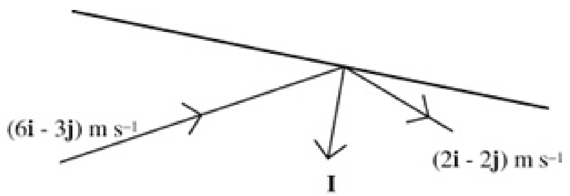
Elastic collisions in two dimensions
Exercise C, Question 4

Question:

A small smooth sphere of mass m is moving with velocity $(6\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ when it hits a smooth wall. It rebounds from the wall with velocity $(2\mathbf{i}-2\mathbf{j})\text{m s}^{-1}$. Find

a the magnitude and direction of the impulse received by the sphere,
b the coefficient of restitution between the sphere and the wall.

Solution:



a $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$

$= m\{(2\mathbf{i}-2\mathbf{j})-(6\mathbf{i}+3\mathbf{j})\}$

$= m(-4\mathbf{i}-5\mathbf{j})$

The impulse has magnitude $m\sqrt{16+25} = m\sqrt{41}$ Ns in the direction parallel to the unit vector $\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$.

b Component of $(6\mathbf{i}+3\mathbf{j})$ parallel to the impulse

$= [(6\mathbf{i}+3\mathbf{j})\cdot\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})]\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

$= \frac{1}{\sqrt{41}}(-24-15)\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

Component of $(2\mathbf{i}-2\mathbf{j})$ parallel to the impulse

$= [(2\mathbf{i}-2\mathbf{j})\cdot\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})]\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

$= \frac{1}{\sqrt{41}}(-8+10)\times\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$

law of restitution

$\frac{2}{\sqrt{41}} = e\times\frac{39}{\sqrt{41}}$

$e = \frac{2}{39}$

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Elastic collisions in two dimensions
Exercise C, Question 5

Question:

Two small smooth spheres A and B have equal radii. The mass of A is $4m$ kg and the mass of B is m kg. The spheres are moving on a smooth horizontal plane and they collide. Immediately before the collision the velocity of A is $(2\mathbf{i}+3\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Immediately after the collision the velocity of A is $(3\mathbf{i}+2\mathbf{j})\text{m s}^{-1}$. Find

a the velocity of B immediately after the collision,

b a unit vector parallel to the line of centres of the spheres at the instant of the collision.

Solution:

a Conservation of momentum $\Rightarrow$

$$4m(2\mathbf{i}+3\mathbf{j})+m(3\mathbf{i}-\mathbf{j})=4m(3\mathbf{i}+2\mathbf{j})+m\mathbf{v}$$

$$\mathbf{v}=\mathbf{i}(4\times2+1\times3-4\times3)+\mathbf{j}(4\times3-1\times1-4\times2)=-\mathbf{i}+3\mathbf{j}$$

b Impulse on $A=4m((3\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+3\mathbf{j}))=4m(\mathbf{i}-\mathbf{j})$

$\Rightarrow \frac{\sqrt{2}}{2}(\mathbf{i}-\mathbf{j})$ is a unit vector parallel to the line of centres.

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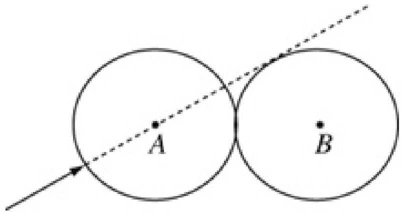
Elastic collisions in two dimensions

Exercise C, Question 6

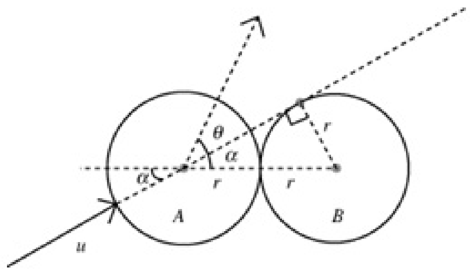
Question:



A smooth uniform sphere B is at rest on a smooth horizontal plane, when it is struck by an identical sphere A moving on the plane. Immediately before the impact, the line of motion of the centre of A is tangential to the sphere B, as shown in the diagram above. The coefficient of restitution between the spheres is $\frac{2}{3}$. The direction of motion of A is turned through an angle θ by the impact.

