Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 1

Question:

A particle P moves in a straight line. At time t seconds, the acceleration of P is e^{2t} m s^{-2}, where $t \geq 0$. When $t = 0$, P is at rest. Show that the speed, v m s^{-1}, of P at time t seconds is given by

$v = \frac{1}{2}(e^{2t} - 1)$ **[E]**

Solution:

$$a = \frac{dv}{dt} = e^{2t}$$

$$v = \int e^{2t}\, dt = \frac{1}{2}e^{2t} + A$$

To find the acceleration, you integrate the velocity with respect to time. Remember to include a constant of integration.

When $t = 0, v = 0$

$$0 = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2}$$

Hence $v = \frac{1}{2}(e^{2t} - 1)$, as required

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Review Exercise 1
Exercise A, Question 2

Question:

A particle P moves along the x-axis in such a way that when its displacement from the origin O is x m, its velocity is v m s^{-1} and its acceleration is $4x$ m s^{-2}. When $x = 2$, $v = 4$.
Show that $v^2 = 4x^2$. **[E]**

Solution:

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x$

When the acceleration is a function of the displacement, x metres, you write $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and integrate both sides of the equation with respect to x.

$\frac{1}{2}v^2 = \int 4x \, dx = 2x^2 + A$

$v^2 = 4x^2 + B$, where $B = 2A$

At $x = 2, v = 4$

$16 = 16 + B \Rightarrow B = 0$

Even when, as here, the constant of integration is 0, it is essential for you to show how this follows from the information given in the question to gain full marks.

Hence

$v^2 = 4x^2$

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Review Exercise 1
Exercise A, Question 3

Question:

A particle P moves along the x-axis in the positive direction. At time t seconds, the velocity of P is v m s^{-1} and its acceleration is $\frac{1}{2}e^{-\frac{1}{6}t}$ m s^{-1}. When $t = 0$ the speed of P is 10 m s^{-1}.

a Express v in terms of t.

b Find, to 3 significant figures, the speed of P when $t = 3$.

c Find the limiting value of v. **[E]**

Solution:

a $a = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = -3e^{-\frac{1}{6}t} + A$

When $t = 0, v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

Using $\int e^{kt}\,dt = \frac{1}{k}e^{kt} + A$, then

$\int \frac{1}{2}e^{-\frac{1}{6}t}\,dt = \frac{1}{2\times(-\frac{1}{6})}e^{-\frac{1}{6}t} + A = -\frac{1}{\frac{1}{3}}e^{-\frac{1}{6}t} + A$

$= -3e^{-\frac{1}{6}t} + A.$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180\ldots$

The speed of P when $t = 3$ is 11.2 m s^{-1} (3 s.f.).

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if $t = 120$, then $e^{-\frac{1}{6}t} \approx 2.06\times10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

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Review Exercise 1
Exercise A, Question 4

Question:

A particle P moves on the positive x-axis. When $OP = x$ metres, where O is the origin, the acceleration of P is directed away from O and has magnitude $\left(1-\frac{4}{x^2}\right)\text{ms}^{-2}$. When $OP = x$ metres, the velocity of P is $v\text{ ms}^{-1}$. Given that when $x = 1$, $v = 3\sqrt{2}$ show that when $x = \frac{3}{2}, v^2 = \frac{49}{3}$. **[E]**

Solution:

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 - \frac{4}{x^2} = 1 - 4x^{-2}$$

$$\frac{1}{2}v^2 = \int (1-4x^{-2})\,dx$$

$$= x - \frac{4x^{-1}}{-1} + A = x + \frac{4}{x} + A$$

$$v^2 = 2x + \frac{8}{x} + B, \text{ where } B = 2A$$

Multiplying the equation $\frac{1}{2}v^2 = 2x + \frac{4}{x} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 1, v = 3\sqrt{2}$

$$18 = 2 + 8 + B \Rightarrow B = 8$$

Hence $v^2 = 2x + \frac{8}{x} + 8$

At $x = \frac{3}{2}$

$$v^2 = 2\times\frac{3}{2} + 8\times\frac{2}{3} + 8 = 11 + \frac{16}{3} = \frac{49}{3}, \text{ as required.}$$

You use the information that at $x = 1, v = 3\sqrt{2}$ to evaluate the constant of integration B. You then substitute $x = \frac{3}{2}$ into the resulting equation and show that $v^2 = \frac{49}{3}$.

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Review Exercise 1
Exercise A, Question 5

Question:

A particle P is moving in a straight line. When P is at a distance x metres from a fixed point O on the line, the acceleration of P is $(5+3\sin 3x)\text{m s}^{-2}$ in the direction OP.
Given that P passes through O with speed 4 m s^{-1}, find the speed of P at $x=6$
Give your answer to 3 significant figures.

Solution:

$a=\frac{\text{d}}{\text{d}x}\left(\frac{1}{2}v^2\right)=5+3\sin 3x$

$\frac{1}{2}v^2=\int(5+3\sin 3x)\text{d}x=5x-\cos 3x+A$

$v^2=10x-2\cos 3x+B$, where $B=2A$

At $x=0, v=4$

$16=0-2+B \Rightarrow B=18$

You use the information that at $x=0, v=4$ to evaluate the constant of integration B. You then substitute $x=6$ into the resulting equation and use your calculator to find v.

Hence $v^2=10x-2\cos 3x+18$

At $x=6$

$v^2=60-2\cos 18+18=76.679\ldots$

When calculus has been used, it is assumed that all angles are measured in radians and you must make sure that your calculator is in the correct mode.

$v=\sqrt{76.679\ldots}=8.756\ldots$

The speed of P at $x=6$ is 8.76 m s^{-1} (3 s.f.)

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Review Exercise 1
Exercise A, Question 6

Question:

A particle P is moving along the positive x-axis in the direction of x increasing. When $OP = x$ metres, the velocity of P is v m s^{-1} and the acceleration of P is $\frac{4k^2}{(x+1)^2}$ m s^{-2}, where k is a positive constant. At $x = 1$, $v = 0$.

a Find v^2 in terms of x and k.

b Deduce that v cannot exceed $2k$.

[E]

Solution:

a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{4k^2}{(x+1)^2} = 4k^2(x+1)^{-2}$

$\frac{1}{2}v^2 = \int 4k^2(x+1)^{-2}\,\mathrm{d}x = \frac{4k^2(x+1)^{-1}}{-1} + A$

$v^2 = B - \frac{8k^2}{x+1}$, where $B = 2A$

At $x = 1, v = 0$

$0 = B - \frac{8k^2}{2} \Rightarrow B = 4k^2$

Hence $v^2 = 4k^2 - \frac{8k^2}{x+1} = 4k^2\left(1 - \frac{2}{x+1}\right)$

b $v = 2k\sqrt{\left(1 - \frac{2}{x+1}\right)}$

As P is moving on the positive x-axis in the direction of x increasing, you need not consider the possibility of a negative square root.

As x is positive, $1 - \frac{1}{x+1} < 1$

Hence $v < 2k$ and v cannot exceed $2k$.

As x is positive, $\frac{1}{1+x}$ is positive and one minus a positive number must be less than one.

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Review Exercise 1
Exercise A, Question 7

Question:

A particle P moves along the x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x-direction. At time t seconds, the velocity of P is v m s^{-1} and $OP = x$ metres. The acceleration of P is $\frac{1}{12}(30 - x)$m s^{-2}, measured in the positive x- direction.

a Give a reason why the maximum speed of P occurs when $x = 30$.

Given that the maximum speed of P is 10 m s^{-1},

b find an expression for v^2 in terms of x. **[E]**

Solution:

a At the maximum value of v, $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$.

As $a = \frac{\mathrm{d}v}{\mathrm{d}t}$, the maximum speed of P occurs when $a = \frac{1}{12}(30 - x) = 0 \Rightarrow x = 30$.

b $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{1}{12}(30 - x)$

$\frac{1}{2}v^2 = \int \frac{1}{12}(30 - x)\mathrm{d}x = \int\left(\frac{5}{2} - \frac{x}{12}\right)\mathrm{d}x$

$= \frac{5x}{2} - \frac{x^2}{24} + A$

$v^2 = 5x - \frac{x^2}{12} + B$, where $B = 2A$ ←

Multiplying the equation $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{24} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 30, v = 10$

$100 = 5 \times 30 - \frac{900}{12} + B$

$B = 100 + \frac{900}{12} - 150 = 25$

Hence $v^2 = 5x - \frac{x^2}{12} + 25$ ←

An alternative form of this answer, completing the square, is $v^2 = 100 - \frac{1}{12}(30 - x)^2$. This confirms that the speed has a maximum at $x = 30$.

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Review Exercise 1
Exercise A, Question 8

Question:

A particle P moves along the x-axis. At time t seconds the velocity of P is v m s^{-1} and its acceleration is $2\sin\frac{1}{2}t$ m s^{-2}, both measured in the direction Ox. Given that $v = 4$ when $t = 0$,

a find v in terms of t,

b calculate the distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$. **[E]**

Solution:

a $a = \frac{dv}{dt} = 2\sin\frac{1}{2}t$

$v = \int 2\sin\frac{1}{2}t\, dt = -4\cos\frac{1}{2}t + A$

Using the formula $\int \sin at\, dt = -\frac{1}{a}\cos at + A$,
$\int 2\sin\frac{1}{2}t\, dt = -\frac{2}{\frac{1}{2}}\cos\frac{1}{2}t + A = -4\cos\frac{1}{2}t + A$

When $t = 0, v = 4$

$4 = -4 + A \Rightarrow A = 8$

Hence $v = 8 - 4\cos\frac{1}{2}t$

b The distance, s metres, travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is given by

$s = \int_0^{\frac{\pi}{2}} \left(8 - 4\cos\frac{1}{2}t\right) dt$

The change in the displacement of P between any two times, say t_1 and t_2, can be found by calculating the definite integral of the velocity between the limits t_1 and t_2. If P has not turned round, this will also give the distance travelled by P. The particle in this question does turn round when $\frac{dv}{dt} = a = 0$ but that does not happen until $t = 2\pi$, so P does not turn round in the interval $0 \le t \le \frac{\pi}{2}$.

$= \left[8t - 8\sin\frac{1}{2}t\right]_0^{\frac{\pi}{2}}$

$= 4\pi - 8\sin\frac{\pi}{4} = 4\pi - \frac{8}{\sqrt{2}}$

$= 4\pi - 4\sqrt{2} = 4(\pi - \sqrt{2})$

The distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is $4(\pi - \sqrt{2})$m .

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Review Exercise 1
Exercise A, Question 9

Question:

A particle P moves along the x-axis. At time t seconds its acceleration is $(-4e^{-2t})\text{m s}^{-2}$ in the direction of x increasing. When $t = 0$, P is at the origin O and is moving with speed 1 m s^{-1} in the direction of x increasing.

a Find an expression for the velocity of P at time t.
b Find the distance of P from O when P comes to instantaneous rest. **[E]**

Solution:

a $a = \frac{dv}{dt} = -4e^{-2t}$

$v = -\int 4e^{-2t}\,dt = 2e^{-2t} + A$

At $t = 0, v = 1$

$1 = 2 + A \Rightarrow A = -1$

Hence $v = 2e^{-2t} - 1$

b P is instantaneously at rest when $v = 0$.

$0 = 2e^{-2t} - 1$

To find the speed when P is instantaneously at rest, you will need to know the value of t when $v = 0$.

$e^{-2t} = \frac{1}{2} \Rightarrow e^{2t} = 2$

Take natural logarithms of both sides of this equation and use the property that, for any x, $\ln(e^x) = x$

$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$

$x = \int v\,dt = \int (2e^{-2t} - 1)\,dt$

$= -e^{-2t} - t + B$

When $t = 0, x = 0$

$0 = -1 - 0 + B \Rightarrow B = 1$

Using $e^0 = 1$. It is a common error to obtain $B = 0$ by carelessly writing $e^0 = 0$.

Hence $x = 1 - e^{-2t} - t$

When $t = \frac{1}{2}\ln 2$

$x = 1 - e^{-2(\frac{1}{2}\ln 2)} - \frac{1}{2}\ln 2 = 1 - e^{-\ln 2} - \frac{1}{2}\ln 2$

$= 1 - e^{\ln\frac{1}{2}} - \frac{1}{2}\ln 2 = 1 - \frac{1}{2} - \frac{1}{2}\ln 2$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

Using the law of logarithms, $\ln a^n = n\ln a$ with $n = -1$, $-\ln 2 = (-1)\ln 2 = \ln 2^{-1} = \ln\frac{1}{2}$. Then as for any x, $e^{\ln x} = x$, $e^{\ln\frac{1}{2}} = \frac{1}{2}$.

The distance of P from O when P comes to instantaneous rest is $\frac{1}{2}(1 - \ln 2)\text{m}$.

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Review Exercise 1
Exercise A, Question 10

Question:

At time $t = 0$, a particle P is at the origin O moving with speed 18 m s^{-1} along the x-axis in the positive x-direction. At time t seconds $(t > 0)$ the acceleration of P has magnitude $\frac{3}{\sqrt{(t+4)}} \text{ m s}^{-2}$ and is directed towards O.

a Show that, at time t seconds, the velocity of P is $\left[30 - 6\sqrt{(t+4)}\right] \text{ m s}^{-1}$. **[E]**

b Find the distance of P from O when P comes to instantaneous rest.

Solution:

a $a = \frac{dv}{dt} = -\frac{3}{\sqrt{(t+4)}} = -3(t+4)^{-\frac{1}{2}}$

As the acceleration is towards O, $\frac{dv}{dt}$, which is always measured in the direction of x increasing, is negative.

$v = -3\int (t+4)^{-\frac{1}{2}}\, dt = \frac{-3(t+4)^{\frac{1}{2}}}{\frac{1}{2}} + A = A - 6(t+4)^{\frac{1}{2}}$

When $t = 0, v = 18$

$18 = A - 6 \times 2 \Rightarrow A = 30$

Hence

$v = 30 - 6(t+4)^{\frac{1}{2}}$

The velocity of P is $\left[30 - 6\sqrt{(t+4)}\right] \text{ m s}^{-1}$, as required.

b $0 = 30 - 6(t+4)^{\frac{1}{2}}$

$(t+4)^{\frac{1}{2}} = 5 \Rightarrow t + 4 = 25 \Rightarrow t = 21$

There are three steps needed to solve part **b**. First you must find the value of t for which P is instantaneously at rest; that is when $v = 0$. You must also find x in terms of t by integrating the expression you proved in part **a**. Finally you substitute your value of t into your expression for x. It is a characteristic of harder questions at this level that you often have to construct for yourself the steps needed to solve a problem.

$v = \frac{dx}{dt} = 30 - 6(t+4)^{\frac{1}{2}}$

$x = \int \left(30 - 6(t+4)^{\frac{1}{2}}\right) dt = 30t - \frac{6(t+4)^{\frac{3}{2}}}{\frac{3}{2}} + B$

$= 30t - 4(t+4)^{\frac{3}{2}} + B$

When $t = 0, x = 0$

$0 = 0 - 4 \times 4^{\frac{3}{2}} + B$

$B = 4 \times 4^{\frac{3}{2}} = 4 \times 8 = 32$

Hence $x = 30t - 4(t+4)^{\frac{3}{2}} + 32$

When $t = 21$

$x = 30 \times 21 - 4(25)^{\frac{3}{2}} + 32 = 630 - 500 + 32 = 162$

The distance of P from O when P comes to instantaneous rest is 162 m.

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Review Exercise 1
Exercise A, Question 11

Question:

A particle P starts at rest and moves in a straight line. The acceleration of P initially has magnitude 20 m s^{-2} and, in a first model of the motion of P, it is assumed that this acceleration remains constant.

a For this model, find the distance moved by P while accelerating from rest to a speed of 6 m s^{-1}.

The acceleration of P when it is x metres from its initial position is $a\text{ m s}^{-2}$ and it is then established that $a = 12$ when $x = 2$. A refined model is proposed in which $a = p - qx$, where p and q are constants.

b Show that, under the refined model, $p = 20$ and $q = 4$.

c Hence find, for this model, the distance moved by P in first attaining a speed of 6 m s^{-1}. **[E]**

Solution:

a $u = 0, a = 20, v = 6, s = ?$

$v^2 = u^2 + 2as$

$36 = 0 + 2 \times 20 \times s$

$s = \frac{36}{40} = 0.9$

For the first model, the distance moved by P while accelerating from rest to 6 m s^{-1} is 0.9 m.

> The model in part **a** is that of constant acceleration, which you studied in module M1. The specification for M3 includes 'a knowledge of the specifications for M1 and M2 and their prerequisites and associated formulae ... is assumed and may be tested'.

b $a = p - qx$

At $x = 0, a = 20$

$20 = p - 0 \Rightarrow p = 20$

Hence $a = 20 - qx$

At $x = 2, a = 12$

$12 = 20 - 2q \Rightarrow q = \frac{20-12}{2} = 4$

$p = 20, q = 4$, as required.

> The initial acceleration is 20 m s^{-2}. This applies to all parts of the question. Additionally in part **b**, you are given that $a = 12$ when $x = 2$. The two conditions enable you to find the two unknowns p and q.

c $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 20 - 4x$

$\frac{1}{2}v^2 = \int (20 - 4x)\,dx = 20x - 2x^2 + A$

$v^2 = 40x - 4x^2 + B$, where $B = 2A$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

Hence $v^2 = 40x - 4x^2$

When $v = 6$

$36 = 40x - 4x^2 \Rightarrow 4x^2 - 40x + 36 = 0$

$x^2 - 10x + 9 = (x-1)(x-9) = 0$

$x = 1, 9$

The distance moved by P in first attaining a speed of 6 m s^{-1} is 1 m.

> Divide this equation throughout by 4 and factorise.

> Comparing this with result in part **a**, the revised model predicts that P moves a little further before reaching the speed of 6 m s^{-1}.

