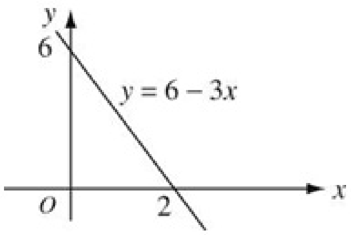
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 1

Question:

Find, by integration, the centre of mass of the uniform triangular lamina enclosed by the lines $y = 6 - 3x, x = 0$ and $y = 0$.

Solution:



Mass $= M = \frac{1}{2}\rho \times 2 \times 6 = 6\rho$, where ρ is the mass per unit area.

$$M\bar{x} = \int_0^2 \rho x(6-3x)\,dx$$
$$= \rho \int_0^2 6x - 3x^2\,dx$$
$$= \rho\left[3x^2 - x^3\right]_0^2$$
$$= \rho[4-0]$$
$$= 4\rho$$
$$\therefore \bar{x} = \frac{4\rho}{M} = \frac{4\rho}{6\rho}$$
$$= \frac{2}{3}$$

$$M\bar{y} = \int_0^2 \rho \frac{1}{2}(6-3x)^2\,dx$$
$$= \frac{1}{2}\rho \int_0^2 36 - 36x + 9x^2\,dx$$
$$= \frac{1}{2}\rho\left[36x - 18x^2 + 3x^3\right]_0^2$$
$$= \frac{1}{2}\rho[24-0]$$
$$= 12\rho$$
$$\therefore \bar{y} = \frac{12\rho}{M} = \frac{12\rho}{6\rho}$$
$$= 2$$

The centre of mass is at the point with coordinates $\left(\frac{2}{3}, 2\right)$.

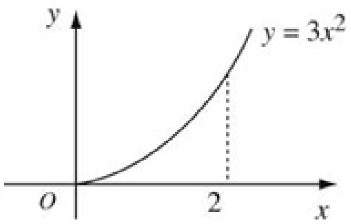
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 2

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = 3x^2$, the x-axis and the line $x = 2$.

Solution:



$$M = \int_0^2 \rho y \,dx$$
$$= \rho \int_0^2 3x^2 \,dx$$
$$= \rho \left[x^3\right]_0^2$$
$$= 8\rho$$

$$M\bar{x} = \int_0^2 \rho x 3x^2 \,dx$$
$$= \rho \int_0^2 3x^3 \,dx$$
$$= \rho \left[\frac{3}{4}x^4\right]_0^2$$
$$= 12\rho$$
$$\therefore \bar{x} = \frac{12\rho}{M} = \frac{12\rho}{8\rho}$$
$$= \frac{3}{2} = 1.5$$

$$M\bar{y} = \int_0^2 \frac{1}{2}\rho(3x^2)^2 \,dx$$
$$= \frac{1}{2}\rho \int_0^2 9x^4 \,dx$$
$$= \frac{9}{2}\rho \left[\frac{x^5}{5}\right]_0^2$$
$$= \frac{9 \times 32}{10}\rho$$
$$= \frac{144}{5}\rho$$
$$\therefore \bar{y} = \frac{144\rho}{5M} = \frac{144\rho}{40\rho}$$
$$= 3.6$$

The centre of mass is at the point with coordinates (1.5, 3.6).

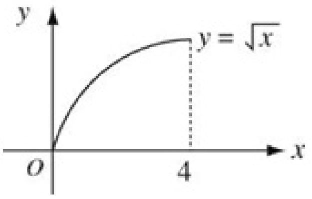
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 3

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y=\sqrt{x}$, the x-axis and the line $x=4$.

Solution:



$$M=\rho\int_0^4 y\,\mathrm{d}x$$
$$=\rho\int_0^4 x^{\frac{1}{2}}\,\mathrm{d}x$$
$$=\left[\frac{2}{3}\rho x^{\frac{3}{2}}\right]_0^4$$
$$=\frac{16\rho}{3}$$

$$M\bar{x}=\rho\int_0^4 xx^{\frac{1}{2}}\mathrm{d}x$$
$$=\rho\int_0^4 x^{\frac{3}{2}}\,\mathrm{d}x$$
$$=\left[\frac{2}{5}\rho x^{\frac{5}{2}}\right]_0^4$$
$$=\frac{64\rho}{5}$$
$$\therefore \bar{x}=\frac{64\rho}{5M}$$
$$=\frac{64\rho}{5}\div\frac{16\rho}{3}$$
$$=\frac{64\rho}{5}\times\frac{3}{16\rho}$$
$$=\frac{12}{5}=2.4$$

$$M\bar{y}=\frac{1}{2}\rho\int_0^4\left(x^{\frac{1}{2}}\right)^2\,\mathrm{d}x$$
$$=\frac{1}{2}\rho\int_0^4 x\,\mathrm{d}x$$
$$=\frac{1}{2}\rho\left[\frac{x^2}{2}\right]_0^4$$
$$=4\rho$$
$$\therefore \bar{y}=\frac{4\rho}{M}$$
$$=4\rho\div\frac{16\rho}{3}$$
$$=4\rho\times\frac{3}{16\rho}$$
$$=\frac{3}{4}=0.75$$

The centre of mass is at the point with coordinates (2.4, 0.75).