Show that $\theta = \tan^{-1}\frac{5\sqrt{3}}{9}$.

Solution:



Tangent perpendicular to radius

$\Rightarrow \sin\alpha = \dfrac{1}{2}$

Initial components of velocity of A are
$u\cos\alpha$ parallel to the line of centres, and
$u\sin\alpha$ perpendicular to the line of centres.

$$\text{Momentum} \Rightarrow mu\cos\alpha = mv + mw$$
$$u\cos\alpha = v + w$$
$$\text{Impact} \Rightarrow w - v = eu\cos\alpha$$

where v is the velocity of A along the line of centres and w the velocity of B along the line of centres immediately after the collision.

Subtracting gives $2v = u\cos\alpha - eu\cos\alpha$, $v = \dfrac{u\cos\alpha\left(1-\dfrac{2}{3}\right)}{2} = \dfrac{u\times\dfrac{\sqrt{3}}{2}\times\dfrac{1}{3}}{2} = \dfrac{u\sqrt{3}}{12}$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{v} = \frac{\left(\dfrac{u}{2}\right)}{\left(\dfrac{u\sqrt{3}}{12}\right)} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

$$\tan\theta = \frac{\tan(\theta+\alpha)-\tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{2\sqrt{3}-\dfrac{1}{\sqrt{3}}}{1+2\sqrt{3}\times\dfrac{1}{\sqrt{3}}} = \frac{\left(\dfrac{6-1}{\sqrt{3}}\right)}{(1+2)} = \frac{5}{3\sqrt{3}} = \frac{5\sqrt{3}}{9}$$

$$\theta = \tan^{-1}\frac{5\sqrt{3}}{9}$$

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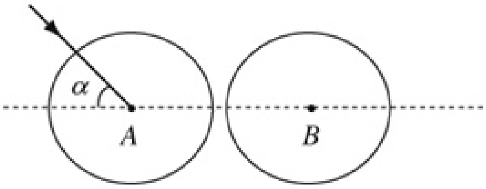
Elastic collisions in two dimensions
Exercise C, Question 7

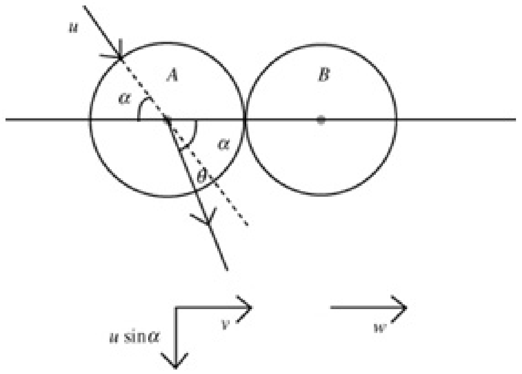
Question:



A smooth uniform sphere A, moving on a smooth horizontal table, collides with a second identical sphere B which is at rest on the table. When the spheres collide the line joining their centres makes an angle of α with the direction of motion of A, as shown in the diagram above. The direction of motion of A is deflected through an angle θ by the collision. Given that $\alpha = \tan^{-1}\frac{3}{4}$ and that the coefficient of restitution between the spheres is e,

show that $\tan\theta = \frac{6+6e}{17-8e}$.

Solution:



Parallel to the line of centres, using conservation of momentum and the impact law gives

$mu\cos\alpha = mv + mw$

and $w - v = eu\cos\alpha$

By subtracting,

$$2v = u\cos\alpha \times (1-e)$$

$$v = \frac{4u(1-e)}{10} = \frac{2u(1-e)}{5}$$

$$\Rightarrow \tan(\theta+\alpha) = \frac{u\sin\alpha}{\left(\frac{2u(1-e)}{5}\right)}$$

$$= \frac{3}{2(1-e)}$$

$$\tan\theta = \frac{\tan(\theta+\alpha) - \tan\alpha}{1+\tan(\theta+\alpha)\tan\alpha} = \frac{\frac{3}{2(1-e)} - \frac{3}{4}}{1+\frac{3}{2(1-e)}\times\frac{3}{4}} = \frac{12-6(1-e)}{8(1-e)+9} = \frac{6+6e}{17-8e}$$