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Review Exercise 1
Exercise A, Question 12

Question:

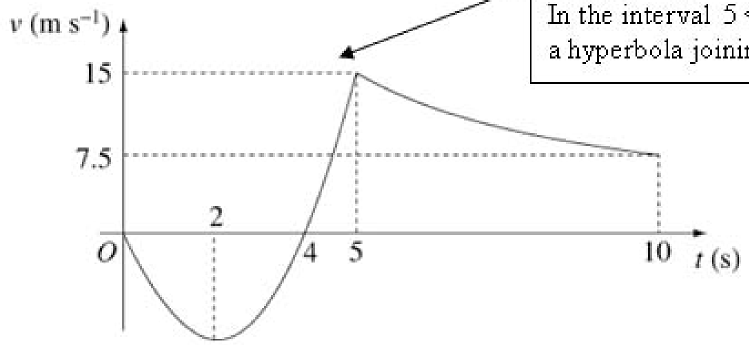
A particle moving in a straight line starts from rest at a point O at time $t = 0$.
At time t seconds, the velocity v m s^{-1} is given by

$$v = \begin{cases} 3t(t-4), & 0 \le t \le 5 \\ 75t^{-1}, & 5 < t \le 10 \end{cases}$$

a Sketch a velocity - time graph for the particle for $0 \le t \le 10$.
b Find the set of values of t for which the acceleration of the particle is positive.
c Show that the total distance travelled by the particle in the interval $0 \le t \le 5$ is 39 m.
d Find, to 3 significant figures, the value of t at which the particle returns to O. **[E]**

Solution:

a



In the interval $0 \le t \le 5$, the graph is part of a parabola which meets the t-axis at the origin and where $t = 4$.
In the interval $5 < t \le 10$, the graph is a segment of a hyperbola joining (5, 15) to (10, 7.5).

b The set of values of t for which the acceleration is positive is $2 < t < 5$.

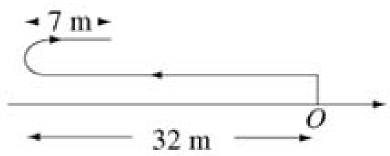
The acceleration is positive when the velocity–time graph has a positive gradient. By the symmetry of a parabola, the graph has a minimum when $t = 2$ and the set of values of t for which the gradient is positive can be written down by inspecting the graph.

c

$$\int_0^4 3t(t-4)\,dt = \int_0^4 (3t^2 - 12t)\,dt$$
$$= \left[t^3 - 6t^2\right]_0^4$$
$$= (64 - 96) - 0 = -32$$
$$\int_4^5 3t(t-4)\,dt = \int_4^5 (3t^2 - 12t)\,dt$$
$$= \left[t^3 - 6t^2\right]_4^5$$
$$= (125 - 150) - (64 - 96)$$
$$= 7$$

The distance travelled by P in the interval $0 \le t \le 5$ is $(32 + 7)\text{m} = 39\text{ m}$.

Taking the direction of v increasing as positive, for the first 4 seconds the particle travels 32 m in the negative direction. In the next second, it travels 7 m in the positive direction. So in 5 seconds, it travels a total of $(32 + 7)\text{m}$ ending at a point which is $(32 - 7)\text{m}$ from O in the negative direction.



d For $t > 5$

$$x = \int v\,dt = \int 75t^{-1}\,dt$$
$$= 75\ln t + A$$

At time $t = 5$, the particle is $(32 - 7)\text{m} = 25\text{ m}$ from O in the negative direction.

So when $t = 5, x = -25$

$-25 = 75\ln 5 + A \Rightarrow A = -75\ln 5 - 25$

Hence

$$x = 75\ln t - 75\ln 5 - 25 = 75\ln\left(\frac{t}{5}\right) - 25$$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$,

$75\ln t - 75\ln 5 = 75(\ln t - \ln 5) = 75\ln\left(\frac{t}{5}\right)$.

At $x = 0$

$$0 = 75\ln\left(\frac{t}{5}\right) - 25 \Rightarrow \ln\left(\frac{t}{5}\right) = \frac{1}{3}$$

$$\frac{t}{5} = e^{\frac{1}{3}} \Rightarrow t = 5e^{\frac{1}{3}} = 6.98\text{ (3 s.f.)}$$

You solve this equation for t by taking exponentials of both sides of the equation and using $e^{\ln\left(\frac{t}{5}\right)} = \frac{t}{5}$.

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Review Exercise 1
Exercise A, Question 13

Question:

A particle P moves along the positive x-axis. When $OP = x$ metres, the velocity of P is v m s^{-1} and its acceleration is $\frac{72}{(2x+1)^2}$ m s^{-2} in the direction of x increasing.

Initially $x = 1$ and P is moving toward O with speed 6 m s^{-1}. Find

a v in terms of x,

b the minimum distance of P from O.

Solution:

a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{72}{(2x+1)^2} = 72(2x+1)^{-2}$

$\frac{1}{2}v^2 = \int 72(2x+1)^{-2}\ dx = \frac{72(2x+1)^{-1}}{2\times(-1)} + A$

$= A - \frac{36}{2x+1}$

$v^2 = B - \frac{72}{2x+1}$, where $B = 2A$

At $x = 1, v = -6$

$36 = B - \frac{72}{3} \Rightarrow B = 60$

Hence $v^2 = 60 - \frac{72}{2x+1}$

To integrate $(2x+1)^{-2}$, you can use the formula $\int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + A$.

b At a minimum value of x, $\frac{dx}{dt} = 0$

and so $v = 0$.

Substituting $v = 0$ into the result of part **a**

$0 = 60 - \frac{72}{2x+1} \quad \Rightarrow 60 = \frac{72}{2x+1}$

$2x+1 = \frac{72}{60} = 1.2 \Rightarrow x = \frac{1.2-1}{2} = 0.1$

The minimum distance of P from O is 0.1 m.

In this question, it is not practical to find x in terms of t. However, to find the minimum value of x, this is not necessary. The minimum is a stationary value and at a stationary value $\frac{dx}{dt}$, which is v, is zero

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Review Exercise 1
Exercise A, Question 14

Question:

A particle moves on the positive x-axis. The particle is moving towards the origin O when it passes through the point A, where $x = 2a$, with speed $\sqrt{\left(\frac{k}{a}\right)}$, where k is a constant. Given that the particle experiences an acceleration $\frac{k}{2x^2} + \frac{k}{4a^2}$ in a direction away from O,

a show that it comes instantaneously to rest at a point B, where $x = a$.

As soon as the particle reaches B the acceleration changes to $\frac{k}{2x^2} - \frac{k}{4a^2}$ in a direction away from O.

b Show that the particle next comes instantaneously to rest at A. **[E]**

Solution:

a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} + \frac{k}{4a^2} = \frac{k}{2}x^{-2} + \frac{k}{4a^2}$

$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} + \frac{k}{4a^2}\right)dx = \frac{k}{2}\times\frac{x^{-1}}{-1} + \frac{kx}{4a^2} + A$

$= -\frac{k}{2x} + \frac{kx}{4a^2} + A$

$v^2 = -\frac{k}{x} + \frac{kx}{2a^2} + B$, where $B = 2A$

At $x = 2a, v = -\sqrt{\left(\frac{k}{a}\right)}$

$\frac{k}{a} = -\frac{k}{2a} + \frac{2ka}{2a^2} + B$

$B = \frac{k}{a} + \frac{k}{2a} - \frac{k}{a} = \frac{k}{2a}$

Hence $v^2 = -\frac{k}{x} + \frac{kx}{2a^2} + \frac{k}{2a}$

At $x = a$

$v^2 = -\frac{k}{a} + \frac{ka}{2a^2} + \frac{k}{2a} = -\frac{k}{a} + \frac{k}{2a} + \frac{k}{2a} = 0$

The particle comes to instantaneous rest where $x = a$.

To show that the particle comes to rest at A, you use integration to obtain v^2 in terms of x, and then substitute $x = a$ into your expression and show that $v = 0$.

b $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} - \frac{k}{4a^2} = \frac{k}{2}x^{-2} - \frac{k}{4a^2}$

$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} - \frac{k}{4a^2}\right)dx = \frac{k}{2}\times\frac{x^{-1}}{-1} - \frac{kx}{4a^2} + C$

$= -\frac{k}{2x} - \frac{kx}{4a^2} + C$

$v^2 = -\frac{k}{x} - \frac{kx}{2a^2} + D$, where $D = 2C$

At $x = a, v = 0$

Although the acceleration changes at B you can assume that the velocity is continuous and that the final velocity, 0, in part **a** is the initial velocity in part **b**.

$0 = -\frac{k}{a} - \frac{k}{2a} + D \Rightarrow D = \frac{3k}{2a}$

Hence

$v^2 = -\frac{k}{x} - \frac{kx}{2a^2} + \frac{3k}{2a} = \frac{-2ka^2 - kx^2 + 3kax}{2a^2x}$

$= -\frac{k}{2a^2x}(x^2 - 3ax + 2a^2) = -\frac{k}{2a^2x}(x-a)(x-2a)$

There are a number of different ways of completing this question. The solution shown here puts all of the terms on the right of the equation over a common denominator and factorises the resulting expression.

When $v = 0, (x-a)(x-2a) = 0$

$x = a, 2a$

After leaving B, the particle next comes to rest at A, where $x = 2a$.

$x = a$ corresponds to the point B and $x = 2a$ corresponds to the point A.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 15

Question:

A car is travelling along a straight horizontal road. As it passes a point O on the road, the engine is switched off. At time t seconds after the car has passed O, it is at a point P, where $OP = x$ metres, and its velocity is v m s^{-1}. The motion of the car is modelled by

$$v = \frac{1}{p + qt}$$

where p and q are positive constants.

a Show that, with this model, the retardation of the car is proportional to the square of the speed.

When $t = 0$, the retardation of the car is 0.75 m s^{-2} and $v = 20$.

Using the model, find

b the value of p and the value of q,

c x in terms of t.

Solution:

a $v = \dfrac{1}{p+qt} = (p+qt)^{-1}$

$a = \dfrac{dv}{dt} = (-1)q(p+qt)^{-2}$

$= -\dfrac{q}{(p+qt)^2} = -qv^2$

So the deceleration is proportional to the square of the speed.

The deceleration is the negative of the acceleration. If $d = kv^n$, for any constant k, then d is proportional to v^n.

The square of the speed and the square of the velocity are identical because, for example, $(-20)^2 = 20^2$.

b When $t = 0, a = -0.75$ and $v = 20$

$a = -qv^2$

$-0.75 = -q \times 20^2$

$q = \dfrac{3}{4} \times \dfrac{1}{20^2} = \dfrac{3}{1600}$

The exact decimal answers $p = 0.05$ and $q = 0.001\,875$ are also acceptable.

$v = \dfrac{1}{p+qt}$

When $t = 0, v = 20$

$20 = \dfrac{1}{p} \Rightarrow p = \dfrac{1}{20}$

$p = \dfrac{1}{20}, q = \dfrac{3}{1600}$

c $v = \dfrac{dx}{dt} = \dfrac{1}{p+qt}$

$x = \displaystyle\int \frac{1}{p+qt}\,dt = \frac{1}{q}\ln(p+qt) + A$

$= \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) + A$

When $t = 0$, $x = 0$

$0 = \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right) + A \Rightarrow A = -\dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

Hence $x = \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) - \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

This expression can be simplified using the law of logarithms $\ln a - \ln b = \ln\dfrac{a}{b}$. However, as the question specifies no particular form for the answer, an unsimplified answer or an answer with decimals would be accepted.

$= \dfrac{1600}{3}\ln\left(\dfrac{\dfrac{1}{20} + \dfrac{3}{1600}t}{\dfrac{1}{20}}\right)$

$x = \dfrac{1600}{3}\ln\left(1 + \dfrac{3}{80}t\right)$

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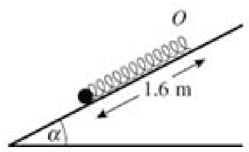
Edexcel AS and A Level Modular Mathematics

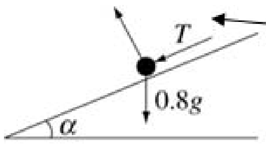
Review Exercise 1
Exercise A, Question 16

Question:



A particle of mass 0.8 kg is attached to one end of a light elastic spring, of natural length 2 m and modulus of elasticity 20 N.
The other end of the spring is attached to a fixed point O on a smooth plane which is inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$.
The particle is held at a point which is 1.6 m down the line of greatest slope of the plane from O, as shown in the figure. The particle is then released from rest. Find the initial acceleration of the particle. **[E]**

Solution:



Initially the spring is in compression and the force of the spring on the particle is acting down the plane.

Let the thrust in the spring be T newtons.

Hooke's law $T = \frac{\lambda x}{l}$

$= \frac{20 \times 0.4}{2} = 4$

The compression is $(2 - 1.6)\text{m} = 0.4\text{ m}$.

R(↙) $\mathbf{F} = m\mathbf{a}$

$mg \sin \alpha + T = ma$

$0.8 \times 9.8 \times \frac{3}{5} + 4 = 0.8a$

$0.8a = 8.704$

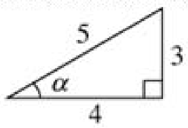
$a = 10.88$

When you know $\tan \alpha$ you can draw a triangle to find $\cos \alpha$ and $\sin \alpha$.

$\tan \alpha = \frac{3}{4}$

$\sin \alpha = \frac{3}{5}$

$\cos \alpha = \frac{4}{5}$



The initial acceleration of the particle is 11 m s^{-1} (2 s.f.)

As you have used an approximate value of g, you should round your answer to a sensible accuracy. Either 2 or 3 significant figures is acceptable.

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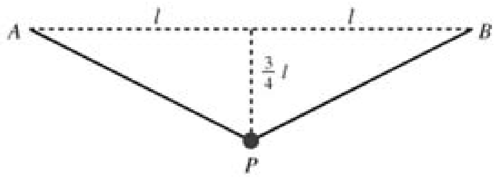
Review Exercise 1
Exercise A, Question 17

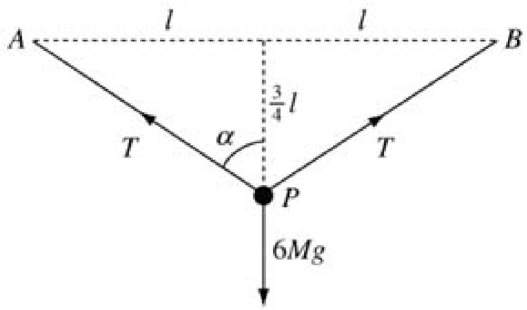
Question:



The figure shows a particle P, of mass $6M$, suspended by two light elastic strings from points A and B which are fixed and at a horizontal distance $2l$ apart. Each string has natural length l and P rests in equilibrium at a vertical distance $\frac{3}{4}l$ below the level of AB. Determine

a the tension in either string,
b the modulus of elasticity of either string. **[E]**

Solution:



a Let the angle between PA and the vertical be α.

$\tan\alpha = \frac{l}{\frac{3}{4}l} = \frac{4}{3} \Rightarrow \sin\alpha = \frac{4}{5}, \cos\alpha = \frac{3}{5}$

Let the tensions in AP and BP be T newtons.

By symmetry, the tension in both strings is the same.

$R(\uparrow)\ 2T\cos\alpha = 6Mg$

$2T \times \frac{3}{5} = 6Mg$

$T = 6Mg \times \frac{5}{6} = 5Mg$

The tension in both strings is $5Mg$.

b $AP^2 = l^2 + \left(\frac{3}{4}l\right)^2 = \frac{25}{16}l^2 \Rightarrow AP = \frac{5}{4}l$

You use Pythagoras' theorem to find the length of one of the strings and use this length to find the extension of the string.

The extension of AP is $\left(\frac{5}{4}l - l\right) = \frac{1}{4}l$

Hooke's law $T = \frac{\lambda x}{l}$

$5Mg = \frac{\lambda \times \frac{1}{4}l}{l} = \frac{\lambda}{4}$

As you know the tension in the string from part **a**, you can use Hooke's law to find the modulus of elasticity, λ.

The modulus of elasticity in either string is $20Mg$.

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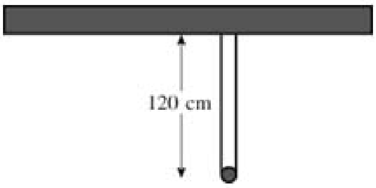
Edexcel AS and A Level Modular Mathematics

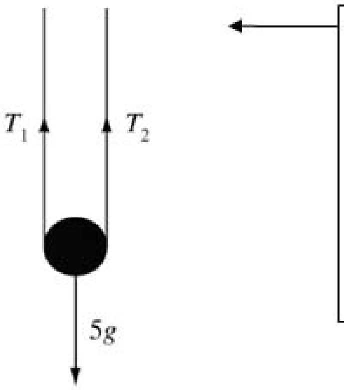
Review Exercise 1
Exercise A, Question 18

Question:



A particle of mass 5 kg is attached to one end of two light elastic strings. The other ends of the string are attached to a hook on a beam. The particle hangs in equilibrium at a distance 120 cm below the hook with both strings vertical, as shown in the figure. One string has natural length 100 cm and modulus of elasticity 175 N. The other string has natural length 90 cm and modulus of elasticity λ newtons.
Find the value of λ. **[E]**

Solution:



You must consider the two strings separately. Here the string of natural length 100 cm is drawn on the left. The extension of this string is $(120-100)\text{cm} = 20$ cm and the tension in this string is taken as T_1 newtons. The string of natural length 90 cm is drawn on the right. The extension of this string is $(120-90)\text{cm} = 30$ cm and the tension in this string is taken as T_2 newtons.