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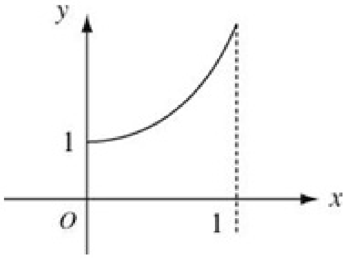
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 4

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = x^3 + 1$, the x-axis and the line $x = 1$.

Solution:



$$M = \rho\int_0^1 (x^3+1)\,dx = \rho\left[\frac{1}{4}x^4 + x\right]_0^1 = \frac{5}{4}\rho$$

$$M\bar{x} = \rho\int_0^1 x(x^3+1)\,dx = \rho\int_0^1 x^4 + x\,dx = \rho\left[\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_0^1 = \frac{7}{10}\rho$$

$$\therefore \bar{x} = \frac{7}{10}\rho \div M = \frac{7}{10}\rho \div \frac{5}{4}\rho = \frac{7\rho}{10}\times\frac{4}{5\rho} = \frac{28}{50} = \frac{14}{25}$$

$$M\bar{y} = \frac{1}{2}\rho\int_0^1 (x^3+1)^2\,dx = \frac{1}{2}\rho\int_0^1 x^6 + 2x^3 + 1\,dx = \frac{1}{2}\rho\left[\frac{1}{7}x^7 + \frac{1}{2}x^4 + x\right]_0^1 = \frac{\rho}{2}\times\frac{23}{14} = \frac{23\rho}{28}$$

$$\therefore \bar{y} = \frac{23\rho}{28} \div M = \frac{23\rho}{28} \div \frac{5}{4}\rho = \frac{23\rho}{28}\times\frac{4}{5\rho} = \frac{23\rho}{35\rho} = \frac{23}{35}$$

The centre of mass is at the point with coordinates $\left(\frac{14}{25}, \frac{23}{35}\right)$.

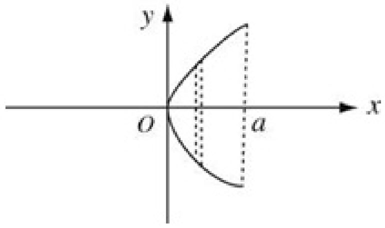
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 5

Question:

Use integration to find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y^2 = 4ax$, and the line $x = a$, where a is a positive constant.

Solution:



$$M = \rho\int_0^a 2y\, dx$$
$$= \rho\int_0^a 2\times 2a^{\frac{1}{2}}x^{\frac{1}{2}}\, dx$$
$$= 4\rho a^{\frac{1}{2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$
$$= \frac{8}{3}\rho a^2$$

$$M\,\bar{x} = \rho\int_0^a x\times 4a^{\frac{1}{2}}x^{\frac{1}{2}}\, dx$$
$$= \rho\int_0^a 4a^{\frac{1}{2}}x^{\frac{3}{2}}\, dx$$
$$= \rho\times 4a^{\frac{1}{2}}\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^a$$
$$= \frac{8\rho}{5}a^3$$
$$\therefore \bar{x} = \frac{8\rho a^3}{5}\div M$$
$$= \frac{8\rho a^3}{5}\div\frac{8\rho a^2}{3}$$
$$= \frac{3}{5}a$$

From symmetry $\bar{y} = 0$.

$\therefore$ The centre of mass is at the point with coordinates $\left(\frac{3}{5}a, 0\right)$.

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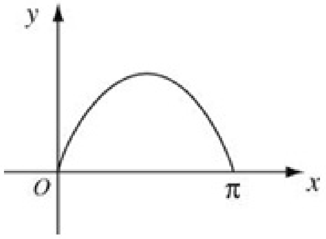
Statics of rigid bodies
Exercise A, Question 6

Question:

Find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = \sin x, (0 \le x \le \pi)$ and the line $y = 0$.

Solution:



$$y = \sin x$$
$$M = \rho \int_0^{\pi} \sin x \, dx$$
$$= \rho [-\cos x]_0^{\pi}$$
$$= 2\rho$$

From symmetry $\bar{x} = \frac{\pi}{2}$

$$M\bar{y} = \frac{1}{2}\rho \int_0^{\pi} \sin^2 x \, dx$$
$$= \frac{1}{2}\rho \times \frac{1}{2}\int_0^{\pi} 1 - \cos 2x \, dx$$
$$= \frac{1}{4}\rho \left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$= \frac{1}{4}\rho\pi$$
$$\therefore \bar{y} = \frac{1}{4}\rho\pi \div M = \frac{1}{4}\rho\pi \div 2\rho$$
$$= \frac{1}{8}\pi$$

The centre of mass is at the point with coordinates $\left(\frac{\pi}{2}, \frac{\pi}{8}\right)$.

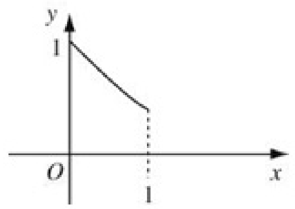
Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 7

Question:

Find the centre of mass of the uniform lamina occupying the finite region bounded by the curve with equation $y = \frac{1}{1+x}, (0 < x < 1)$ and the lines $x = 0, x = 1$ and $y = 0$.