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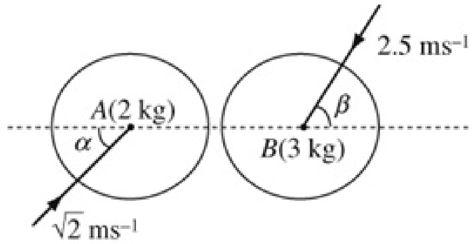
Elastic collisions in two dimensions
Exercise C, Question 8

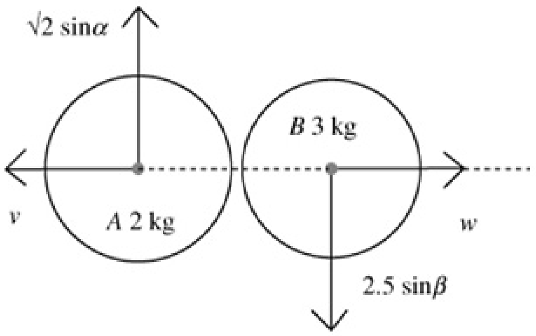
Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 3 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is $\sqrt{2}$ m s^{-1} and the speed of B is 2.5 m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan\alpha = 1$ and $\tan\beta = \frac{3}{4}$ as shown in the diagram. The coefficient of restitution between A and B is $\frac{2}{3}$.

Find the speed of each sphere after the collision.

Solution:



Before the collision, the components of the velocity of A are 1 m s^{-1} perpendicular to the line of centres and 1 m s^{-1} parallel to the line.

The components of the velocity of B are 1.5 m s^{-1} perpendicular to the line, and 2 m s^{-1} parallel to it.

Conservation of momentum: $2\times1-3\times2=3\times w-2\times v, -4=3w-2v$

Law of restitution: $w+v=e(1+2), w+v=3e=2$

Solving the simultaneous equations $-4=3w-2v$ and $4=2w+2v$

$\Rightarrow w=0, v=2$

$\Rightarrow$ speed of A is $\sqrt{1^2+2^2}=\sqrt{5}$ m s^{-1} and speed of B is 1.5 m s^{-1}

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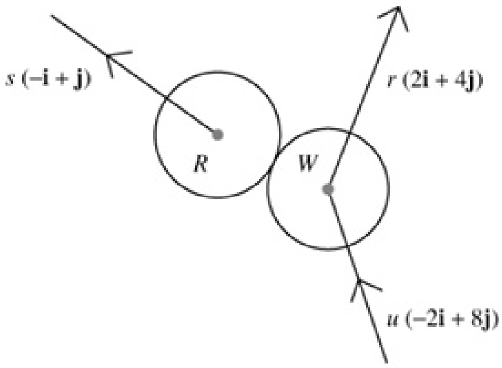
Elastic collisions in two dimensions
Exercise C, Question 9

Question:

A red ball is stationary on a rectangular billiard table $OABC$. It is then struck by a white ball of equal mass and equal radius moving with velocity $u(-2\mathbf{i}+8\mathbf{j})$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors parallel to OA and OC respectively. After the impact the velocity of the red ball is parallel to the vector $(-\mathbf{i}+\mathbf{j})$ and the velocity of the white ball is parallel to the vector $(2\mathbf{i}+4\mathbf{j})$. Prove that the coefficient of restitution between the two balls is $\frac{3}{5}$.

Solution:



Conservation of momentum:

$u(-2\mathbf{i}+8\mathbf{j}) = s(-\mathbf{i}+\mathbf{j}) + r(2\mathbf{i}+4\mathbf{j})$

$\Rightarrow -2u = -s + 2r$ and $8u = s + 4r$

Adding $\Rightarrow 6u = 6r, r = u, s = 4u$

Line of centres is parallel to $-\mathbf{i}+\mathbf{j}$ (as this is the direction of the impulse on the red ball).