For the string of natural length 100 cm

Hooke's law $T = \frac{\lambda x}{l}$

$$T_1 = \frac{175 \times 20}{100} = 35$$

For the string of natural length 90 cm

Hooke's law $T = \frac{\lambda x}{l}$

$$T_2 = \frac{\lambda \times 30}{90} = \frac{\lambda}{3}$$

R($\uparrow$) $T_1 + T_2 = 5g$

$$35 + \frac{\lambda}{3} = 5 \times 9.8$$

$$\lambda = 3(5 \times 9.8 - 35) = 42$$

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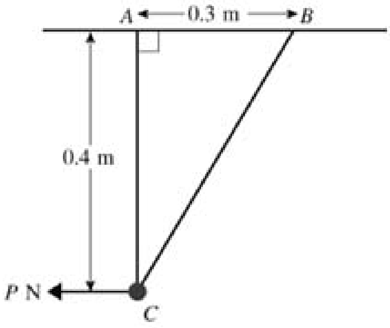
Review Exercise 1
Exercise A, Question 19

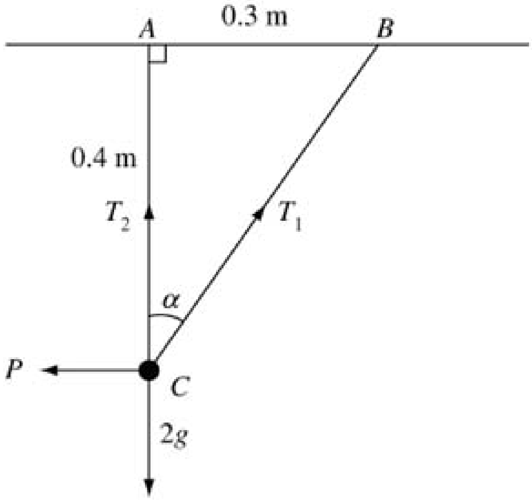
Question:



The figure shows a particle C of mass 2 kg suspended by two strings. The strings are fixed to two points A and B on a horizontal ceiling, where $AB = 0.3$ m . The string AC is light and inextensible, with length 0.4 m, while the string BC is light and elastic with natural length 0.4 m and modulus of elasticity 32 N. A horizontal force of magnitude P N holds the system in equilibrium with AC vertical.

a Show that the tension in BC is 8 N.
b Find the value of P.
c Find the tension in AC. **[E]**

Solution:



a Let $BC = x$ m

$$x^2 = 0.3^2 + 0.4^2 = 0.25 \Rightarrow x = 0.5$$

The extension of BC is $(0.5 - 0.4)\text{m} = 0.1\text{ m}$.

Let the tension in BC be T_1 newtons

Hooke's Law $T = \dfrac{\lambda x}{l}$

$$T_1 = \frac{32 \times 0.1}{0.4} = 8$$

The tension in BC is 8 N, as required.

> BC is an elastic string and, to find its tension, you need to know its extension. You use Pythagoras' theorem to find the length of BC and subtract the natural length to find the extension.

b Let $\angle ACB = \alpha$

$$\tan\alpha = \frac{0.3}{0.4} = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$$

$R(\rightarrow)$ $P = T_1 \sin\alpha$

$$= 8 \times \frac{3}{5} = 4.8$$

> To resolve, you need to know the sines and cosines of an appropriate angle. The dimensions of the triangle ABC enable you to find these.

c Let the tension in AC be T_2 newtons

$R(\uparrow)$ $T_2 + T_1\cos\alpha = 2g$

$$T_2 = 2 \times 9.8 - 8 \times \frac{4}{5} = 13.2$$

The tension is AC is 13.2 N.

> When a numerical value of g is used, 2 or 3 significant figures is acceptable.

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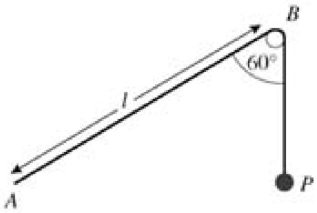
Review Exercise 1
Exercise A, Question 20

Question:



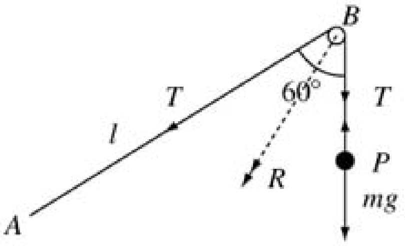
A light elastic string, of natural length l and modulus of elasticity $4mg$, has one end tied to a fixed point A. The string passes over a fixed small smooth peg B and at the other end a particle P, of mass m, is attached. The particle hangs in equilibrium. The distance between A and B is l and AB is inclined at $60°$ to the vertical, as shown in the figure.

a Find, in terms of l, the length of the vertical portion BP of the string.

b Show that the magnitude of the force exerted by the string on the peg is $mg\sqrt{3}$.

[E]

Solution:



As the string is light and the peg is smooth, the tension, T say, is the same throughout the string.

a Let $BP = x$

For the particle P

$R(\uparrow)\quad T = mg$

Hooke's law

$$T = \frac{\lambda x}{l}$$

$$mg = \frac{4mg \times x}{l}$$

$$x = \frac{l}{4}$$

$$BP = \frac{l}{4}$$

As the natural length of the string is l and $AB = l$, the length BP is the extension of the string.

b Resolving along the bisector of the angle between the two portions of the string

$R(\swarrow)\quad R = 2T\cos 30°$

$$= 2mg \times \frac{\sqrt{3}}{2}$$

$$= mg\sqrt{3}, \text{ as required}$$

As both parts of the string exert the same force T on the peg, by symmetry, the resultant force on the peg acts along the angle bisector of the angle between the two portions of the string. You find the magnitude of the resultant force, R say, by resolving along this angle bisector.

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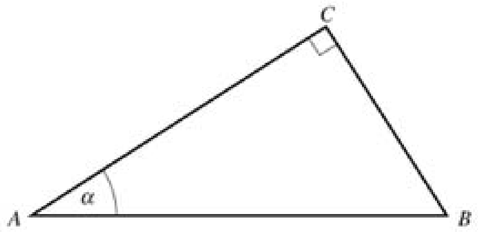
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 21

Question:



A rod AB, of mass $2m$ and length $2a$, is suspended from a fixed point C by two light strings AC and BC. The rod rests horizontally in equilibrium with AC making an angle α with the rod, where $\tan \alpha = \frac{3}{4}$, and with AC perpendicular to BC, as shown in the figure.

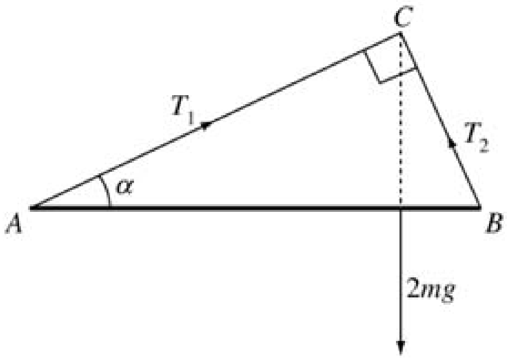
a Give a reason why the rod cannot be uniform.

b Show that the tension in BC is $\frac{8}{5}mg$ and find the tension in AC.

The string BC is elastic, with natural length a and modulus of elasticity kmg, where k is a constant.

c Find the value of k. **[E]**

Solution:



a The line of action of the weight must pass through C which is not above the centre of the rod.

For three forces to be in equilibrium the lines of action of all three forces must pass through the same point. As the lines of action of both tensions pass through C, the line of action of the weight has to pass through C as well and so the rod cannot be uniform.

b $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

Let the tension in AC be T_1 newtons and the tension in BC be T_2 newtons.

$R(\rightarrow) \quad T_1\cos\alpha = T_2\sin\alpha$

$\frac{4}{5}T_1 = \frac{3}{5}T_2 \Rightarrow T_1 = \frac{3}{4}T_2$

$R(\uparrow) \quad T_1\sin\alpha + T_2\cos\alpha = 2mg$

$\frac{3}{4}T_2 \times \frac{3}{5} + T_2 \times \frac{4}{5} = 2mg$

$\left(\frac{9}{20} + \frac{4}{5}\right)T_2 = \frac{5}{4}T_2 = 2mg$

$T_2 = \frac{8}{5}mg$

You substitute $T_1 = \frac{3}{4}T_2$ and the values of $\sin\alpha$ and $\cos\alpha$ into this equation and solve for T_2.

The tension in BC is $\frac{8}{5}mg$, as required.

$T_1 = \frac{3}{4}T_2 = \frac{3}{4} \times \frac{8}{5}mg = \frac{6}{5}mg$

The tension in AC is $\frac{6}{5}mg$.

c $BC = AB\sin\alpha = 2a \times \frac{3}{5} = \frac{6}{5}a$

For BC

Hooke's law $\quad T_2 = \frac{\lambda x}{l}$

$\frac{8}{5}mg = \frac{kmg \times \frac{1}{5}a}{a} \Rightarrow k = 8$

You find the length of BC by trigonometry. Then the extension of the elastic string BC is $\frac{6}{5}a - a = \frac{1}{5}a$.

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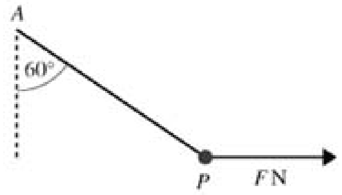
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 22

Question:



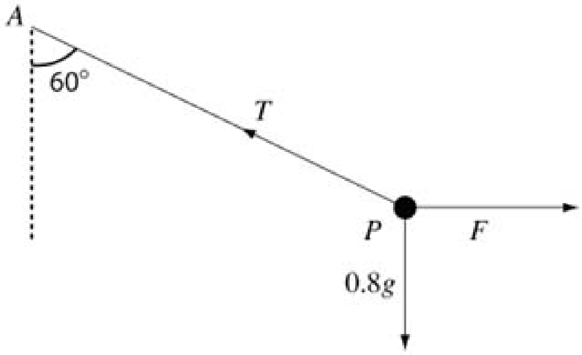
A particle of mass 0.8 kg is attached to one end of a light elastic string, of natural length 1.2 m and modulus of elasticity 24 N. The other end of the string is attached to a fixed point A. A horizontal force of magnitude F newtons is applied to P. The particle is in equilibrium with the string making an angle $60°$ with the downward vertical as shown in the figure. Calculate

a the value of F,

b the extension of the string,

c the elastic energy stored in the string. **[E]**

Solution:



R(↑) $T\cos 60° = 0.8\,g$

$\frac{1}{2}T = 0.8\,g \Rightarrow T = 1.6\,g$

Resolving vertically gives you the tension in the string.

R(←) $F = T\cos 30° = 1.6\,g \times \frac{\sqrt{3}}{2}$

$= 13.579\ldots$

$= 14$ (2 s.f.)

Substituting for the tension into the equation obtained by resolving horizontally gives the value of F.

b Hooke's law $T = \frac{\lambda x}{l}$

$1.6\,g = \frac{24x}{1.2}$

$x = \frac{1.6\,g \times 1.2}{24} = 0.784$

Substituting for the tension into Hooke's Law gives you an equation for the extension.

The extension of the string is 0.78 m (2 s.f.).

c The elastic energy stored in the string is given by

$E = \frac{\lambda x^2}{2l}$

You need to remember the formula for the energy stored in an elastic string.

$= \frac{24 \times (0.784)^2}{2 \times 1.2} = 6.146\,56$

The elastic energy stored in the string is 6.1 J (2 s.f.)

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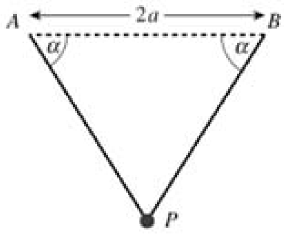
Edexcel AS and A Level Modular Mathematics

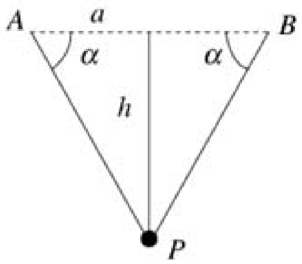
Review Exercise 1
Exercise A, Question 23

Question:



Two light elastic strings each have natural length a and modulus of elasticity λ. A particle P of mass m is attached to one end of each string. The other ends of the string are attached to points A and B, where AB is horizontal and $AB = 2a$. The particle is held at the mid-point of AB and released from rest. It comes to rest for the first time in its subsequent motion when PA and PB make angles α with AB, where $\tan\alpha = \frac{4}{3}$, as shown in the figure. Find λ in terms of m and g. **[E]**

Solution:



When P comes instantaneously to rest, it is not in equilibrium and so the question cannot easily be solved by resolving. It is a common error to attempt the solution of this, and similar questions, by resolving.

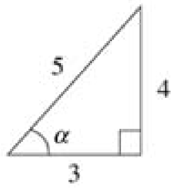
$\tan\alpha = \frac{4}{3} \Rightarrow \cos\alpha = \frac{3}{5}$

Let the distance fallen by P be h.

When you know $\tan\alpha$ you can draw a triangle to find $\cos\alpha$.

$\tan\alpha = \frac{4}{3}$

$\cos\alpha = \frac{3}{5}$



$$h = a\tan\alpha = \frac{4a}{3}$$

$$AP^2 = h^2 + a^2 = \left(\frac{4a}{3}\right)^2 + a^2 = \frac{25a^2}{9}$$

$$AP = \frac{5a}{3}$$

When P first comes to rest the energy stored in one string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{\lambda\left(\frac{2a}{3}\right)^2}{2a} = \frac{2\lambda a}{9}$$

The extension in one string is

$\text{AP} - \text{natural length} = \frac{5a}{3} - a$

$= \frac{2a}{3}$

When P first comes to rest the potential energy lost is given by

$$mgh = mg \times \frac{4}{3}a$$

Conservation of energy

Elastic energy gained = potential energy lost

$$\frac{4\lambda a}{9} = \frac{4mga}{3}$$

$$\lambda = \frac{4mga}{3} \times \frac{9}{4a} = 3mg$$

Initially P is at rest and, when it has fallen $\frac{5a}{3}$, it is at rest again. So there is no change in kinetic energy. Elastic energy is gained by both strings and potential energy is lost by the particle.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 24

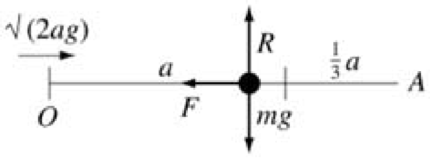
Question:

A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $3.6mg$. The other end of the string is fixed at a point O on a rough horizontal table. The particle is projected along the surface of the table from O with speed $\sqrt{(2ag)}$. At its furthest point from O, the particle is at the point A, where $OA = \frac{4}{3}a$.

a Find, in terms of m, g and a, the elastic energy stored in the string when P is at A.

b Using the work–energy principle, or otherwise, find the coefficient of friction between P and the table. **[E]**

Solution:



a At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$
$$= \frac{3.6mg \times (\frac{1}{3}a)^2}{2a}$$
$$= 0.2mga$$

At A, the extension of the string is $\frac{4}{3}a - a = \frac{1}{3}a$.

b The total energy lost is

$$\frac{1}{2}mu^2 - 0.2mga = \frac{1}{2} \times 2ag - 0.2mga$$
$$= 0.8mga$$

As P is at rest at A, the net loss of energy is the loss in kinetic energy minus the gain in elastic energy.

At any point in the motion

R(↑) $R = mg$

The friction is given by

$F = \mu R = \mu mg$

By the work–energy principle

$$0.8mg = \mu mg \times \frac{4}{3}a$$
$$\mu = 0.8 \times \frac{3}{4} = 0.6$$

By the work–energy principle, the net loss in energy is equal to the work done by friction. You find the work done by friction by multiplying the magnitude of the friction, μmg, by the distance the particle moves, $\frac{4}{3}a$. This gives you an equation in μ, which you solve.

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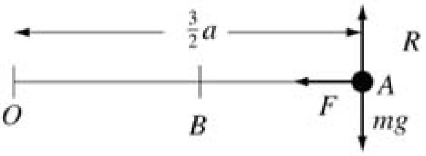
Review Exercise 1
Exercise A, Question 25

Question:

A particle P of mass m is held at a point A on a rough horizontal plane. The coefficient of friction between P and the plane is $\frac{2}{3}$. The particle is attached to one end of a light elastic string, of natural length a and modulus of elasticity $4mg$. The other end of the string is attached to a fixed point O on the plane, where $OA = \frac{3}{2}a$. The particle P is released from rest and comes to rest at a point B, where $OB < a$. Using the work–energy principle, or otherwise, calculate the distance AB. **[E]**

Solution:



At any point in the motion

R(↑) $R = mg$

The friction is given by

$F = \mu R = \frac{2}{3}mg$

At A, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{4mg \times (\frac{1}{2}a)^2}{2a}$$

$$= \frac{1}{2}mga$$

At A, the extension of the string is $\frac{3}{2}a - a = \frac{1}{2}a$.

By the work–energy principle

$$\frac{1}{2}mga = \frac{2}{3}mg \times AB$$

$$AB = \frac{3}{4}a$$

When P comes to rest, as $OB < a$, the string is slack so all of the elastic energy has been lost. This lost energy must equal the work done by friction, which is the magnitude of the friction, $\frac{2}{3}mg$, multiplied by the distance moved by P, which is AB.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 26

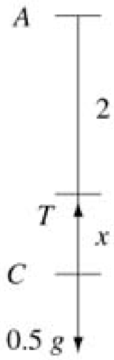
Question:

One end of an light elastic string, of natural length 2 m and modulus of elasticity 19.6 N, is attached to a fixed point A. A small ball B of mass 0.5 kg is attached to the other end of the string. The ball is released from rest at A and first comes to instantaneous rest at the point C, vertically below A.

a Find the distance AC.

b Find the instantaneous acceleration of B at C. **[E]**

Solution:



In solving nearly all questions involving elastic strings and springs you need to find the value of, or an expression for, the extension. If no symbol is given in the question, you should introduce a symbol, here x m, yourself.

a Let the extension of string when B is at C be x m.
Conservation of energy
elastic energy gained = potential energy lost

$$\frac{\lambda x^2}{2l} = mgh$$

$$\frac{19.6x^2}{4} = 0.5 \times 9.8\,(2+x)$$

In falling from A to C, the ball moves a distance of $(2+x)$m and so the potential energy lost is, in Joules, $mg(2+x)$.

$$4.9x^2 = 4.9(2+x)$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2$$

$$AC = 4\text{ m}$$

For there to be elastic energy in a string, the extension must be positive, so you can discard the solution $x = -1$.

b At C
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{19.6 \times 2}{2} = 19.6$$

R($\downarrow$) $\quad \mathbf{F} = m\mathbf{a}$

$$mg - T = m\mathbf{a}$$

$$0.5 \times 9.8 - 19.6 = 0.5\mathbf{a}$$

$$\mathbf{a} = \frac{0.5 \times 9.8 - 19.6}{0.5} = -29.4$$

The instantaneous acceleration of B at C is 29.4 m s^{-2} directed towards A.