Solution:



$$M = \rho \int_0^1 y \, dx = \rho \int_0^1 \frac{1}{1+x} \, dx = \rho [\ln(1+x)]_0^1 = \rho \ln 2$$

$$M\bar{x} = \rho \int_0^1 \frac{x}{1+x} \, dx = \rho \int_0^1 1 - \frac{1}{1+x} \, dx = \rho [x - \ln(1+x)]_0^1 = \rho [1 - \ln 2]$$

$$\therefore \bar{x} = \rho \frac{[1-\ln 2]}{M} = \rho \frac{[1 - \ln 2]}{\rho \ln 2} = \frac{1-\ln 2}{\ln 2}$$

$$M\bar{y} = \frac{1}{2}\rho \int_0^1 \frac{1}{(1+x)^2} \, dx = \frac{1}{2}\rho \left[-(1+x)^{-1}\right]_0^1 = \frac{1}{2}\rho\left[\frac{-1}{2} + 1\right] = \frac{1}{4}\rho$$

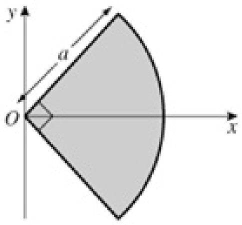
$$\therefore \bar{y} = \frac{1}{4}\rho \div M = \frac{\frac{1}{4}\rho}{\rho \ln 2} = \frac{1}{4 \ln 2}$$

The centre of mass is at the point with coordinates $\left(\frac{1-\ln 2}{\ln 2}, \frac{1}{4\ln 2}\right)$.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

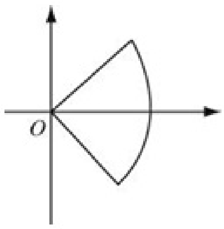
Statics of rigid bodies
Exercise A, Question 8

Question:



Find, by integration, the centre of mass of a uniform lamina in the shape of a quadrant of a circle of radius r as shown.

Solution:



$M=\frac{1}{4}\rho\pi r^2$ as this is a quarter of a circle.

$$M\bar{x}=\rho\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}r^2\times\frac{2}{3}r\cos\theta\,\mathrm{d}\theta$$

$$=\rho\times\frac{1}{3}r^3\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\cos\theta\,\mathrm{d}\theta$$

$$=\frac{1}{3}\rho r^3\left[\sin\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$=\frac{1}{3}\rho r^3\left[\frac{1}{\sqrt{2}}-\left(\frac{-1}{\sqrt{2}}\right)\right]$$

$$=\frac{2\rho r^3}{3\sqrt{2}}$$

$$\therefore\bar{x}=\frac{2\rho r^2}{3\sqrt{2}}\div\frac{1}{4}\rho\pi r^2$$

$$=\frac{8r}{3\pi\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{8\sqrt{2}r}{6\pi}=\frac{4\sqrt{2}r}{3\pi}$$

Also $\bar{y}=0$ from symmetry.

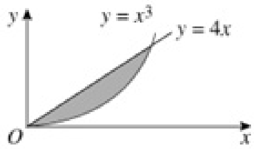
The centre of mass is at the point with coordinates $\left(\frac{4\sqrt{2}r}{3\pi},0\right)$.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

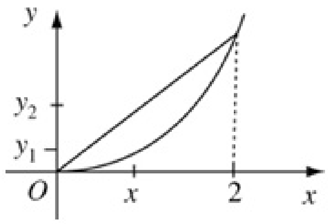
Statics of rigid bodies
Exercise A, Question 9

Question:

The figure shows a uniform lamina bounded by the curve $y = x^3$ and the line with equation $y = 4x$, where $x > 0$. Find the coordinates of the centre of mass of the lamina.



Solution:



$y = x^3$ meets $y = 4x$ when $x^3 = 4x$ i.e. $x = \pm 2$.
$\therefore$ when $x > 0, x = 2$

The small strip shown has dimensions $(y_2 - y_1)$ by δx and centre of mass at $\left(x, \frac{1}{2}(y_1 + y_2)\right)$.

$$M = \rho \int_0^2 (4x - x^3)\,dx$$
$$= \rho \left[2x^2 - \frac{1}{4}x^4\right]_0^2$$
$$= 4\rho$$

$$M\bar{x} = \rho \int_0^2 x(4x - x^3)\,dx$$
$$= \rho \int_0^2 4x^2 - x^4\,dx$$
$$= \rho \left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$
$$= \rho \left[\frac{32}{3} - \frac{32}{5}\right]$$
$$= \frac{64\rho}{15}$$
$$\therefore \bar{x} = \frac{64\rho}{15} \div M = \frac{64\rho}{15} \div 4\rho$$
$$= \frac{16}{15}$$

$$M\bar{y} = \frac{1}{2}\rho \int_0^2 (y_1 + y_2)(y_2 - y_1)\,dx$$
$$= \frac{1}{2}\rho \int_0^2 (4x + x^3)(4x - x^3)\,dx$$
$$= \frac{1}{2}\rho \int_0^2 16x^2 - x^6\,dx$$
$$= \frac{\rho}{2}\left[\frac{16}{3}x^3 - \frac{1}{7}x^7\right]_0^2$$
$$= \frac{\rho}{2}\left[\frac{128}{3} - \frac{128}{7}\right]$$
$$= \frac{256}{21}\rho$$
$$\therefore \bar{y} = \frac{256}{21}\rho \div M = \frac{256}{21}\rho \div 4\rho$$
$$= \frac{64}{21}$$

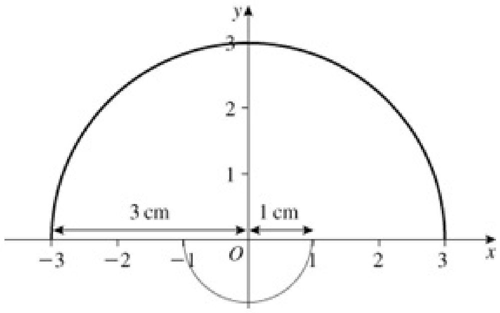
The centre of mass is at the point with coordinates $\left(\frac{16}{15}, \frac{64}{21}\right)$.