In the direction of the line of centres

component of $(-2\mathbf{i}+8\mathbf{j})$ is $\frac{(-2\mathbf{i}+8\mathbf{j})\cdot(-\mathbf{i}+\mathbf{j})}{|(-\mathbf{i}+\mathbf{j})|} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$

component of $(2\mathbf{i}+4\mathbf{j})$ is $\frac{2\sqrt{2}}{2} = \sqrt{2}$ and component of $(-\mathbf{i}+\mathbf{j})$ is $\sqrt{2}$

so using law of restitution:

$4u\sqrt{2} - u\sqrt{2} = e \times 5u\sqrt{2}, 3\sqrt{2} = 5\sqrt{2}e$

$e = \frac{3}{5}$

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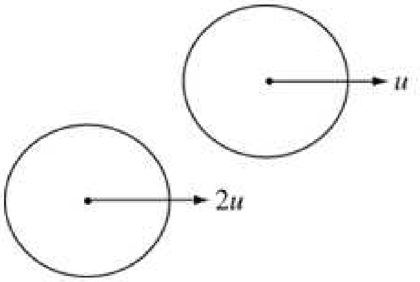
Elastic collisions in two dimensions
Exercise C, Question 10

Question:

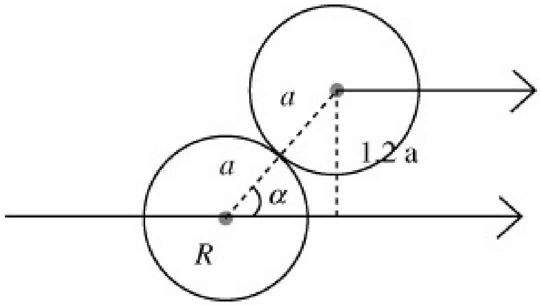


Two uniform spheres, each of mass m and radius a, collide when moving on a horizontal plane. Before the impact the spheres are moving with speeds $2u$ and u, as shown in the diagram.

The centres of the spheres are moving on parallel paths distance $\frac{6a}{5}$ apart.

The coefficient of restitution between the spheres is $\frac{3}{4}$. Find the speeds of the spheres just after the impact, and show that the angle between their paths is then equal to $\tan^{-1}\frac{14}{23}$.

Solution:



Before impact the balls are moving at angle α to the line of centres.

$\alpha = \sin^{-1}\frac{1.2}{2} = \sin^{-1}\frac{3}{5}$

momentum:

$2u \times \frac{4}{5} + u \times \frac{4}{5} = v + w = \frac{12u}{5}$

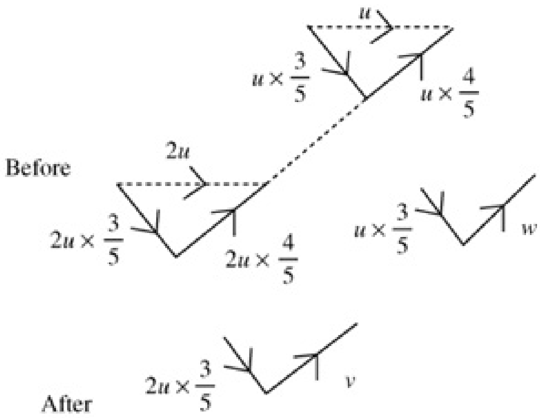
law of restitution:

$w - v = \frac{3}{4}\left(\frac{8u}{5} - \frac{4u}{5}\right) = \frac{3u}{5}$

Adding:

$2w = \frac{15u}{5} = 3u, w = \frac{3u}{2}$

$\Rightarrow v = \frac{12u}{5} - \frac{3u}{2} = \frac{9u}{10}$



Speeds are $u\sqrt{\frac{81}{100} + \frac{36}{25}} = u\sqrt{\frac{225}{100}} = \frac{3u}{2}$

and $u\sqrt{\frac{9}{4} + \frac{9}{25}} = u\sqrt{\frac{9 \times 29}{100}} = \frac{3\sqrt{29}}{10}u$

Directions relative to the line of centres are $\tan^{-1}\left(\frac{\frac{6}{5}}{\frac{9}{10}}\right) = \tan^{-1}\frac{4}{3}$ and

$\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{3}{2}}\right) = \tan^{-1}\frac{2}{5}$, so the angle between the paths is

$\tan^{-1}\left(\frac{\frac{4}{3} - \frac{2}{5}}{1 + \frac{4}{3} \times \frac{2}{5}}\right) = \tan^{-1}\left(\frac{20-6}{15+8}\right) = \tan^{-1}\frac{14}{23}$