The negative acceleration shows you that the acceleration is in the direction of x decreasing, that is towards A.

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Review Exercise 1
Exercise A, Question 27

Question:

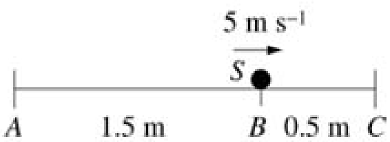
A light elastic string AB of natural length 1.5 m has modulus of elasticity 20 N. The end A is fixed to a point on a smooth horizontal table. A small ball S of mass 0.2 kg is attached to the end B. Initially S is at rest on the table with $AB = 1.5$ m. The ball S is then projected horizontally directly away from A with a speed of 5 m s^{-1}. By modelling S as a particle

a Find the speed of S when $AS = 2$ m.

When the speed of S is 1.5 m s^{-1}, the string breaks.

b Find the tension in the string immediately before the string breaks. **[E]**

Solution:



a Let $AC = 2\text{ m}$. When S is at C, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l}$$

$$= \frac{20 \times (0.5)^2}{2 \times 1.5} = \frac{5}{3}$$

Let the speed of S at C be $v\text{ m s}^{-1}$.

Conservation of energy

Kinetic energy lost = elastic energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{5}{3}$$

$$\frac{1}{2} \times 0.2 \times 5^2 - \frac{1}{2} \times 0.2v^2 = \frac{5}{3}$$

$$0.1v^2 = 0.1 \times 25 - \frac{5}{3} = \frac{5}{6}$$

$$v^2 = \frac{25}{3} \Rightarrow v = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3} \approx 2.886\ldots$$

The exact answer $\frac{5\sqrt{3}}{3}\text{ m s}^{-1}$ is also accepted.

The speed of S when $AS = 2\text{ m}$ is 2.89 m s^{-1} (3 s.f.)

b Let the extension of the string immediately before the string breaks be x m.

To find the tension in the string when the speed of S is 1.5 m s^{-1}, you first need to find the extension of the string at this speed. The extension is found using conservation of energy.

When the extension in the string is x m, the elastic energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{20x^2}{3}$$

Conservation of energy

Kinetic energy lost = elastic energy gained

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{20x^2}{3}$$

$$\frac{1}{2} \times 0.2 \times 5^2 - \frac{1}{2} \times 0.2 \times 1.5^2 = \frac{20x^2}{3}$$

$$\frac{20x^2}{3} = 2.275 \Rightarrow x^2 = 0.34125$$

$$x = \sqrt{(0.34125)}$$

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{20\sqrt{(0.34125)}}{1.5} = 7.788\ldots$$

The tension in the string immediately before the string breaks is 7.79 N (3 s.f.).

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Review Exercise 1
Exercise A, Question 28

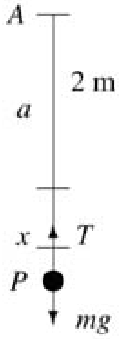
Question:

One end of a light elastic string of natural length a and modulus of elasticity $3mg$, is fixed at a point A and the other end carries a particle P of mass m. The particle is held at A and then projected vertically downwards with speed $\sqrt{(3ga)}$.

a Find the distance AP when the acceleration of the particle is instantaneously zero.

b Find the maximum speed attained by the particle during its motion. **[E]**

Solution:



a Let the extension of the string when the acceleration of P is zero be x m.
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{3mgx}{a}$$

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m \times 0$$

$$mg - \frac{3mgx}{a} = 0 \Rightarrow x = \frac{a}{3}$$

$$AP = a + \frac{a}{3} = \frac{4a}{3}$$

It is usually easier, when using Hooke's law, to work in terms of the extension of the string than to work with the total length of the string. Here, first find the extension when the acceleration is zero and, then, find the total length by adding the extension to the natural length.

b When $AP = \frac{4a}{3}$, the energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{3mg\left(\frac{a}{3}\right)^2}{2a} = \frac{1}{6}mga$$

Let the maximum speed of P be v m s^{-1}

Using conservation of energy

The maximum speed occurs when the acceleration is zero. This is the equivalent of the calculus property that at a maximum value of v, $\frac{dv}{dt} = 0$. In part **a**, you showed that $a = 0$ when $AP = \frac{4a}{3}$. Using conservation of energy, you can find the speed of P when $AP = \frac{4a}{3}$.

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 + \frac{\lambda x^2}{2l} = mgh$$

$$\frac{1}{2}mv^2 - \frac{1}{2}m \times 3ga + \frac{1}{6}mga = mg \times \frac{4a}{3}$$

$$\frac{1}{2}mv^2 = \frac{4mga}{3} + \frac{3mga}{2} - \frac{mga}{6} = \frac{8mga}{3}$$

$$v^2 = \frac{16mga}{3m} \Rightarrow v = 4\left(\frac{ga}{3}\right)^{\frac{1}{2}}$$

The maximum speed attained by P during its motion is $4\left(\frac{ga}{3}\right)^{\frac{1}{2}}$.

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Review Exercise 1
Exercise A, Question 29

Question:

A light elastic string of natural length 30 cm is placed on a smooth horizontal table with one end attached to a fixed point P on the table. The other end of the string is attached to a fixed point Q on the table such that $PQ = 80$ cm .

a Given that the tension in the string is 175 N, find, in newtons, the modulus of elasticity of the string.

The mid-point of the string is pulled a distance 30 cm along the perpendicular bisector of PQ in the plane of the table.

b Find the increase of tension in the string.

c Find, in joules, the corresponding increase in the elastic energy of the string. **[E]**

Solution:

a Hooke's law $T = \frac{\lambda x}{l}$

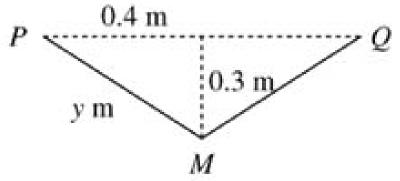
$175 = \frac{\lambda \times 0.5}{0.3}$

$\lambda = 175 \times \frac{0.3}{0.5} = 105$

The modulus of elasticity of the string is 105 N

It is always a good idea to convert all units to base SI units. Here 30 cm = 0.3 m, and the extension 50 cm = 0.5 m. In this question, using centimetres could be very misleading as you will get the correct answers in parts **a** and **b** but the incorrect answer in part **c**, where the extension is squared.

b



Let the mid-point of the string be M and $PM = y$ m.

$y^2 = 0.4^2 + 0.3^2 = 0.25 \Rightarrow y = 0.5$

The new tension in the string is given by Hooke's law

$T = \frac{\lambda x}{l}$

$= \frac{105(0.5 - 0.15)}{0.15} = 245$

Here, the tension is calculated using half the string PM. You could use the whole of the string when the working is $T = \frac{105(1-0.3)}{0.3} = 245$.

The increase in the tension in the string is $(245 - 105)$ N $= 140$ N

c The initial energy, E_1 Joules, say, is given by

$E_1 = \frac{\lambda x^2}{2l} = \frac{105 \times (0.5)^2}{0.6} = 43.75$

The final energy, E_2 Joules, say, is given by

$E_2 = 2 \times \frac{\lambda x^2}{2l} = 2 \times \frac{105 \times (0.35)^2}{0.3} = 85.75$

The final energy has been calculated as twice the energy in half of the string PM.

The increase in the elastic energy of the string is $(85.75 - 43.75)$ J $= 42$ J

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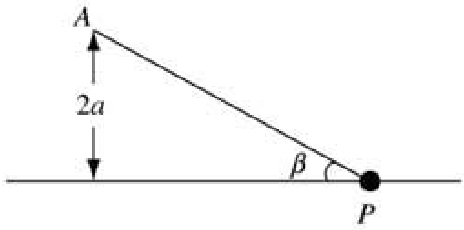
Review Exercise 1
Exercise A, Question 30

Question:

A particle P of mass m is attached to one end of a light elastic string of length a and modulus of elasticity $\frac{1}{2}mg$. The other end of the string is fixed at a point A which is at a height $2a$ above a smooth horizontal table. The particle is held on the table with the string making an angle β with the table, where $\tan\beta = \frac{3}{4}$.

a Find the elastic energy stored in the string.
The particle is now released from rest. Assuming that P remains on the table.
b Find the speed of P when the string is vertical.
By finding the vertical component of the tension in the string when P is on the table and AP makes an angle θ with the horizontal,
c show that the assumption that P remains in contact with the table is justified. **[E]**

Solution:



a $\tan\beta = \frac{3}{4} \Rightarrow \sin\beta = \frac{3}{5}$

$\frac{2a}{AP} = \sin\beta$

$AP = \frac{2a}{\sin\beta} = \frac{2a}{\frac{3}{5}} = \frac{10a}{3}$

The extension of the string is $\frac{10a}{3} - a = \frac{7a}{3}$

You find the extension of the string by calculating the length of AP using trigonometry and subtracting the natural length of the string a.

The energy stored in the string is given by

$$E = \frac{\lambda x^2}{2l} = \frac{\frac{1}{2}mg\left(\frac{7a}{3}\right)^2}{2a} = \frac{49}{36}mga$$

b When the particle is vertically below A, the energy stored in the string, E' say, is given by

$$E' = \frac{\lambda x^2}{2l} = \frac{\frac{1}{2}mg(a)^2}{2a} = \frac{1}{4}mga$$

When P is vertically below A, $AP = 2a$ and the extension of the string is $2a - a = a$.

Let v m s^{-1} be the velocity of P when it is vertically below A.

Conservation of energy

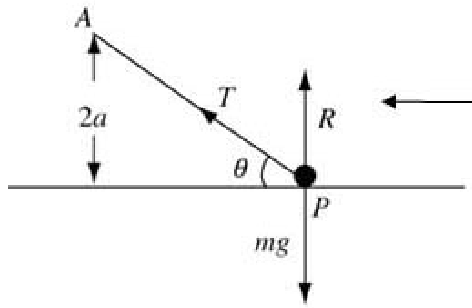
kinetic energy gained = elastic energy lost

$$\frac{1}{2}mv^2 = \frac{49}{36}mga - \frac{1}{4}mga = \frac{10}{9}mga$$

$$v^2 = \frac{20ga}{9} \Rightarrow v = \frac{2}{3}\sqrt{(5ga)}$$

The speed of P when the string is vertical is $\frac{2}{3}\sqrt{(5ga)}$.

c



For P to remain in contact with the table, the normal reaction between P and the table, here called R, must remain positive throughout the motion. If $R = 0$, the particle loses contact with the table

$$\frac{2a}{AP} = \sin\theta \Rightarrow AP = \frac{2a}{\sin\theta}$$

The extension in the string is $\frac{2a}{\sin\theta} - a$

You must consider the forces on P at a general point in its motion. As P moves from its starting position to the point below A, $\sin\theta$ varies from $\frac{3}{5}$ to 1.

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{\frac{1}{2}mg\left(\frac{2a}{\sin\theta} - a\right)}{a}$$

$$= \frac{mg}{\sin\theta} - \frac{1}{2}mg$$

The vertical component of the tension, T_V say, is given by

$$T_V = T\sin\theta = \left(\frac{mg}{\sin\theta} - \frac{1}{2}mg\right)\sin\theta = mg - \frac{1}{2}mg\sin\theta$$

The question requires you to consider the vertical component of the tension in the string. You find an expression for the vertical component and then resolve vertically to obtain the normal reaction.

R(↑) $T_V + R = mg$

$$R = mg - T_v$$

$$= mg - \left(mg - \frac{1}{2}mg\sin\theta\right)$$

$$= \frac{1}{2}mg\sin\theta > 0$$

So P remains in contact with the table.

As $\sin\theta$ varies from $\frac{3}{5}$ to 1, the normal reaction R is always positive and so the assumption that P remains in contact with the table is justified.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 31

Question:

A light elastic string has natural length 4 m and modulus of elasticity 58.8 N. A particle P of mass 0.5 kg is attached to one end of the string. The other end of the string is attached to a fixed point A. The particle is released from rest at A and falls vertically.

a Find the distance travelled by P before it comes to instantaneous rest for the first time.

The particle is now held at a point 7 m vertically below A and released from rest.

b Find the speed of the particle when the string first becomes slack. **[E]**

Solution:

a When P comes to rest for the first time, let the extension of the string be x m
Conservation of energy
elastic energy gained = potential energy lost

$$\frac{\lambda x^2}{2l} = mgh$$

$$\frac{58.8x^2}{8} = 0.5 \times 9.8 \times (4+x)$$

$$7.35x^2 = 19.6 + 4.9x$$

Divide this equation throughout by 2.45 and rearrange the terms. If you cannot see this simplification, you can use the quadratic formula but you would be expected to obtain an exact answer.

$$3x^2 - 2x - 8 = 0$$

$$(x-2)(3x+4) = 0$$

$$x = 2$$

For the string to have elastic energy, it has to be stretched so you can ignore the negative solution $-\frac{4}{3}$.

The distance fallen by P is $(4+2)\text{m} = 6\text{ m}$.

b P will first become slack when it has moved 3 m vertically.
Let the velocity at this point be $v \text{ m s}^{-1}$.
Conservation of energy
kinetic energy gained + potential energy gained = elastic energy lost

Initially P is at rest and then rises 3 m. So both kinetic and potential energy are gained. Initially the string is stretched but, after rising 3 m, it is slack. So elastic energy is lost. By conservation of energy, the net gain of kinetic and potential energies must equal the elastic energy lost.

$$\frac{1}{2}mv^2 + mgh = \frac{\lambda x^2}{2l}$$

$$\frac{1}{2}0.5v^2 + 0.5 \times 9.8 \times 3 = \frac{58.8 \times 3^2}{8}$$

$$0.25v^2 + 14.7 = 66.15$$

$$v^2 = \frac{66.15 - 14.7}{0.25} = 205.8$$

$$v = \sqrt{(205.8)} = 14.345\ldots$$

The speed of the particle when the string first becomes slack is 14 m s^{-1} (2 s.f.).

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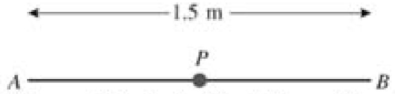
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 32

Question:

Two light elastic strings each have natural length 0.75 m and modulus of elasticity 49 N. A particle P of mass 2 kg is attached to one end of each string. The other ends of the strings are attached to fixed points A and B, where AB is horizontal and $AB = 1.5\text{ m}$.



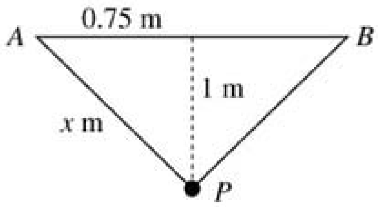
The particle is held at the mid-point of AB. The particle is released from rest, as shown in the figure.

a Find the speed of P when it has fallen a distance of 1 m.

b Given instead that P hangs in equilibrium vertically below the mid-point of AB with $\angle APB = 2\alpha$
show that $\tan\alpha + 5\sin\alpha = 5$. **[E]**

Solution:

a



When P has fallen 1 m, let $AP = x$ m.

$x^2 = 0.75^2 + 1^2 = 1.5625 \Rightarrow x = 1.25$

At this point the extension of the string AP is $(1.25 - 0.75)\text{m} = 0.5\text{ m}$

To find the elastic energy in the string, you need to calculate the extension in the string. First find AP (you could just use the 3, 4, 5 triangle) and then subtract the natural length, 0.75 m.

Let the velocity of P when it has fallen 1 m be $v\text{ m s}^{-1}$.
Conservation of energy

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mv^2 + 2\times\frac{\lambda x^2}{2l} = mgh$$

Both strings have elastic energy stored in them. By symmetry, the energy in both strings is the same.

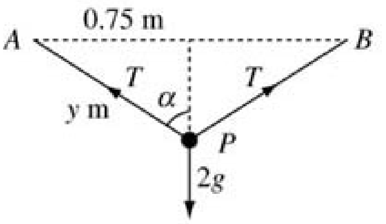
$$\frac{1}{2}2\times v^2 + 2\times\frac{49\times 0.5^2}{2\times 0.75} = 2\times 9.8\times 1$$

$$v^2 + \frac{49}{3} = 19.6 \Rightarrow v^2 = 3.2\dot{6}$$

$$v = 1.807\ldots$$

The speed of P when it has fallen 1 m is 1.8 m s^{-1} (2 s.f.).

b



Let $AP = y$ m and the angle AP makes with the vertical be α.

By trigonometry

$$\sin\alpha = \frac{0.75}{y} \Rightarrow y = \frac{0.75}{\sin\alpha}$$

R(↑) $2T\cos\alpha = 2g$

$$T = \frac{g}{\cos\alpha} = \frac{9.8}{\cos\alpha}$$

Hooke's law

$$T = \frac{\lambda x}{l}$$

$$= \frac{49}{0.75}(y - 0.75) = \frac{49}{0.75}\left(\frac{0.75}{\sin\alpha} - 0.75\right)$$

$$= 49\left(\frac{1}{\sin\alpha} - 1\right) = 49\left(\frac{1-\sin\alpha}{\sin\alpha}\right)$$

You find two separate expression for T, one by resolving vertically and the other from Hooke's law. Equating the two expressions gives you an equation in α.

$$\frac{9.8}{\cos\alpha} = 49\left(\frac{1-\sin\alpha}{\sin\alpha}\right)$$

$$\frac{\sin\alpha}{\cos\alpha} = \frac{49}{9.8}(1 - \sin\alpha) = 5(1 - \sin\alpha)$$

$$\tan\alpha = 5 - 5\sin\alpha$$

$\tan\alpha + 5\sin\alpha = 5$, as required.