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 10

Question:

The figure shows a badge cut from a uniform sheet of fabric. The badge is formed from one semi-circle of radius 1 cm and a semi-circle of radius 3 cm joined as shown in the figure to make a plane lamina. Both semi-circles have the same centre O. Determine, in terms, of pi, the distance from O of the centre of mass.



Solution:

This question may be answered using M2 techniques. List the shapes with their masses in a table.

Shape	Mass	Position of centre of mass
Large semi-circle	$\frac{9\pi\rho}{2}$	$\left(0, \frac{4}{\pi}\right)$
Small semi-circle	$\frac{\pi\rho}{2}$	$\left(0, \frac{-4}{3\pi}\right)$
Total	$5\pi\rho$	$(0, \bar{y})$

From symmetry, the centre of mass lies on the axis of symmetry, taken as the y-axis. The common diameter is taken as the x-axis.

Then use $\sum m_i y_i = \bar{y} \sum m_i$ to give

$$\frac{9\pi\rho}{2} \times \frac{4}{\pi} + \frac{\pi\rho}{2} \times \frac{-4}{3\pi} = 5\pi\rho\bar{y}$$

i.e. $18 - \frac{2}{3} = 5\pi\bar{y}$

i.e. $\frac{52}{15\pi} = \bar{y}$

The centre of mass is on the axis of symmetry at a distance $\frac{52}{15\pi}$ above the common diameter.

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Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise A, Question 11

Question:

The figure shows a uniform lamina made from a sector of a circle with radius 5 cm from which a similar sector of radius 2.5 cm has been removed. The sector is three quarters of the original circle in each case, and both circles have the same centre O. Find the distance of the centre of mass of the lamina from the point O.

Solution:

The centre of mass lies an the axis of symmetry.

Centre of mass of a sector is $\frac{2r\sin\alpha}{3\alpha}$, where the angle of the sector is 2α.

In the given shape $2\alpha = \frac{3\pi}{2} \Rightarrow \alpha = \frac{3\pi}{4}$ and centre of mass of the sectors is at a

distance from the centre O of $\frac{2r \times \frac{1}{\sqrt{2}}}{\frac{9\pi}{4}} = \frac{8r}{9\pi\sqrt{2}} = \frac{4r\sqrt{2}}{9\pi}$.

Shape	Mass	Distance from O of centre of mass
Large sector	$\frac{3}{4}\times 25\pi\rho$	$\frac{20\sqrt{2}}{9\pi}$
Small sector	$\frac{3}{4}\times 6.25\pi\rho$	$\frac{10\sqrt{2}}{9\pi}$
Remainder	$\frac{3}{4}\times 18.75\pi\rho$	$\bar{y}$

From the moments equation

$$\frac{3}{4}\times 25\pi\rho\times\frac{20\sqrt{2}}{9\pi} - \frac{3}{4}\times 6.25\pi\rho\times\frac{10\sqrt{2}}{9\pi} = \frac{3}{4}\times 18.75\pi\rho\bar{y}$$

$$\therefore 4\times\frac{20\sqrt{2}}{9\pi} - 1\times\frac{10\sqrt{2}}{9\pi} = 3\bar{y}$$

Divide each term by $\frac{3}{4}\times 6.25\pi\rho$.

$$\therefore \bar{y} = \frac{70\sqrt{2}}{27\pi} \approx 1.2$$

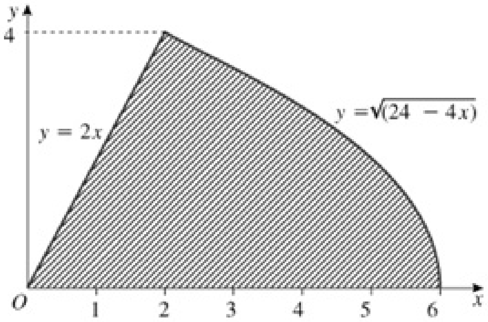
The distance of the centre of mass from O is 1.2 cm (2 s.f.).

Solutionbank M3
Edexcel AS and A Level Modular Mathematics

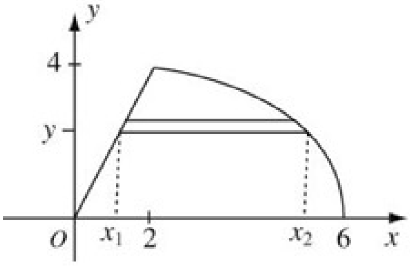
Statics of rigid bodies
Exercise A, Question 12

Question:

The figure shows a uniform lamina occupying the finite region bounded by the x-axis, the curve $y=\sqrt{(24-4x)}$, where $2 \leq x \leq 6$, and the line with equation $y=2x$, where $0 \leq x \leq 2$. Find the coordinates of the centre of mass of the lamina.



Solution:



Divide the region into horizontal strips of dimensions $(x_2 - x_1)$ by δy.
The centre of mass of such strips lies at $\left(\frac{x_1 + x_2}{2}, y\right)$
where $x_1 = \frac{y}{2}$ and $x_2 = \frac{24 - y^2}{4}$.

Using $M = \rho\int_0^4 (x_2 - x_1)\mathrm{d}y$

$$\therefore M = \rho\int_0^4 \left(6 - \frac{1}{4}y^2 - \frac{y}{2}\right)\mathrm{d}y$$
$$= \rho\left[6y - \frac{1}{12}y^3 - \frac{1}{4}y^2\right]_0^4$$
$$= \rho\left[24 - \frac{16}{3} - 4\right]$$
$$= \frac{44}{3}\rho$$

Using $M\bar{y} = \rho\int_0^4 y(x_2 - x_1)\mathrm{d}y$

$$M\bar{y} = \rho\int_0^4 6y - \frac{1}{4}y^3 - \frac{y^2}{2}\,\mathrm{d}y$$
$$= \rho\left[3y^2 - \frac{1}{16}y^4 - \frac{1}{6}y^3\right]_0^4$$
$$= \rho\left[48 - 16 - \frac{64}{6}\right]$$
$$= \frac{64}{3}\rho$$
$$\therefore \bar{y} = \frac{64}{3}\rho \div M = \frac{64}{3} \times \frac{3}{44}$$
$$= \frac{16}{11}$$
$$= 1.5\ (2\text{ s.f.})$$

Using $M\bar{y} = \frac{\rho}{2}\int_0^4 (x_1 + x_2)(x_2 + x_1)\,dy$

$$= \frac{\rho}{2}\int_0^4 (x_2^2 - x_1^2)\,dy$$

$$= \frac{\rho}{2}\int_0^4 \left[\left(6 - \frac{1}{4}y^2\right)^2 - \frac{1}{4}y^2\right]dy$$

$$= \frac{\rho}{2}\int_0^4 36 - 3y^2 + \frac{1}{16}y^4 - \frac{1}{4}y^2\ \,dy$$

$$= \frac{\rho}{2}\int_0^4 36 - \frac{13}{4}y^2 + \frac{1}{16}y^4\ \,dy$$

$$= \frac{\rho}{2}\left[36y - \frac{13}{12}y^3 + \frac{1}{80}y^4\right]_0^4$$

$$= \frac{\rho}{2}\left[144 - \frac{13}{12}\times 64 + \frac{1024}{80}\right]$$

$$= \frac{\rho}{2}\left[87\frac{7}{15}\right]$$

$$\therefore\quad = \frac{1}{2}\times 87\frac{7}{15} \div \frac{44}{3}$$

$$= 2\frac{54}{55}$$

$$= 3.0\ (2\text{ s.f.})$$

The centre of mass is at the point with coordinates $\left(2\frac{54}{55}, 1\frac{5}{11}\right)$.

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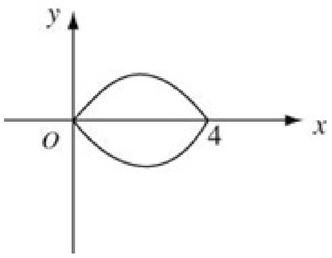
Statics of rigid bodies
Exercise B, Question 1

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $y = x^2 - 4x$ and the x-axis is rotated through $360°$ about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:



From symmetry the centre of mass lies an the x-axis.

As $y = x^2 - 4x$ meets the x axis when

$$x^2 - 4x = 0$$

i.e. $x(x-4) = 0$

$$\therefore x = 0 \text{ and } x = 4$$

Again from symmetry the centre of mass lies at $x = 2$.

The coordinates of the centre of mass are (2, 0).

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Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise B, Question 2

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $(x-1)^2+y^2=1$ is rotated through $180°$ about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:

The curve with equation
$(x-1)^2+y^2=1$

is a circle, centre (1, 0) radius 1.

It is rotated about the x-axis to form a sphere – centre (1, 0).

The centre of mass is at (1, 0).

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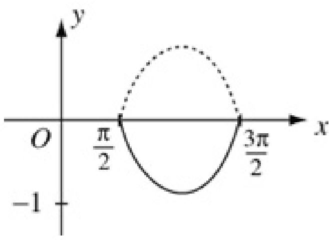
Statics of rigid bodies
Exercise B, Question 3

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.

The finite region bounded by the curve $y = \cos x$ $\frac{\pi}{2} \le x \le \frac{3x}{2}$, and the x-axis, is rotated through $360°$ about the x-axis to form a solid of revolution. Find the coordinates of its centre of mass.

Solution:



From symmetry the centre of mass of the solid of revolution is at the point with coordinates $(\pi, 0)$.

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Statics of rigid bodies
Exercise B, Question 4

Question:

Use symmetry to find the coordinates of the centre of mass of the solid.
The finite region bounded by the curve $y^2 + 6y = x$ and the y-axis, is rotated through $360°$ about the y-axis to form a solid of revolution. Find the coordinates of its centre of mass.

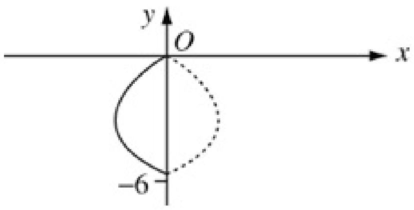
Solution:

The curve with equation $y^2 + 6y = x$ meets the x-axis at $x = 0$, and meets the y-axis when $y^2 + 6y = 0$

i.e. $y(y + 6) = 0$

$\therefore y = 0$ or -6

The curve is shown in the diagram, and is rotated about the y-axis through $360°$.



From symmetry the centre of mass lies at the point $(0, -3)$.