Eliminating T gives an equation in α. You have to manipulate this equation to obtain the printed answer.

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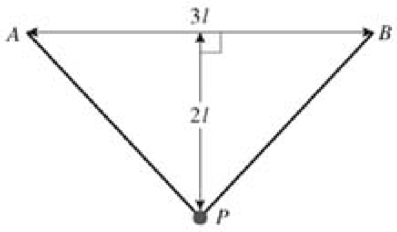
Review Exercise 1
Exercise A, Question 33

Question:



A light elastic string, of natural length $3l$ and modulus of elasticity λ, has ends attached to two points A and B where $AB = 3l$ and AB is horizontal. A particle P of mass m is attached to the mid-point of the string. Given that P rests in equilibrium at a distance $2l$ below AB, as shown in the figure,

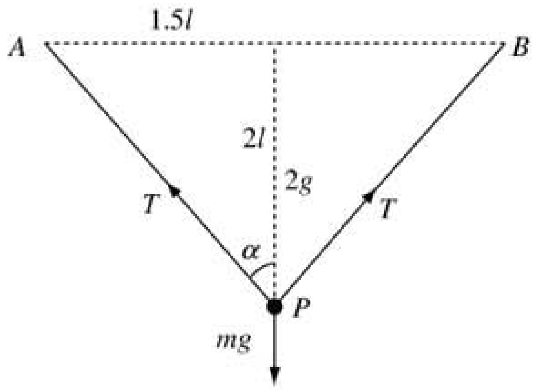
a show that $\lambda = \frac{15mg}{16}$.

The particle is pulled vertically downwards from its equilibrium position until the total length of the elastic string is $7.8l$. The particle is released from rest.

b Show that P comes to instantaneous rest on the line AB. **[E]**

Solution:

a



$AP^2 = (1.5l)^2 + (2l)^2 = 6.25l^2 \Rightarrow AP = 2.5l$

Let α be the angle between AP and the vertical.

$\cos\alpha = \frac{2l}{2.5l} = \frac{4}{5}$

The extension of half of the string, AP, is $2.5l - 1.5l = l$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}}$$

$$= \frac{\lambda l}{1.5l} = \frac{2\lambda}{3} \quad ①$$

R(↑) $\quad 2T\cos\alpha = mg$

$$2T \times \frac{4}{5} = mg$$

$$T = \frac{5mg}{8} \quad ②$$

Eliminating T between ① and ②

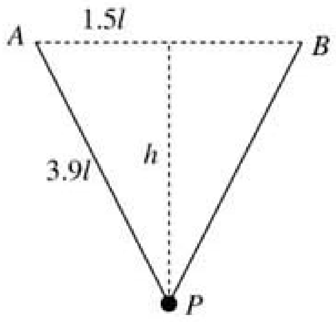
$$\frac{2\lambda}{3} = \frac{5mg}{8}$$

$\lambda = \frac{5mg}{8} \times \frac{3}{2} = \frac{15mg}{16}$, as required.

It is acceptable just to write down $AP = 2.5l$, using the 3, 4, 5 triangle.

You find two equations in T and λ by resolving vertically and using Hooke's law. Eliminating T between the two equations gives λ.

b



Let the perpendicular distance from the original position of P to AB be h.

$h^2 = (3.9l)^2 - (1.5l)^2 = 12.96l^2 \Rightarrow h = 3.6l$

Let the speed of P as it reaches AB be v m s^{-1}

To prove that the speed of P at AB is zero, the speed of P at AB is taken as v m s^{-1}. You then use conservation of energy to obtain an equation for v. You complete the proof by solving the equation for v and showing the solution is zero.

Conservation of energy

kinetic energy gained + potential energy gained = elastic energy lost

$$\frac{1}{2}mv^2 + mgh = 2 \times \frac{\lambda x^2}{2 \times \text{natural length}}$$

Each of the two halves of the string have natural length $1.5l$ and extension $2.4l$.

$$\frac{1}{2}mv^2 + mg \times 3.6l = 2 \times \frac{\left(\frac{15mg}{16}\right)(3.9l - 1.5l)^2}{2 \times 1.5l}$$

$$\frac{1}{2}mv^2 + 3.6mgl = \frac{5mg}{8l} \times (2.4l)^2 = 3.6mgl$$

Hence $\frac{1}{2}mv^2 = 0 \Rightarrow v = 0$

P comes to instantaneous rest on the line AB, as required.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 34

Question:

A particle P, of mass m, is attached to one end of a light elastic string, of natural length l and modulus of elasticity $8mg$. The other end of the string is attached at a point A to a horizontal ceiling which is at a height $2l$ above a horizontal floor. The particle P is held at A and projected vertically downwards with speed u.

a Find the least possible value of u for P to reach the floor.

Given that $u^2 = 16gl$ and that when P strikes the floor its speed is halved, find

b the speed of P when it hits the ceiling after striking the floor once,

c the maximum speed of P during its motion. **[E]**

Solution:

a Conservation of energy

elastic energy gained = potential energy lost + kinetic energy lost

$$\frac{\lambda x^2}{2l} = mgh + \frac{1}{2}mu^2$$

$$\frac{8mg \times l^2}{2l} = mg \times 2l + \frac{1}{2}mu^2$$

$$4mgl = 2mgl + \frac{1}{2}mu^2$$

$$\frac{1}{2}mu^2 = 2mgl \Rightarrow u^2 = 4gl$$

$$u = 2\sqrt{(gl)}$$

The least possible value of u to reach the floor is $2\sqrt{(gl)}$.

> The least value of u occurs when P just reaches the floor. That is when the speed of P becomes zero just as it reaches the floor. At that point, P has lost all its kinetic energy $\left(\frac{1}{2}mu^2\right)$, P has fallen a distance $2l$ and the extension of the string is l.

b Let the speed of P as it strikes the floor be v m s^{-1}

Conservation of energy

elastic energy gained = potential energy lost + kinetic energy lost

$$\frac{\lambda x^2}{2l} = mgh + \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$

$$4mgl = 2mgl + \frac{1}{2}m(16gl) - \frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2 = 2mgl + 8mgl - 4mgl = 6mgl$$

$$v^2 = 12gl \Rightarrow v = 2\sqrt{(3gl)}$$

> In part **b**, you first find the speed of P immediately before it strikes the ground. In the downward motion, the gain in elastic energy and the loss in potential energy are the same as in part **a**.

P rebounds from the floor with speed, v', say,
where $v' = \sqrt{(3gl)}$

> As P strikes the ground, its velocity is halved.

Let the speed with which P returns to the ceiling be w m s^{-1}.

Conservation of energy

elastic energy lost + kinetic energy lost = potential energy gained

$$\frac{\lambda x^2}{2l} + \frac{1}{2}mv'^2 - \frac{1}{2}mw^2 = mgh$$

$$4mgl + \frac{1}{2}m(3gl) - \frac{1}{2}mw^2 = 2mgl$$

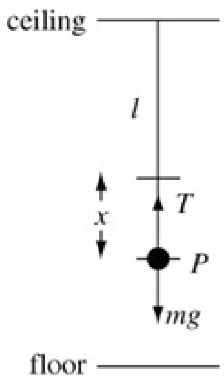
$$\frac{1}{2}mw^2 = 4mgl + \frac{3}{2}mgl - 2mgl = \frac{7}{2}mgl$$

$$w^2 = 7gl \Rightarrow w = \sqrt{(7gl)}$$

> Now you use conservation of energy for the motion of P as it moves from the floor to the ceiling. As P rises, elastic and kinetic energy are lost; potential energy is gained.

The speed of P when it hits the ceiling after striking the floor once is $\sqrt{(7gl)}$.

c



The maximum speed occurs when the acceleration is zero. Let the extension of the string at this point be x.

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{8mgx}{l}$$

$$\text{R}(\downarrow)\ mg - T = m \times 0$$

$$mg - \frac{8mgx}{l} = 0$$

$$x = \frac{l}{8}$$

> The maximum value of v is when $a = \frac{\mathrm{d}v}{\mathrm{d}t} = 0$. You use Newton's second law and Hooke's law to find the extension of the string when the maximum occurs.

Let the maximum speed of P be V m s^{-1}.

Considering the motion of P from the ceiling to the point where $x = \frac{l}{8}$.

kinetic energy gained + elastic energy gained = potential energy lost

$$\frac{1}{2}mV^2 - \frac{1}{2}mu^2 + \frac{\lambda x^2}{2l} = mg(l + x)$$

$$\frac{1}{2}mV^2 - \frac{1}{2}m(16gl) + \frac{8mg\left(\frac{1}{8}\right)^2}{2l} = mg \times \frac{9l}{8}$$

$$\frac{1}{2}mV^2 - 8mgl + \frac{mgl}{16} = \frac{9mgl}{8}$$

$$\frac{1}{2}mV^2 = 8mgl - \frac{mgl}{16} + \frac{9mgl}{8} = \frac{145mgl}{16}$$

$$V^2 = \frac{145gl}{8} \Rightarrow V = \left(\frac{145gl}{8}\right)^{\frac{1}{2}}$$

> You consider the fall of P from the point where it first leaves the ceiling to the point where the maximum speed is reached. In that motion kinetic energy and elastic energy are gained, and potential energy is lost.

The maximum speed of P during its motion is $\left(\frac{145gl}{8}\right)^{\frac{1}{2}}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 35

Question:

A particle P of mass 0.2 kg moves away from the origin along the positive x-axis. It moves under the action of a force directed away from the origin O of magnitude $\frac{5}{x+1}$ N, where $OP = x$ m. Given that the speed of P is 5 m s^{-1} when $x = 0$, find the value of x, to 3 significant figures, when the speed of P is 15 m s^{-1}.

Solution:

$$\mathbf{F} = ma$$
$$\frac{5}{x+1} = 0.2a$$
$$a = \frac{5}{0.2(x+1)} = \frac{25}{x+1}$$

You need to remember that $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{25}{x+1}$$

You integrate both sides of this equation with respect to x. It is important to include a constant of integration. Most questions include information which enables you to find the constant.

$$\frac{1}{2}v^2 = \int \frac{25}{x+1}\,\mathrm{d}x = 25\ln(x+1) + A$$
$$v^2 = 50\ln(x+1) + B, \text{ where } B = 2A$$

At $x = 0, v = 5$

You use the information that $v = 5$ when $x = 0$ to evaluate the constant.

$$25 = 50\ln 1 + B \Rightarrow B = 25$$

Hence $v^2 = 50\ln(x+1) + 25$

When $v = 15$

$$15^2 = 50\ln(x+1) + 25$$
$$\ln(x+1) = \frac{225-25}{50} = 4$$

Take exponentials of both sides of this equation and use $e^{\ln(x+1)} = x+1$.

$$x + 1 = e^4$$
$$x = e^4 - 1 = 53.6 \text{ (3 s.f.)}$$

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Review Exercise 1
Exercise A, Question 36

Question:

A particle P of mass 2.5 kg moves along the positive x-axis. It moves away from a fixed origin O, under the action of a force directed away from O. When $OP = x$ metres, the magnitude of the force is $2e^{-0.1x}$ N and the speed of P is v ms^{-1}. When $x = 0, v = 2$, Find

a v^2 in terms of x,

b the value of x when $v = 4$.

c Give a reason why the speed of P does not exceed $\sqrt{20}$ m s^{-1}.

Solution:

a $\mathbf{F} = m\mathbf{a}$

$2e^{-0.1x} = 2.5a$

$a = \frac{2}{2.5}e^{-0.1x} = 0.8e^{-0.1x}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.8e^{-0.1x}$

Using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

$\frac{1}{2}v^2 = \int 0.8e^{-0.1x}\,dx = \frac{0.8e^{-0.1x}}{-0.1} + A$

$= -8e^{-0.1x} + A$

$v^2 = B - 16e^{-0.1x}$, where $B = 2A$

Twice one arbitrary constant is another arbitrary constant.

When $x = 0, v = 2$

At $x = 0, e^{-0.1x} = e^0 = 1$.

$4 = B - 16 \Rightarrow B = 20$

Hence $v^2 = 20 - 16e^{-0.1x}$

b When $v = 4$

$16 = 20 - 16e^{-0.1x}$

$e^{-0.1x} = \frac{20-16}{16} = \frac{1}{4}$

Take logarithms of both sides of this equation and use $\ln(e^{-0.1x}) = -0.1x$.

$-0.1x = \ln\left(\frac{1}{4}\right) = -\ln 4$

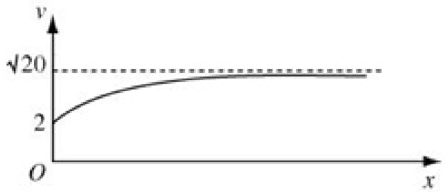
$x = 10\ln 4$

c For all x, $e^{-0.1x} > 0$

So $v^2 = 20 - 16e^{-0.1x} < 20$

and hence $v < \sqrt{20}$

The speed of P does not exceed $\sqrt{20}\text{ m s}^{-1}$.



This sketch, of v against x, shows that as x increases, the velocity of P gets closer to $\sqrt{20}\text{ m s}^{-1}$ but never reaches it. $\sqrt{20}\text{ m s}^{-1}$ is the **terminal** or **limiting speed** of P.

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Review Exercise 1
Exercise A, Question 37

Question:

A toy car of mass 0.2 kg is travelling in a straight line on a horizontal floor. The car is modelled as a particle. At time $t = 0$ the car passes through a fixed point O. After t seconds the speed of the car is v m s^{-1} and the car is at a point P with $OP = x$ metres. The resultant force on the car is modelled as $\frac{1}{10}x(4-3x)$N in the direction OP. The car comes to instantaneous rest when $x = 6$. Find

a an expression for v^2 in terms of x,

b the initial speed of the car. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$\frac{1}{10}x(4-3x) = 0.2a$$

$$a = \frac{1}{0.2\times 10}x(4-3x) = \frac{1}{2}x(4-3x) = 2x - \frac{3x^2}{2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - \frac{3x^2}{2}$$

Integrate both sides of this equation with respect to x. Remember to include a constant of integration.

$$\frac{1}{2}v^2 = \int\left(2x - \frac{3x^2}{2}\right)dx = x^2 - \frac{x^3}{2} + A$$

$$v^2 = 2x^2 - x^3 + B\text{, where } B = 2A$$

At $x = 6, v = 0$

The car comes to instantaneous rest when $x = 6$. So $v = 0$ at $x = 6$.

$0 = 2\times 36 - 216 + B \Rightarrow B = 144$

Hence $v^2 = 2x^2 - x^3 + 144$

b When $x = 0$

$v^2 = 144 \Rightarrow v = \pm 12$

Both signs are possible as the car could pass through O in either direction when $t = 0$. However, in either case, the speed of the car, which is the magnitude of the velocity, is 12 m s^{-1}.

The initial speed of the car is 12 m s^{-1}.

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Review Exercise 1
Exercise A, Question 38

Question:

A particle P of mass 0.6 kg is moving along the positive x-axis under the action of a force which is directed away from the origin O. At time t seconds, the force has magnitude $3e^{-0.5t}$ N. When $t = 0$, the particle P is at O and moving with speed 2 m s^{-1} in the direction of x increasing. Find

a the value of t when the speed is 8 m s^{-1},

b the distance of P from O when $t = 2$. **[E]**

Solution:

a $F = ma$

$3e^{-0.5t} = 0.6a$

$a = \frac{3e^{-0.5t}}{0.6} = 5e^{-0.5t}$

$\frac{dv}{dt} = 5e^{-0.5t}$

$v = \int 5e^{-0.5t}\,dt = \frac{5e^{-0.5t}}{-0.5} + A = A - 10e^{-0.5t}$

When the acceleration is a function of time, you use $a = \frac{dv}{dt}$. When the acceleration is a function of distance, you can use $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

When $t = 0, v = 2$

$2 = A - 10 \Rightarrow A = 12$

Hence $v = 12 - 10e^{-0.5t}$

When $v = 8$

$8 = 12 - 10e^{-0.5t} \Rightarrow e^{-0.5t} = \frac{12-8}{10} = \frac{2}{5}$

$-0.5t = \ln\left(\frac{2}{5}\right)$

$t = -2\ln\left(\frac{2}{5}\right) = 2\ln\left(\frac{5}{2}\right)$

No particular form of the answer is asked for in the question and an approximate answer, such as $t = 1.83$, would be accepted.

b $v = \frac{dx}{dt} = 12 - 10e^{-0.5t}$

$x = \int (12 - 10e^{-0.5t})\,dt = 12t + 20e^{-0.5t} + B$

When $t = 0, x = 0$

$0 = 0 + 20 + B \Rightarrow B = -20$

$x = 12t + 20e^{-0.5t} - 20$

Using $e^0 = 1$. Carelessly writing $e^0 = 0$ is a common error.

When $t = 2$

$x = 24 + 20e^{-1} - 20 = 4 + 20e^{-1}$

The distance of P from O when $t = 2$ is $(4 + 20e^{-1})$ m.

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Review Exercise 1
Exercise A, Question 39

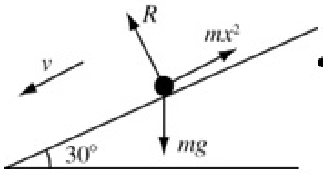
Question:

A particle P of mass m kg slides from rest down a smooth plane inclined at $30°$ to the horizontal. When P has moved a distance x metres down the plane, the resistance to motion from non-gravitational forces has magnitude mx^2 N. Find

a the speed of P when $x = 2$,
b the distance P has moved when it comes to rest for the first time. **[E]**

Solution:

a



Resistance opposes motion, so as P moves down the plane, resistance acts up the plane.