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Statics of rigid bodies
Exercise B, Question 5

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3x^2$, the line $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies on the x-axis, from symmetry.

Using the formula

$$\bar{x} = \frac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}, \text{ as } y = 3x^2$$

$$\bar{x} = \frac{\int_0^1 \pi \times 9x^5 \, dx}{\int_0^1 \pi \times 9x^4 \, dx}$$

$$= \frac{\pi \left[\frac{9}{6}x^6\right]_0^1}{\pi \left[\frac{9}{5}x^5\right]_0^1}$$

$$= \frac{9}{6} \div \frac{9}{5}$$

$$= \frac{5}{6}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{5}{6}, 0\right)$.

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Statics of rigid bodies
Exercise B, Question 6

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = \sqrt{x}$, the line $x = 4$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis, from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = \sqrt{x}$,

$$\bar{x} = \frac{\int_0^4 \pi x \times x \, dx}{\int_0^4 \pi x \, dx} = \frac{\int_0^4 \pi x^2 \, dx}{\int_0^4 \pi x \, dx}$$

then $$= \frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4}$$

$$= \frac{64}{3} \div 8$$

$$= \frac{8}{3} = 2\frac{2}{3}$$

$\therefore$ The centre of mass lies at the point $\left(2\frac{2}{3}, 0\right)$.

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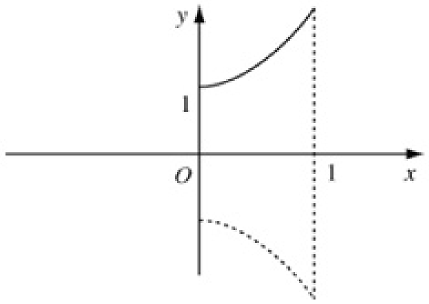
Statics of rigid bodies
Exercise B, Question 7

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3x^2 + 1$, the lines $x = 0$, $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:



The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, as $y = 3x^2 + 1$,

$$\bar{x} = \frac{\int_0^1 \pi (3x^2+1)^2 x \, dx}{\int_0^1 \pi (3x^2+1)^2 \, dx}$$

$$= \frac{\pi \int_0^1 9x^5 + 6x^3 + x \, dx}{\pi \int_0^1 9x^4 + 6x^2 + 1 \, dx}$$

$$= \frac{\left[\frac{9}{6}x^6 + \frac{6}{4}x^4 + \frac{1}{2}x^2\right]_0^1}{\left[\frac{9}{5}x^5 + \frac{6}{3}x^3 + x\right]_0^1}$$

$$= \frac{\frac{9}{6} + \frac{6}{4} + \frac{1}{2}}{\frac{9}{5} + \frac{6}{3} + 1} = \frac{3\frac{1}{2}}{4\frac{4}{5}}$$

$$= \frac{35}{48}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{35}{48}, 0\right)$.

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Statics of rigid bodies
Exercise B, Question 8

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y=\frac{3}{x}$, the lines $x=1$, $x=3$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis from symmetry.

Using the formula $\bar{x}=\frac{\int \pi y^2 x \,\mathrm{d}x}{\int \pi y^2 \,\mathrm{d}x}$, with $y=\frac{3}{x}$, then

$$\bar{x}=\frac{\int_1^3 \pi\left(\frac{9}{x^2}\right)x\,\mathrm{d}x}{\int_1^3 \pi\left(\frac{9}{x^2}\right)\mathrm{d}x}$$

$$=\frac{\pi\int_1^3 \frac{9}{x}\,\mathrm{d}x}{\pi\int_1^3 9x^{-2}\,\mathrm{d}x}$$

$$=\frac{\left[9\ln x\right]_1^3}{\left[-9x^{-1}\right]_1^3}$$

$$=\frac{9\ln 3}{9-3}$$

$$=\frac{3}{2}\ln 3.$$

$\therefore$ The centre of mass lies at the point $\left(\frac{3}{2}\ln 3, 0\right)=(1.65, 0)$ (3 s.f.)

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Statics of rigid bodies
Exercise B, Question 9

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 2e^x$, the lines $x = 0$, $x = 1$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies on the x-axis from symmetry.

Using the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = 2e^x$, then

$$\bar{x} = \frac{\int_0^1 \pi \times 4e^{2x} \times x \, dx}{\int_0^1 \pi \times 4e^{2x}}$$

$$= \frac{2\pi \left\{ \left[xe^{2x} \right]_0^1 - \int_0^1 e^{2x} \, dx \right\}}{2\pi \int_0^1 2e^{2x} \, dx}$$

$$= \frac{\left[xe^{2x} - \frac{1}{2} e^{2x} \right]_0^1}{\left[e^{2x} \right]_0^1}$$

$$= \frac{e^2 - \frac{1}{2} e^2 + \frac{1}{2}}{e^2 - 1}$$

$$= \frac{1}{2} \frac{(e^2 + 1)}{(e^2 - 1)}$$

$\therefore$ The centre of mass lies at the point $\left(\frac{1}{2} \frac{(e^2 + 1)}{(e^2 - 1)}, 0 \right)$.

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Statics of rigid bodies
Exercise B, Question 10

Question:

Use integration to the find the position of the centre of mass of the solid.
Find, by integration, the coordinates of the centre of mass of the solid formed when the finite region bounded by the curve $y = 3e^{-x}$, the lines $x = 0, x = 2$ and the x-axis is rotated through $360°$ about the x-axis.

Solution:

The centre of mass lies an the x-axis from symmetry.

Use the formula $\bar{x} = \dfrac{\int \pi y^2 x \, dx}{\int \pi y^2 \, dx}$, with $y = 3e^{-x}$.

$$\text{Then } \bar{x} = \frac{\int_0^2 \pi \times 9e^{-2x} x \, dx}{\int_0^2 \pi \times 9e^{-2x} \, dx}$$

$$= \frac{9\pi \int_0^2 xe^{-2x} \, dx}{9\pi \int_0^2 e^{-2x} \, dx}$$

$$= \frac{\left[-\frac{1}{2}xe^{-2x}\right] + \int_0^2 \frac{1}{2}e^{-2x} \, dx}{\left[-\frac{1}{2}e^{-2x}\right]_0^2}$$

$$= \frac{\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^2}{-\frac{1}{2}e^{-4} + \frac{1}{2}}$$

$$= \frac{\left[-e^{-4} - \frac{1}{4}e^{-4}\right] + \frac{1}{4}}{-\frac{1}{2}e^{-4} + \frac{1}{2}}$$

$$= \frac{1 - 5e^{-4}}{2(1 - e^{-4})} \quad \text{or} \quad \frac{e^4 - 5}{2(e^4 - 1)} = 0.46 \text{ (2 s.f.)}$$

$\therefore$ The centre of mass lies at the point with coordinates $\left(\dfrac{e^4 - 5}{2(e^4 - 1)}, 0\right)$.

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Statics of rigid bodies

Exercise B, Question 11

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A uniform solid right circular cone of height 10 cm and base radius 5 cm is joined at its base to the base of a uniform solid hemisphere. The centres of their bases coincide and their axes are collinear. The radius of the hemisphere is also 5 cm. Find the position of the centre of mass of the composite body,

a when both the cone and the hemisphere have the same density,

b when the hemisphere has density twice that of the cone.

Solution:

a Let O be the point at the centre of the plane circular face of the cone and of the hemisphere. Let positive displacement be towards the vertex of the cone

Shape	Mass	Units of mass	Distance from O to centre of mass
Cone	$\frac{1}{3}\pi\rho \times 25 \times 10$	1	$\frac{10}{4}$
Hemisphere	$\frac{2}{3}\pi\rho \times 5^3$	1	$-\frac{15}{8}$
Composite body	$\frac{250}{3}\pi\rho + \frac{250}{3}\pi\rho$	2	$\bar{x}$

Take moments about O:

$$1 \times \frac{10}{4} + 1 \times \left(\frac{-15}{8}\right) = 2\bar{x}$$

$$\therefore \bar{x} = \frac{5}{16}$$

The centre of mass lies an the axis of symmetry at a point $\frac{5}{16}$ cm from O towards the vertex of the cone.

b If the hemisphere has twice the density of the cone then the ratio of the masses becomes cone 1, hemisphere 2, composite body 3 so the moments equation becomes

$$1 \times \frac{10}{4} + 2 \times \frac{-15}{8} = 3\bar{x}$$

$$\therefore \bar{x} = \frac{-15}{12}$$

The centre of mass lies on the axis of symmetry at a point $\frac{5}{12}$ cm from O towards the rim of the hemisphere.

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Edexcel AS and A Level Modular Mathematics

Statics of rigid bodies
Exercise B, Question 12

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A solid is composed of a uniform solid right circular cylinder of height 10 cm and base radius 6 cm joined at its top plane face to the base of a uniform hemisphere of the same radius. The centres of their adjoining circular faces coincide at point *O* and their axes are collinear. The radius of the hemisphere is also 6 cm. Find the position of the centre of mass of the composite body,

a if the cylinder and hemisphere are of the same density,

b if the hemisphere has three times the density of the cylinder.

Solution:

Let ρ be the mass per unit volume of the cylinder.

a The centre of mass lies an the axis of symmetry.

Shape	Mass	Units of mass	Distance of centre of mass from O (+ve towards top of hemisphere)
Cylinder	$\rho\pi\times36\times10$	5	-5
Hemisphere	$\rho\frac{2}{3}\pi\times6^3$	2	$\frac{3}{8}\times6$
Composite body	$360\pi\rho+144\pi\rho$	7	$\bar{x}$

Take moments about O.

$$5\times-5+2\times\frac{18}{8}=7\bar{x}$$

$$\text{i.e. } 9-50=14\bar{x}$$

$$\therefore \bar{x}=-\frac{41}{14}$$

$\therefore$ Centre of mass is an axis of symmetry at a distance $2\frac{13}{14}$ cm away from base of hemisphere.

b Redraw the table noting that the masses of the hemisphere and of the composite body have changed.

Shape	Mass	Units of mass	Distance of centre of mass from O (+ve as before)
Cylinder	$\rho\pi\times36\times10$	5	-5
Hemisphere	$3\rho\times\frac{2}{3}\pi\times6^3$	6	$\frac{3}{8}\times6$
Composite body	$360\pi\rho+432\pi\rho$	11	$\bar{x}$

Take moments about O:

$$-25+6\times\frac{18}{8}=11\bar{x}$$

$$\therefore \bar{x}=\frac{-23}{22}$$

$\therefore$ Centre of mass is on the axis of symmetry at a distance $1\frac{1}{22}$ cm away from the base of the hemisphere.