$R(\nearrow)\quad \mathbf{F} = ma$

$mg\sin 30^\circ - mx^2 = ma$

$a = g\sin 30^\circ - x^2$

Divide this equation throughout by m.

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}g - x^2$

Using $\sin 30^\circ = \frac{1}{2}$.

$\frac{1}{2}v^2 = \int\left(\frac{1}{2}g - x^2\right)dx = \frac{1}{2}gx - \frac{x^3}{3} + A$

$v^2 = gx - \frac{2x^3}{3} + B$, where $B = 2A$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

P starts from rest and so at $x = 0, v = 0$.

Hence $v^2 = gx - \frac{2x^3}{3}$

When $x = 2$

$v^2 = 9.8\times 2 - \frac{16}{3} = 14.2\dot{6}$

$v = \sqrt{14.2\dot{6}} = 3.777\ldots$

As a numerical value of g has been used, you should round your answer to 2 or 3 significant figures.

The speed of P when $x = 2$ is $3.78\ \text{m s}^{-1}$ (3 s.f.)

b $v^2 = gx - \frac{2x^3}{3}$

When $v = 0$

$0 = gx - \frac{2x^3}{3} = x\left(g - \frac{2x^2}{3}\right)$

$x = 0$ is the starting position and this root of the equation can be ignored.

P comes to rest when

$g - \frac{2x^2}{3} = 0 \Rightarrow x^2 = \frac{3g}{2} = 14.7$

$x = \sqrt{14.7} = 3.834\ldots$

The distance moved before P first comes to rest is 3.83 m (3 s.f.).

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Review Exercise 1
Exercise A, Question 40

Question:

Above the Earth's surface, the magnitude of the force on a particle due to the Earth's gravity is inversely proportional to the square of the distance of the particle from the centre of the Earth. Assuming that the Earth is a sphere of radius R, and taking g as the acceleration due to gravity at the surface of the Earth,

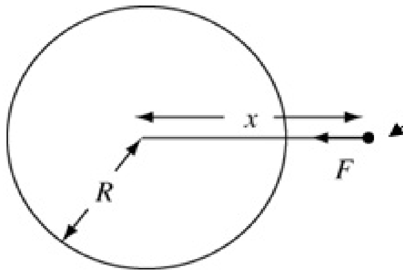
a prove that the magnitude of the gravitational force on a particle of mass m when it is a distance x (where $x \geq R$) from the centre of the Earth is $\frac{mgR^2}{x^2}$.

A particle is fired vertically upwards from the surface of the Earth with initial speed u, where $u^2 = \frac{3}{2}gR$. Ignoring air resistance.

b Find, in terms of g and R, the speed of the particle when it is at a height $2R$ above the surface of the Earth. **[E]**

Solution:

a



The gravitational force is directed towards the centre of the Earth and so is in the direction of x decreasing.

As $F \propto \frac{1}{x^2}$, $F = -\frac{k}{x^2}$

At $x = R, F = -mg$

$-mg = -\frac{k}{R^2} \Rightarrow k = mgR^2$

Hence $F = -\frac{mgR^2}{x^2}$

You introduce a constant of proportionality k and use the fact, that the force due to gravity at the surface of the Earth is known to have magnitude mg, to find k.

The magnitude of the force is $\frac{mgR^2}{x^2}$, as required.

b $\mathbf{F} = m\mathbf{a}$

$$-\frac{mgR^2}{x^2} = ma$$

$$a = -\frac{gR^2}{x^2}$$

In this equation, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$$

$$\frac{1}{2}v^2 = -\int gR^2x^{-2}\,\mathrm{d}x = -\frac{gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{ where } B = 2A$$

At $x = R, v^2 = \frac{3}{2}gR$

$$\frac{3}{2}gR = \frac{2gR^2}{R} + B \Rightarrow B = \frac{3}{2}gR - 2gR = -\frac{1}{2}gR$$

The question gives the velocity of the particle as it is fired from the surface of the Earth. That is the velocity when $x = R$, the radius of the Earth.

Hence $v^2 = \frac{2gR^2}{x} - \frac{1}{2}gR$

When $x = 3R$

$$v^2 = \frac{2gR^2}{3R} - \frac{1}{2}gR = \frac{gR}{6} \Rightarrow v = \sqrt{\left(\frac{gR}{6}\right)}$$

When the particle is at a height of $2R$ above the surface of the Earth, it is $2R + R = 3R$ from the centre of the Earth.

The speed of the particle when it is $2R$ above the surface of the Earth is $\sqrt{\left(\frac{gR}{6}\right)}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 41

Question:

A rocket is fired vertically upwards with speed U from a point on the Earth's surface. The rocket is modelled as a particle P of constant mass m, and the Earth as a fixed sphere of radius R. At a distance x from the centre of the Earth, the speed of P is v. The only force acting on P is directed towards the centre of the Earth and has magnitude $\frac{cm}{x^2}$, where c is a constant.

a Show that $v^2 = U^2 + 2c\left(\frac{1}{x} - \frac{1}{R}\right)$.

The kinetic energy of P at $x = R$ is half of the kinetic energy at $x = R$.

b Find c in terms of U and R. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$-\frac{cm}{x^2} = ma$$

In the equation of motion, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{c}{x^2} = -cx^{-2}$$

$$\frac{1}{2}v^2 = -\int cx^{-2}\,dx = \frac{c}{x} + A$$

Using $\int x^{-2}\,dx = \frac{x^{-1}}{-1} + k = -\frac{1}{x} + k$.

$$v^2 = \frac{2c}{x} + B, \text{ where } B = 2A$$

When $x = R, v = U$

$$U^2 = \frac{2c}{R} + B \Rightarrow B = U^2 - \frac{2c}{R}$$

Hence $v^2 = \frac{2c}{x} + U^2 - \frac{2c}{R}$

$$= U^2 + 2c\left(\frac{1}{x} - \frac{1}{R}\right), \text{ as required.}$$

b At $x = R, v = U$ and the kinetic energy of P is $\frac{1}{2}mU^2$

At $x = 2R$, using the result of part **a**

$$v^2 = U^2 + 2c\left(\frac{1}{2R} - \frac{1}{R}\right) = U^2 + 2c\left(-\frac{1}{2R}\right)$$

$$v^2 = U^2 - \frac{c}{R}$$

and the kinetic energy of P is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(U^2 - \frac{c}{R}\right)$

(kinetic energy at $x = 2R$) $= \frac{1}{2}$ (kinetic energy at $x = R$)

$$\frac{1}{2}m\left(U^2 - \frac{c}{R}\right) = \frac{1}{2}\left(\frac{1}{2}mU^2\right)$$

Divide this equation throughout by $\frac{1}{2}m$ and then make c the subject of the formula.

$$U^2 - \frac{c}{R} = \frac{1}{2}U^2$$

$$\frac{1}{2}U^2 = \frac{c}{R}$$

$$c = \frac{1}{2}RU^2$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 42

Question:

A projectile P is fired vertically upwards from a point on the Earth's surface. When P is at a distance x from the centre of the Earth its speed is v. Its acceleration is directed towards the centre of the Earth and has magnitude $\frac{k}{x^2}$, where k is a constant. The Earth is assumed to be a sphere of radius R.

a Show that the motion of P may be modelled by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{gR^2}{x^2}$$

The initial speed of P is U, where $U^2 < 2gR$. The greatest distance of P from the centre of the Earth is X.

b Find X in terms of U, R and g. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$-\frac{k}{x^2} = ma$ ①

$a = -\frac{k}{mx^2}$

At $x = R, a = -g$

$-g = -\frac{k}{mR^2}$

$k = mgR^2$

The force is negative in equation ① as the force on P due to gravity is directed towards the centre of the Earth and that is the direction of x decreasing.

You know that the acceleration due to gravity at the surface of the Earth is g and that the direction of the acceleration is towards the centre of the Earth. Substituting $a = -g$ into ① gives you k in terms of m, g and R.

Substituting $k = mgR^2$ into ①

$-\frac{mgR^2}{x^2} = ma$

$a = v\frac{dv}{dx} = -\frac{gR^2}{x^2}$, as required.

$a = v\frac{dv}{dx}$ is one of the alternative forms of the acceleration.

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$ and you must pick out the form of a which you need in any particular question.

b Separating the variables in the printed answer to part **a** and integrating

$$\int v\,dv = -\int \frac{gR^2}{x^2}\,dx = -\int gR^2x^{-2}\,dx$$

$$\frac{1}{2}v^2 = \frac{-gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{ where } B = 2A$$

At $x = R, v = U$

$$U^2 = \frac{2gR^2}{R} + B \Rightarrow B = U^2 - 2gR$$

The projectile is fired from a point on the Earth's surface with speed U. This gives you that at $x = R, v = U$.

Hence $v^2 = \frac{2gR^2}{x} + U^2 - 2gR$

When $v = 0, x = X$

$$0 = \frac{2gR^2}{X} + U^2 - 2gR$$

$$0 = 2gR^2 + U^2X - 2gRX$$

$$X(2gR - U^2) = 2gR^2$$

$$X = \frac{2gR^2}{2gR - U^2}$$

Multiply this equation throughout by X and then make X the subject of the formula.

Solutionbank M3

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Review Exercise 1
Exercise A, Question 43

Question:

A car of mass 800 kg moves along a horizontal straight road. At time t seconds, the resultant force on the car has magnitude $\frac{48\,000}{(t+2)^2}$ N, acting in the direction of motion of the car. When $t = 0$, the car is at rest.

a Show that the speed of the car approaches a limiting value as t increases and find this value.

b Find the distance moved by the car in the first 6s of its motion. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

$$\frac{48\,000}{(t+2)^2} = 600a$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{60}{(t+2)^2} = 60(t+2)^{-2}$$

When the acceleration is a function of t, the velocity can be found by writing $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ and integrating with respect to t.

$$v = \int 60(t+2)^{-2}\,\mathrm{d}t = \frac{60(t+2)^{-1}}{-1} + A$$

$$= A - \frac{60}{t+2}$$

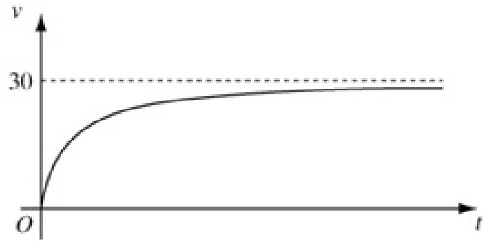
When $t = 0, v = 0$

$$0 = A - \frac{60}{2} \Rightarrow A = 30$$

Hence $v = 30 - \frac{60}{t+2}$

As $t \to \infty, \frac{60}{t+2} \to 0$ and $v \to 30$

As t increases, the car approaches a limiting speed of $30\ \mathrm{m\ s^{-1}}$.



As the value of t increases, the value of $\frac{60}{t+2}$ decreases and so $30 - \frac{60}{t+2}$ gets closer to 30. The graph of v against t approaches $v = 30$ as an asymptote.

b The distance moved in the first 6 s is given by

$$x = \int_0^6 \left(30 - \frac{60}{t+2}\right)\mathrm{d}t$$

$$= \left[30t - 60\ln(t+2)\right]_0^6$$

$$= (180 - 60\ln 8) - (0 - 60\ln 2)$$

$$= 180 - 60\ln 2^3 + 60\ln 2$$

$$= 180 - 180\ln 2 + 60\ln 2$$

$$= 180 - 120\ln 2$$

The car is always travelling in the same direction. It does not turn round and so the distance moved in the interval $0 \le t \le 6$ can be found by evaluating the definite integral $\int_0^6 v\,\mathrm{d}t$.

The distance moved by the car in the first 6 s of its motion is $(180 - 120\ln 2)$ m.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 44

Question:

A particle P of mass $\frac{1}{3}$ kg moves along the positive x-axis under the action of a single force. The force is directed towards the origin O and has magnitude $\frac{k}{(x+1)^2}$ N, where $OP = x$ metres and k is a constant. Initially P is moving away from O.
At $x = 1$ the speed of P is 4 m s^{-1}, and at $x = 8$ the speed of P is $\sqrt{2} \text{ m s}^{-1}$.

a Find the value of k.

b Find the distance of P from O when P first comes to instantaneous rest. **[E]**

Solution:

a $\mathbf{F} = m\mathbf{a}$

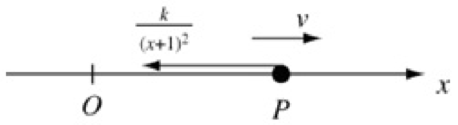
$$-\frac{k}{(x+1)^2}f = \frac{1}{3}a$$

$$a = -\frac{3k}{(x+1)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -3k(x+1)^{-2}$$

$$\frac{1}{2}v^2 = \frac{-3k(x+1)^{-1}}{-1} + A = \frac{3k}{x+1} + A$$

$$v^2 = \frac{6k}{x+1} + B, \text{ where } B = 2A$$



The particle is moving along the positive x-axis and the force is directed toward O. So the force is in the direction of x decreasing and force has a negative sign in this equation.

At $x = 1, v = 4$

$$16 = \frac{6k}{2} + B \Rightarrow 3k + B = 16 \quad ①$$

At $x = 8, v = \sqrt{2}$

$$2 = \frac{6k}{9} + B \Rightarrow \frac{2}{3}k + B = 2 \quad ②$$

The information in the question gives you the values of v at two values of x and you use the information to obtain two simultaneous equations, which you solve.

① − ②

$$3k - \frac{2}{3}k = \frac{7}{3}k = 14$$

$$k = 14 \times \frac{3}{7} = 6$$

b Substituting $k = 6$ into ①

$$18 + B = 16 \Rightarrow B = -2$$

Hence $v^2 = \frac{36}{x+1} - 2$

To find the value of x for which P comes to rest, substitute $v = 0$ into this equation and solve for x.

When $v = 0$

$$0 = \frac{36}{x+1} - 2 \Rightarrow 2(x+1) = 36$$

$$2x + 2 = 36 \Rightarrow x = \frac{36-2}{2} = 17$$

The distance of P from O when P first comes to instantaneous rest is 17 m.

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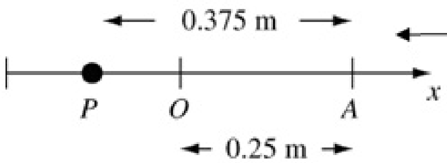
Review Exercise 1
Exercise A, Question 45

Question:

A particle P moves in a straight line with simple harmonic motion about a fixed centre O with period 2 s. At time t seconds the speed of P is v m s^{-1}. When $t = 0$, $v = 0$ and P is at a point A where $OA = 0.25$ m. Find the smallest positive value of t for which $AP = 0.375$ m. **[E]**

Solution:



The motion is simple harmonic with amplitude, a m, given by $a = 0.25$.

At P, $x = 0.25 - 0.375 = -0.125$

When $AP = 0.375$ m, P is 0.125 m from the centre of oscillation O. It is the other side of O from A and, if OA is taken as the direction of x increasing, the displacement of P is -0.125 m.

The period is 2 s

Hence $T = \dfrac{2\pi}{\omega} = 2 \Rightarrow \omega = \pi$

$x = a\cos\omega t$

$-0.125 = 0.25\cos\pi t$

$\cos\pi t = -\dfrac{1}{2}$

If the time, t seconds, is measured from the time when the velocity is zero, that is when the distance of P from O is the amplitude, then $x = a\cos\omega t$ is the appropriate formula connecting the displacement with the time.

The smallest positive value of t is given by

$\pi t = \arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

$t = \dfrac{2}{3}$

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Review Exercise 1
Exercise A, Question 46

Question:

A particle P of mass 0.2 kg oscillates with simple harmonic motion between the points A and B, coming to rest at both points. The distance AB is 0.2 m, and P completes 5 oscillations every second.

a Find, to 3 significant figures, the maximum resultant force exerted on P.

When the particle is at A, it is struck a blow in the direction BA. The particle now oscillates with simple harmonic motion with the same frequency as previously but twice the amplitude.

b Find, to 3 significant figures, the speed of the particle immediately after it has been struck. **[E]**

Solution:

a If P completes 5 oscillations in one second,

then P takes $\frac{1}{5}$ s to complete one oscillation.

$T = \frac{2\pi}{\omega} = \frac{1}{5} \Rightarrow \omega = 10\pi$

The acceleration $\ddot{x}$ m s^{-2} is given by

$\ddot{x} = -\omega^2 x$

In many other topics in Mechanics, it is usual to use the symbol a for the acceleration. With simple harmonic motion, a is often used for the amplitude and it is sensible to use another symbol. Here the calculus symbol $\ddot{x}$, for the acceleration is used.

The amplitude of the oscillation is

$\frac{0.2}{2}$ m $= 0.1$ m

The amplitude is half of the distance between the extreme points of the oscillation.

The greatest magnitude of the acceleration is given by

$|\ddot{x}| = \omega^2 a = (10\pi)^2 \times 0.1 = 10\pi^2$

The greatest magnitude of the acceleration, and hence the force of greatest magnitude, occurs when the displacement is the amplitude.

The maximum magnitude of the force is given by

$|\mathbf{F}| = |ma| = m\omega^2 a$

$= 0.2 \times 10\pi^2 = 2\pi^2 = 19.739\ldots$

The magnitude of the greatest force exerted on P is 19.7 N (3 s.f.)

b $\omega = 10\pi, a = 2 \times 0.1 = 0.2$

$v^2 = \omega^2(a^2 - x^2)$

$= 100\pi^2(0.2^2 - 0.1^2) = 3\pi^2$

$v = \sqrt{3}\pi = 5.441\ldots$

The blow is struck when P is 0.1 m from the centre of oscillation. So $x = 0.1$.

The speed of P immediately after it has been struck is 5.44 m s^{-1} (3 s.f.)

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Review Exercise 1
Exercise A, Question 47

Question:

A piston P in a machine moves in a straight line with simple harmonic motion about a fixed centre O. The period of the oscillations is π s. When P is 0.5 m from O, its speed is 2.4 m s^{-1}. Find

a the amplitude of the motion,
b the maximum speed of P during its motion,
c the maximum magnitude of the acceleration of P during the motion,
d the total time, in seconds to 2 decimal places, in each complete oscillation for which the speed of P is greater than 2.4 m s^{-1}. **[E]**

Solution:

a $T = \frac{2\pi}{\omega} = \pi \Rightarrow \omega = 2$

When $x = 0.5, v = 2.4$

$$v^2 = \omega^2(a^2 - x^2)$$
$$2.4^2 = 2^2(a^2 - 0.5^2)$$
$$a^2 - 0.25 = \frac{2.4^2}{2^2} = 1.44$$
$$a^2 = 1.69 \Rightarrow a = 1.3$$

The amplitude of the motion is 1.3 m.

b The maximum speed is given by

$v = \omega a = 2 \times 1.3 = 2.6$

The maximum speed of P during its motion is 2.6 m s^{-1}.

As $v^2 = \omega^2(a^2 - x^2)$ and x^2 is positive for all x, the greatest value of v^2 is at $x = 0$. So the greatest value of v^2 is $\omega^2 a^2$ and the greatest value of the speed is ωa.

c The maximum magnitude of the acceleration is given by

$|\ddot{x}| = |\omega^2 a| = 4 \times 1.3 = 5.2$

The maximum magnitude of the acceleration is 5.2 m s^{-2}.

The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

d $x = a\cos\omega t$

Differentiating with respect to t

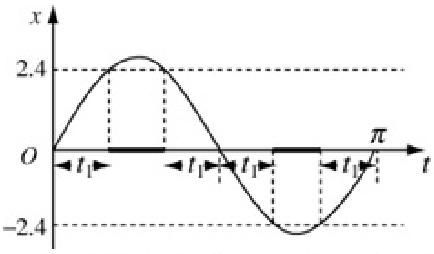
$$\dot{x} = -a\omega\sin\omega t$$
$$|\dot{x}| = |a\omega\sin\omega t|$$
$$2.4 = 1.3 \times 2\sin 2t_1 = 2.6\sin 2t_1$$
$$\sin 2t_1 = \frac{12}{13}$$
$$2t_1 = \arcsin\left(\frac{12}{13}\right) = 1.176\ldots$$
$$t_1 = 0.588\ldots$$

The required time is given by

$T' = \pi - 4t_1 = \pi - 4 \times 0.588\ldots$

$= 0.789\ldots$

The time for which the speed is greater than 2.4 m is 0.79 s (2 d.p.).



This diagram illustrates that if t_1 is the smallest positive solution of $2.6\sin 2t_1 = 2.4$, the time for which the speed is greater than 2.4 is $\pi - 4t_1$.

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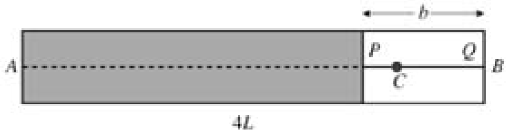
Review Exercise 1
Exercise A, Question 48

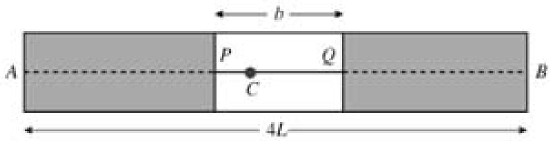
Question:



In a game at a fair, a small target C moves horizontally with simple harmonic motion between the points A and B, where $AB = 4L$. The target moves inside a box and takes 3 s to travel from A to B. A player has to shoot at C, but C is only visible to the player when it passes a window PQ where $PQ = b$. The window is initially placed with Q at the point shown in the figure above. The target takes 0.75 s to travel from Q to P.

a Show that $b = (2 - \sqrt{2})L$.

b Find the speed of C as it passes P.



For advanced players, the window PQ is moved to the centre of AB so that $AP = QB$, as shown in the second figure above.

c Find the time, in seconds to 2 decimal places, taken for C to pass from Q to P in this new position. **[E]**

Solution:

a The period of motion is 6 s.

> The time taken to move from A to B is half of a complete oscillation. So the period is $2\times 3\text{ s} = 6\text{ s}$.

$T = \frac{2\pi}{\omega}$

$6 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{3}$

Measuring the time, t seconds, from an instant when C is at Q and the displacement from the centre of the oscillation, O say

$x = a\cos\omega t$

> The formula for the displacement in terms of time when time is measured from the instant when a particle is at the amplitude is $x = a\cos\omega t$. If time is measured from the instant when a particle is at the centre of oscillation, then the formula is $x = a\sin\omega t$. In part **a**, it is more efficient to use the formula with the cosine but in part **c**, the formula with the sine gives you a quicker solution.

The amplitude is $2L$ and $\omega = \frac{\pi}{3}$

After 0.75 s, C is at P.

$x = 2L\cos\left(\frac{\pi}{3}\times 0.75\right) = 2L\cos\frac{\pi}{4}$

$= 2L\times\frac{1}{\sqrt{2}} = \sqrt{2}L$

$b = a - x = 2L - \sqrt{2}L$

$= (2-\sqrt{2})L$, as required

> From part **a**, when C is at P, its displacement from the centre of oscillation is $\sqrt{2}L$ so $x = \sqrt{2}L$.

b The speed of C at P is given by

$v^2 = \omega^2(a^2 - x^2)$

$= \left(\frac{\pi}{3}\right)^2((2L)^2 - (\sqrt{2}L)^2)$

$= \left(\frac{\pi}{3}\right)^2(4L^2 - 2L^2) = 2\left(\frac{\pi}{3}\right)^2 L^2$

$v = \frac{\sqrt{2}\pi L}{3}$

c If the window is centred the displacement of Q from the centre of oscillation is given by

$x = \frac{b}{2} = \frac{2-\sqrt{2}}{2}L$

Measuring time, t seconds, from the centre of oscillation, at Q

$x = a\sin\omega t$

$\frac{2-\sqrt{2}}{2}L = 2L\sin\left(\frac{\pi}{3}t\right)$

$\sin\left(\frac{\pi}{3}t\right) = \frac{2-\sqrt{2}}{4} = 0.146\,446\ldots$

$\frac{\pi}{3}t = 0.146\,975\ldots \Rightarrow t = 0.140\,35\ldots$

> In all SHM questions, it is assumed that angles are measured in radians. It is important that you make sure your calculator is in the correct mode.

The time from P to Q is given by

$T' = 2t = 0.2807\ldots$

The time taken for C to pass from P to Q is 0.28 (2 d.p.).

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Review Exercise 1
Exercise A, Question 49

Question:

The points O, A, B and C lie in a straight line, in that order, with $OA = 0.6$ m, $OB = 0.8$ m and $OC = 1.2$ m. A particle P, moving in a straight line, has speed $\left(\frac{3}{10}\sqrt{3}\right)$ m s^{-1} at A, $\left(\frac{1}{5}\sqrt{5}\right)$ m s^{-1} at B and is instantaneously at rest at C.

a Show that this information is consistent with P performing simple harmonic motion with centre O.

Given that P is performing simple harmonic motion with centre O,

b show that the speed of P at O is 0.6 m s^{-1},

c find the magnitude of the acceleration of P as it passes A,

d find, to 3 significant figures, the time taken for P to move directly from A to B. **[E]**

Solution:

a At C $v^2 = \omega^2(a^2 - x^2)$

$0^2 = \omega^2(a^2 - 1.2^2) \Rightarrow a = 1.2$

At A $v^2 = \omega^2(a^2 - x^2)$

$$\left(\frac{3}{10}\sqrt{3}\right)^2 = \omega^2(1.2^2 - 0.6^2)$$

$$\frac{27}{100} = \omega^2 \times 1.08$$

$$\omega^2 = \frac{27}{108} = \frac{1}{4} \Rightarrow \omega = \frac{1}{2}$$

> You show that the information is the question is consistent with SHM by taking the information you have been given about two of the points and using it to find the values of a and ω. You then confirm these values are correct using the information about the third point. As the information about C gives you a directly, it is sensible to start with that point.

Checking $a = 1.2$ and $\omega = \frac{1}{2}$ at B

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{1}{4}(1.2^2 - 0.8^2) = 0.2 = \frac{1}{5}$$

$$v = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{1}{5}\sqrt{5}$$

> Using $a = 1.2$ and $\omega = \frac{1}{2}$, you find the speed of P at B. This calculation confirms the speed of P given in the question and you deduce that the information is consistent with P performing simple harmonic motion.

This is consistent with the information in the question. The information is consistent with P performing SHM with centre O.

b At O, $x = 0$. Using $v^2 = \omega^2(a^2 - x^2)$

$$= \frac{1}{4}(1.2^2 - 0^2) = 0.36$$

$$v = \sqrt{0.36} = 0.6$$

The speed of P at O is 0.6 m s^{-1}, as required.

c At A $\ddot{x} = -\omega^2 x = -\frac{1}{4} \times 0.6 = -0.15$

The magnitude of the acceleration at A is 0.15 m s^{-2}.

d At A $x = a\sin\omega t$

$$0.6 = 1.2\sin\frac{1}{2}t_1 \Rightarrow \sin\frac{1}{2}t_1 = \frac{1}{2}$$

$$\frac{1}{2}t_1 = \frac{\pi}{6} \Rightarrow t_1 = \frac{\pi}{3}$$

At B $x = a\sin\omega t$

$$0.8 = 1.2\sin\frac{1}{2}t_2 \Rightarrow \sin\frac{1}{2}t_2 = \frac{2}{3}$$

$$\frac{1}{2}t_2 = 0.729\,727\ldots \Rightarrow t_2 = 1.459\,455\ldots$$

$$t_2 - t_1 = 1.459\,455\ldots - \frac{\pi}{3} = 0.412\,257\ldots$$

> In this question, as you need to find the difference between the times at which P is at A and B, it does not matter which of the formulae $x = a\sin\omega t$ or $x = a\cos\omega t$ you use. If you use the formula with cosine, you obtain $\frac{2\pi}{3}$ s and 1.6821… s as the times. The difference between these times is again 0.412 (3 s.f.).

The time taken to move directly from A to B is 0.412 s (3 s.f.).

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Review Exercise 1
Exercise A, Question 50

Question:

The rise and fall of the water level in a harbour is modelled as simple harmonic motion. On a particular day the maximum and minimum depths of the water in the harbour are 10 m and 4 m and these occur at 1100 hours and 1700 hours respectively.

a Find the speed, in m h^{-1}, at which the water level in the harbour is falling at 1600 hours on this particular day.

b Find the total time, between 1100 hours and 2300 hours on this particular day, for which the depth in the harbour is less than 5.5 m. **[E]**

Solution:

a $a = \frac{10-4}{2} = 3$

The period of motion is 12 hours

The 6 hours, from 1100 to 1700, are half of a complete oscillation.

$T = \frac{2\pi}{\omega} = 12 \Rightarrow \omega = \frac{\pi}{6}$

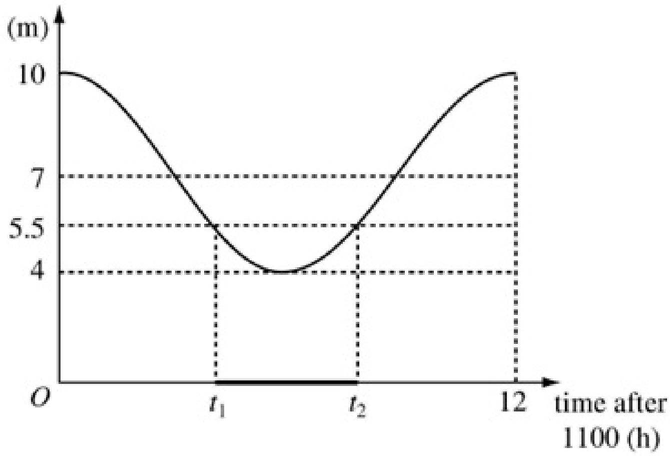
At 1600 hours, $t = 5$

$$x = a\cos\omega t$$
$$= 3\cos\frac{5\pi}{6} = 3\times\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$$
$$v^2 = \omega^2(a^2 - x^2)$$
$$= \left(\frac{\pi}{6}\right)^2\left(3^2 - \left(-\frac{3\sqrt{3}}{2}\right)^2\right) = \left(\frac{\pi}{6}\right)^2\left(9 - \frac{27}{4}\right)$$
$$= \left(\frac{\pi}{6}\right)^2 \times \frac{9}{4}$$
$$v = (-)\frac{\pi}{6}\times\frac{3}{2} = \frac{\pi}{4}$$

The formulae for simple harmonic motion can be used with any consistent set of units. Here metres and hours are used.

At 1600, the water level is falling at a rate of $\frac{\pi}{4}$ m h^{-1}.

b



You find the times, here labelled t_1 and t_2, where the water is at a depth of 5.5 m. The diagram shows that the total time for which the depth of the water is less that 5.5 m is the difference between these times.

5.5 m is 1.5 m below the centre of oscillation

The centre of the oscillation is at a depth of $\frac{10-4}{2}$ m $= 7$ m .

$$x = a\cos\omega t$$
$$-1.5 = 3\cos\left(\frac{\pi}{6}t\right) \Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$$

$\frac{\pi}{6}t = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8$

The time for which the depth of water in the harbour is less than 5.5 m is $(8-4)$hours $= 4$ hours .

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Review Exercise 1
Exercise A, Question 51

Question:

A piston in a machine is modelled as a particle of mass 0.2 kg attached to one end A of a light elastic spring, of natural length 0.6 m and modulus of elasticity 48 N. The other end B of the spring is fixed and the piston is free to move in a horizontal tube which is assumed to be smooth. The piston is released from rest when $AB = 0.9$ m.

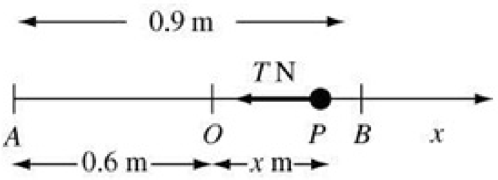
a Prove that the motion of the piston is simple harmonic with period $\frac{\pi}{10}$ s.

b Find the maximum speed of the piston.

c Find, in terms of π, the length of time during each oscillation for which the length of the spring is less than 0.75 m. **[E]**

Solution:

a



Let the piston be modelled by the particle P.
Let O be the point where $AO = 0.6$ m
When P is at a general point in its motion,

let $OP = x$ metres and the force of the spring on P be T newtons.

> Displacements in simple harmonic questions are usually measured from the centre of the motion. At the centre, the acceleration of the particle is zero and the forces on the particle are in equilibrium. In this question, the point of equilibrium, O, is where the spring is at its natural length. No horizontal forces will then be acting on the particle.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{48x}{0.6} = 80x$$

R($\rightarrow$) $\quad \mathbf{F} = m\mathbf{a}$

$$-T = 0.2\ddot{x}$$

$$-80x = 0.2\ddot{x}$$

$$\ddot{x} = -400x = -20^2 x$$

> When x is positive, the tension in the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

> To show that P is moving with simple harmonic motion, you have to show that, at a general point of its motion, the equation of motion of P has the form $\ddot{x} = -kx$, where k is a positive constant. In this case $k = \omega^2 = 100$.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 20$. The period, T seconds, is given by

$$T = \frac{2\pi}{20} = \frac{\pi}{10}, \text{ as required.}$$

b $a = 0.3, \omega = 20$

The maximum speed is given by

$$v = a\omega = 0.3 \times 20 = 6$$

The maximum speed is 6 m s^{-1}.

c When the length of the spring is 0.75 m

$$x = 0.75 - 0.6 = 0.15$$

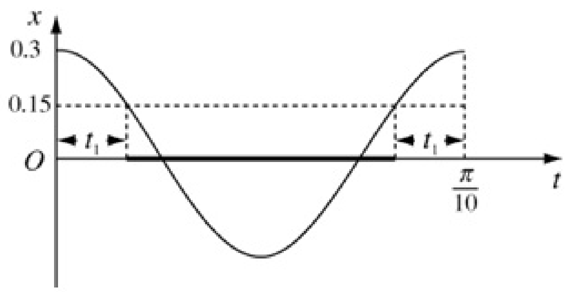
$$x = a\cos\omega t$$

$$0.15 = 0.3\cos 20t_1 \Rightarrow \cos 20t_1 = \frac{1}{2}$$

$$20t_1 = \frac{\pi}{3} \Rightarrow t_1 = \frac{\pi}{60}$$

The total time for which the length of the spring is less that 0.75 m is given by

$$T' = T - 2t_1 = \frac{\pi}{10} - 2\times\frac{\pi}{60} = \frac{\pi}{15}$$



> When the length of the spring is less than 0.75 m, the extension of the spring, x m, is less than 0.15 m. This sketch shows you that if the first time where the extension is 0.15 m is t_1 s, the length of time for which the extension is less than 0.15 m is $\left(\frac{\pi}{10} - 2t_1\right)$s.

The length of time for which the length of the spring is less that 0.75 m is $\frac{\pi}{15}$ s.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 52

Question:

A particle P of mass 0.8 kg is attached to one end A of a light elastic spring OA, of natural length 60 cm and modulus of elasticity 12 N. The spring is placed on a smooth table and the end O is fixed. The particle is pulled away from O to a point B, where $OB = 85$ cm, and is released from rest.

a Prove that the motion of P is simple harmonic motion with period $\frac{2\pi}{5}$ s.

b Find the greatest magnitude of the acceleration of P during the motion.

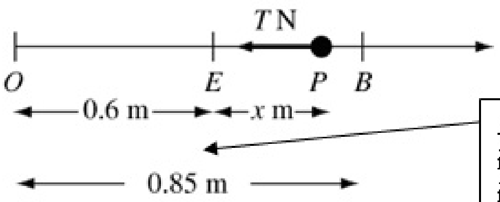
Two seconds after being released from rest, P passes through the point C.

c Find, to 2 significant figures, the speed of P as it passes through C.

d State the direction in which P is moving 2 s after being released. **[E]**

Solution:

a



As you will use Newton's second law in this question, it is safer to use base SI units. So convert the distances in cm to m.

Let E be the point where $OE = 0.6$ m .

When P is at a general point in its motion, let $EP = x$ metres and the force of the spring on P be T newtons.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{12x}{0.6} = 20x$$

$$\text{R}(\rightarrow) \quad \mathbf{F} = ma$$

$$-T = 0.8\ddot{x}$$

$$-20x = 0.8\ddot{x}$$

$$\ddot{x} = -25x = -5^2 x$$

When x is positive, the tension is the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 5$. The period, T seconds, is given by

$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$, as required.

b The amplitude of the motion is 0.25 m.
The maximum magnitude of the acceleration is given by

$$|\ddot{x}| = |\omega^2 a| = 25 \times 0.25 = 6.25$$

The maximum magnitude of the acceleration is 6.25 m s^{-2}.

The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

c $x = a\cos\omega t$

$\dot{x} = -a\omega\sin\omega t$

At $t = 2$

$$\dot{x} = -0.25 \times 5\sin(5 \times 2) = -1.25\sin 10$$

$$= +0.680\ 026\ldots$$

The speed of P as it passes through C is 0.68 m s^{-1} (2 s.f.).

You can derive an equation connecting velocity with time by differentiating $x = a\cos\omega t$ with respect to t. You obtain $v = \dot{x} = \frac{dx}{dt} = -a\omega\sin\omega t$. This equation is particularly useful when you are asked about the direction of motion of a particle. As the v is squared in $v^2 = \omega^2(a^2 - x^2)$, values of v found using this formula have an ambiguous $\pm$ sign.

d As the sign of $\dot{x}$ in part **c** is positive, P is travelling in the direction of x increasing as it passes through C.
As it passes through C, P is moving away from O towards B.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 53

Question:

A light elastic string of natural length l has one end attached to a fixed point A. A particle P of mass m is attached to the other end of the string and hangs in equilibrium at the point O, where $AO = \frac{5}{4}l$.

a Find the modulus of elasticity of the string.

The particle P is then pulled down and released from rest. At time t the length of the string is $\frac{5l}{4} + x$.

b Prove that, while the string is taut, $\frac{d^2x}{dt^2} = -\frac{4gx}{l}$

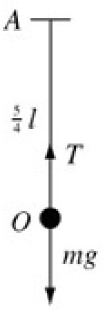
When P is released, $AP = \frac{7}{4}l$. The point B is a distance l vertically below A.

c Find the speed of P at B.

d Describe briefly the motion of P after it has passed through B for the first time until it next passes through O. **[E]**

Solution:

a



At the equilibrium level

R(↑) $T = mg$ ①

Hooke's law

$T = \frac{\lambda e}{l} = \frac{\lambda \times \frac{1}{4}l}{l} = \frac{\lambda}{4}$ ②

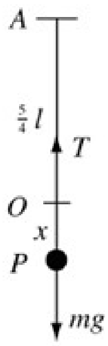
The information you have been given at the equilibrium level enables you to obtain two equations for the tension T, one using Hooke's law and a second by resolving vertically. Eliminating T between the two equations gives you an equation for the modulus of elasticity λ.

Combining ① and ②

$\frac{\lambda}{4} = mg \Rightarrow \lambda = 4mg$

The coefficient of elasticity is $4mg$.

b



Hooke's law

$T = \frac{\lambda e}{l} = \frac{4mg\left(\frac{1}{4}l + x\right)}{l}$

$= \frac{mgl + 4mgx}{l} = mg + \frac{4mgx}{l}$ ③

When $AP = \frac{5}{4}l + x$, the extension is $\left(\frac{5}{4}l + x\right) - l = \frac{1}{4}l + x$.

Newton's second law

R(↓) $\mathbf{F} = m\mathbf{a}$

$mg - T = m\frac{d^2x}{dt^2}$ ④

You take the forces in the direction of x increasing. The weight tends to increase the value of x, so mg is positive. The tension tends to decrease the value of x so, in this equation, T has a negative sign.

Substituting ③ into ④

$$mg - \left(mg + \frac{4mgx}{l}\right) = m\frac{d^2x}{dt^2}$$

$$-\frac{4mgx}{l} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{4gx}{l}, \text{ as required.}$$

c Comparing the result of part **b** with the standard formula $\ddot{x} = -\omega^2 x$, while the string is taut, P moves with SHM about O, with $\omega^2 = \frac{4g}{l}$.

When P is released from the point where $AP = \frac{7}{4}l$, the amplitude, a, is given by

$$a = AP - AO = \frac{7}{4}l - \frac{5}{4}l = \frac{1}{2}l$$

At B, $x = -\frac{1}{4}l$

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{4g}{l}\left(\left(\frac{1}{2}l\right)^2 - \left(-\frac{1}{4}l\right)^2\right)$$

$$= \frac{4g}{l}\left(\frac{1}{4}l^2 - \frac{1}{16}l^2\right) = \frac{4g}{l} \times \frac{3l^2}{16} = \frac{3gl}{4}$$

$$v = \frac{1}{2}\sqrt{(3gl)}$$

The speed of P at B is $\frac{1}{2}\sqrt{(3gl)}$.

Alternatively, you can use conservation of energy between the point of release and B to find the velocity at B.
Loss in elastic energy = gain in kinetic energy + gain in potential energy

$$\frac{4mg(\frac{3}{4}l)^2}{2l} = \frac{1}{2}mv^2 + mg \times \frac{3}{4}l.$$

This, of course, leads to the same answer.

d First P moves freely under gravity until it returns to B. Then it moves with simple harmonic motion about O.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 54

Question:

A light elastic string, of natural length $4a$ and modulus of elasticity $8mg$, has one end attached to a fixed point A. A particle P of mass m is attached to the other end of the string and hangs in equilibrium at the point O.

a Find the distance AO.

The particle is now pulled down to a point C vertically below O, where $OC = d$. It is released from rest. In the subsequent motion the string does not become slack.

b Show that P moves with simple harmonic motion of period $\pi\sqrt{\frac{2a}{g}}$. The greatest speed of P during this motion is $\frac{1}{2}\sqrt{(ga)}$.

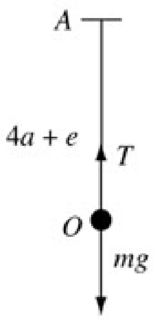
c Find d in terms of a.

Instead of being pulled down a distance d, the particle is pulled down a distance a. Without further calculation,

d Describe briefly the subsequent motion of P. **[E]**

Solution:

a



A particle attached to one end of an elastic string will oscillate about the equilibrium position. When solving problems about vertical oscillations, you often have to begin by finding the point of equilibrium. In this case, the oscillations later in the question have centre O.

At the equilibrium level, let $AO = 4a + e$, where e is the extension of the string.
Hooke's law

$$T = \frac{\lambda e}{l} = \frac{8mge}{4a} = \frac{2mge}{a}$$

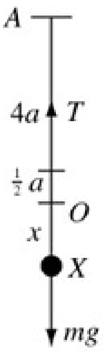
R($\uparrow$) $T = mg$

Hence

$$mg = \frac{2mge}{a} \Rightarrow e = \frac{a}{2}$$

$$AO = 4a + e = 4a + \frac{a}{2} = \frac{9a}{2}$$

b



When P is at a general point, X say, of its motion, let $OX = x$. At this point, the extension of the string is $\frac{1}{2}a + x$

To show that P is moving with simple harmonic motion, you have to show that, at a general point in its motion, the equation of motion of P has the form $\ddot{x} = -\omega^2 x$, where ω is a positive constant.

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{8mg\left(\frac{1}{2}a + x\right)}{4a}$$

$$= \frac{4mga + 8mgx}{4a} = mg + \frac{2mgx}{a} \quad ①$$

Newton's second law

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m\ddot{x} \quad ②$$

Hooke's law and Newton's second law give you two equations from which you eliminate the tension, T.

Substituting ① into ②

$$mg - \left(mg + \frac{2mgx}{a}\right) = m\ddot{x}$$

$$-\frac{2mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a}x$$

Comparing this equation with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P moves with simple harmonic motion about O and

$$\omega = \sqrt{\left(\frac{2g}{a}\right)}.$$

The period of motion T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{a}{2g}\right)} = \pi\sqrt{\left(\frac{2a}{g}\right)}, \text{ as required.}$$

c The maximum speed is given by $v = a\omega$

$$\frac{1}{2}\sqrt{(ga)} = d\sqrt{\left(\frac{2a}{a}\right)}$$

As the particle is pulled down a distance d from the equilibrium position and released from rest, d is the amplitude of the motion.

$$\frac{1}{4}ga = d^2 \times \frac{2g}{a}$$

Squaring both sides of the previous line.

$$d^2 = \frac{1}{8}a^2$$

$$d = \frac{1}{2\sqrt{2}}a$$

d As $a > \frac{1}{2}a$, the string will become slack during its motion. The subsequent motion of P will be partly under gravity, partly simple harmonic motion.

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 55

Question:

A particle P of mass 0.3 kg is attached to one end of a light elastic spring. The other end of the spring is attached to a fixed point O on a smooth horizontal table. The spring has natural length 2 m and modulus of elasticity 21.6 N. The particle P is placed on the table at a point A, where $OA = 2$ m . The particle P is now pulled away from O to the point B, where OAB is a straight line with $OB = 3.5$ m. It is then released from rest.

a Prove that P moves with simple harmonic motion of period $\frac{\pi}{3}$ s .

b Find the speed of P when it reaches A.

The point C is the mid-point of AB.

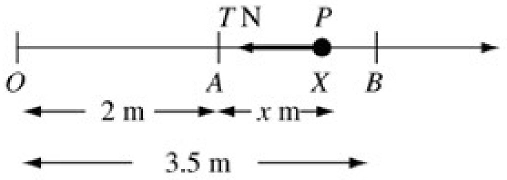
c Find, in terms of π, the time taken for P to reach C for the first time.

Later in the motion, P collides with a particle Q of mass 0.2 kg which is at rest at A. After impact, P and Q coalesce to form a single particle R.

d Show that R also moves with simple harmonic motion and find the amplitude of this motion. **[E]**

Solution:

a



When P is at the point X, where $AX = x$ m, let the tension in the spring be T N.
Hooke's law

$$T = \frac{\lambda x}{l} = \frac{21.6 \times x}{2} = 10.8x$$

R($\rightarrow$) $\mathbf{F} = ma$

$$-T = 0.3\ddot{x}$$

$$-10.8x = 0.3\ddot{x}$$

$$\ddot{x} = -36x = -6^2 x$$

When x is positive, the tension in the spring is acting in the direction of x decreasing, so T has a negative sign in the equation of motion.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P is performing simple harmonic motion about A with $\omega = 6$.

The period of motion T s is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}, \text{ as required.}$$

b At A, $x = 0$

$$v^2 = \omega^2(a^2 - x^2) = 36(1.5^2 - 0^2) = 81$$

$$v = \sqrt{81} = 9$$

The speed of P at A is 9 m s^{-1}.

c At C, $x = \frac{1.5}{2} = 0.75$

$$x = a\cos\omega t$$

$$0.75 = 1.5\cos 6t$$

$$\cos 6t = \frac{1}{2} \Rightarrow 6t = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{18}$$

P reaches C for the first time after $\frac{\pi}{18}$ s.

The time when P first reaches C is the smallest positive value of t for which this equation is true. In principle, in simple harmonic motion, P will reach this point infinitely many times.

d Before impact, the linear momentum of P is
$m_1 u = 0.3 \times 9$ N s $= 2.7$ N s
Let the velocity of the combined particle R immediately after impact be U m s^{-1}.

After impact, the linear momentum of R is
$m_2 U = 0.5U$ N s

Conservation of linear momentum
$0.5U = 2.7 \Rightarrow U = 5.4$

Conservation of linear momentum is an M1 topic and is assumed, and can be tested, in any of the subsequent Mechanics modules.

For R
$$\mathrm{R}(\rightarrow) \quad -T = 0.5\ddot{x}$$
$$-10.8x = 0.5\ddot{x}$$
$$\ddot{x} = -21.6x$$

When R is at X, Hooke's law gives $T = 10.8x$, exactly as in part **a**. There is no need to repeat the working in part **d**.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, after the impact R is performing simple harmonic motion about A with $\omega^2 = 21.6$.
$$v = U = a\omega$$
$$5.4 = a\sqrt{21.6}$$
$$a = \frac{5.4}{\sqrt{21.6}} = 1.161\ldots$$
The amplitude of the motion is 1.16 m (3 s.f.)

As R is performing simple harmonic motion about A, the speed of R immediately after the impact is the maximum speed of R during its motion. The maximum speed during simple harmonic motion is given by $v = a\omega$.

No accuracy is specified in the question and the accurate answer, $\frac{3\sqrt{15}}{10}$ m, or any reasonable approximation would be accepted.

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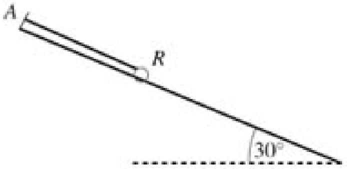
Review Exercise 1
Exercise A, Question 56

Question:



A small ring R of mass m is free to slide on a smooth straight wire which is fixed at an angle of $30°$ to the horizontal. The ring is attached to one end of a light elastic string of natural length a and modulus of elasticity λ. The other end is attached to a fixed point A on the wire, as shown in the figure. The ring rests in equilibrium at the point B, where $AB = \frac{9}{8}a$.

a Show that $\lambda = 4mg$.

The ring is pulled down to a point C, where $BC = \frac{1}{4}a$ and released from rest. At time t after R is released the extension in the string is $\left(\frac{1}{8}a + x\right)$.

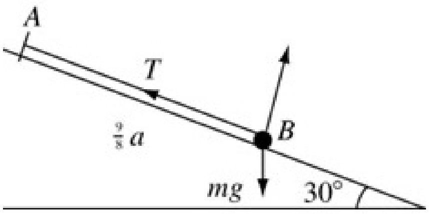
b Obtain a differential equation for the motion of R while the string remains taut, and show that it represents simple harmonic motion with period $\pi\sqrt{\left(\frac{a}{g}\right)}$.

c Find, in terms of g, the greatest magnitude of the acceleration of R while the string remains taut.

d Find, in terms of a and g, the time taken for R to move from the point at which it first reaches a maximum speed to the point where the string becomes slack for the first time. **[E]**

Solution:

a



When R is at B, the extension in the string is

$$\frac{9}{8}a - a = \frac{1}{8}a$$

The extension in the string is AB − the natural length.

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{\lambda \times \frac{1}{8}a}{a} = \frac{\lambda}{8}$$

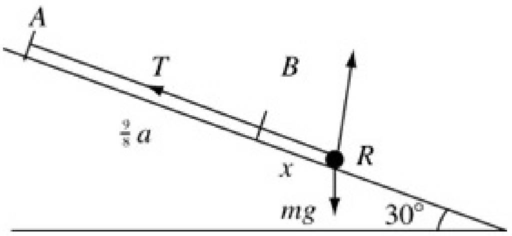
R(↖) $T = mg\sin 30° = \frac{1}{2}mg$

Using $\sin 30° = \frac{1}{2}$.

Hence $\frac{\lambda}{8} = \frac{1}{2}mg \Rightarrow \lambda = 4mg$, as required.

You equate the two expressions for T.

b



When the extension of the string is $\left(\frac{1}{8}a + x\right)$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{4mg \times \left(\frac{1}{8}a + x\right)}{a}$$

$$= \frac{\frac{1}{2}mga + 4mgx}{a} = \frac{mg}{2} + \frac{4mgx}{a} \quad ①$$

R(↘) $\mathbf{F} = m\mathbf{a}$

$$mg\sin 30° - T = m\ddot{x} \quad ②$$

$mg\sin 30°$ is the component of the weight parallel to the string.

Substituting ② into ①

$$\frac{mg}{2}-\left(\frac{mg}{2}+\frac{4mgx}{a}\right)=m\ddot{x}$$

$$-\frac{4mgx}{a}=m\ddot{x}$$

$$\ddot{x}=-\frac{4g}{a}x$$

Comparing with the standard formula for simple harmonic motion, $\ddot{x}=-\omega^2 x$,

R is performing simple harmonic motion about B with $\omega=2\sqrt{\left(\frac{g}{a}\right)}$.

The period of the motion is given by

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{2\sqrt{\left(\frac{g}{a}\right)}}=\pi\sqrt{\left(\frac{a}{g}\right)}$$, as required.

c The maximum magnitude of the acceleration is given by

$$|\ddot{x}|=\omega^2\times\text{amplitude}=\frac{4g}{a}\times\frac{1}{4}a=g$$

The maximum magnitude of the acceleration is g.

As $\ddot{x}=-\omega^2 x$, the magnitude of the acceleration is $\omega^2|x|$. So the maximum magnitude is where x is a large as possible. That is at the amplitude.

d The maximum speed is at B and the string becomes slack after moving a further distance $\frac{1}{8}a$.

$$x=a\sin\omega t$$

$$\frac{1}{8}a=\frac{1}{4}a\sin\omega t$$

$$\sin\omega t=\frac{1}{2}\Rightarrow\omega t=\frac{\pi}{6}$$

$$t=\frac{\pi}{6\omega}=\frac{\pi}{6\times 2\sqrt{\left(\frac{g}{a}\right)}}=\frac{\pi}{12}\sqrt{\left(\frac{a}{g}\right)}$$

You find the shortest time for R to move a distance of $\frac{1}{8}a$ from the centre of the simple harmonic motion.