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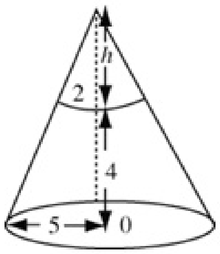
Statics of rigid bodies
Exercise B, Question 13

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
Find the position of the centre of mass of the frustum of a right circular uniform solid cone, where the frustum has end radii 2 cm and 5 cm, and has height 4 cm.

Solution:

A frustum of a cone is obtained by removing a small cone from a large cone.
Draw a diagram, showing the cones and the frustum and let the height of the small cone be h cm.



Using similar triangles:

$$\frac{h}{h+4} = \frac{2}{5}$$

$$\therefore 5h = 2(h+4)$$

$$\therefore 3h = 8$$

$$\text{i.e. } h = \frac{8}{3}$$

The centre of mass lies an the axis of symmetry. Let the centre of the base of the large cone be O.

Shape	Mass	Units of mass	Distance of centre of mass from O
Large cone	$\rho\times\frac{1}{3}\pi\times5^2\times\frac{20}{3}$	500	$\frac{5}{3}$
Small cone	$\rho\times\frac{1}{3}\pi\times2^2\times\frac{8}{3}$	32	$4+\frac{2}{3}$
Frustum	$\frac{500\pi}{9}\rho-\frac{32\pi}{9}\rho$	468	$\bar{x}$

Take moments about O

$$500\times\frac{5}{3}-32\times\left(\frac{14}{3}\right)=468\bar{x}$$

$$\therefore \frac{2052}{3}=468\bar{x}$$

$$\therefore \bar{x}=1\frac{6}{13} \text{ or } 1.46 \quad (3\text{ s.f.})$$

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Statics of rigid bodies
Exercise B, Question 14

Question:

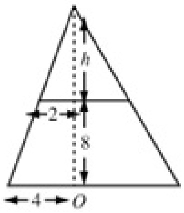
You may quote results for the centres of mass of cones and hemispheres obtained earlier.

a Find the position of the centre of mass of the frustum of a right circular uniform solid cone, where the frustum has end radii 2 cm and 4 cm, and has height 8 cm.

b A cylindrical hole of radius 1 cm with the same axis as that of the frustum is now drilled through the frustum. Find the distance of the new centre of mass from the larger face of the frustum.

Solution:

a



From similar triangles:

$$\frac{h}{h+8} = \frac{2}{4}$$

$$\therefore 4h = 2(h+8)$$

$$\therefore 2h = 16$$

i.e. $h = 8$

Shape	Mass	Units of mass	Distance of centre of mass from O
Large cone	$\rho \times \frac{1}{3}\pi \times 4^2 \times 16$	8	4
Small cone	$\rho \times \frac{1}{3}\pi \times 2^2 \times 8$	1	$8+2=10$
Frustum	$\frac{256\pi}{3}\rho - \frac{32\pi}{3}\rho$	7	$\bar{x}$

Take moments about O:

$$8 \times 4 - 1 \times 10 = 7\bar{x}$$

$$\therefore \bar{x} = \frac{22}{7} = 3.14 \text{ (3 s.f.)}$$

Required distance is 3.14 cm.

b

Shape	Mass	Units of mass	Distance of centre of mass from O
Frustum	$\frac{224\pi}{3}\rho$	28	$\frac{22}{7}$
Cylindrical hole	$\pi \times 1^2 \times 8\rho$	3	4
Remainder	$\frac{200\pi}{3}\rho$	25	$\bar{x}$

Take moments about O:

$$28 \times \frac{22}{7} - 3 \times 4 = 25\bar{x}$$

i.e. $88 - 12 = 25\bar{x}$

$$\therefore \bar{x} = \frac{76}{25} = 3.04$$

Required distance is 3.04 cm.

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Statics of rigid bodies
Exercise B, Question 15

Question:

You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A thin uniform hemispherical shell has a circular base of the same material. Find the position of the centre of mass above the base in terms of its radius r.

Solution:

Let the density of the material be ρ and its thickness be t.
Then the mass will be proportional to the surface area.
The centre of mass will be an the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass above base
Hemispherical shell	$\rho t \times 2\pi r^2$	2	$\frac{r}{2}$
Circular base	$\rho t \times \pi r^2$	1	0
Composite body	$\rho t \times 3\pi r^2$	3	$\bar{x}$

Take moments about centre of base:

$$2 \times \frac{r}{2} + 0 = 3\bar{x}$$

$$\therefore \bar{x} = \frac{1}{3}r$$

So the centre of mass is at a distance $\frac{r}{3}$ above the base.

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Statics of rigid bodies
Exercise B, Question 16

Question:

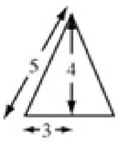
You may quote results for the centres of mass of cones and hemispheres obtained earlier.
A thin uniform hollow cone has a circular base of the same material. Find the position of the centre of mass above the base, given that the radius of the cone is 3 cm and its height is 4 cm.

Solution:

Let the density of the material be ρ and its thickness be t.
Then the mass will be proportional to the surface area.
The centre of mass will be on the axis of symmetry

Shape	Mass	Units of mass	Distance of centre of mass above base
Hollow cone	$\rho t \times \pi \times 3 \times 5$	15	$\frac{1}{3} \times 4$
Circular base	$\rho t \times \pi \times 3^2$	9	0
Composite body	$15\pi\rho t + 9\pi\rho t$	24	$\bar{x}$

[The surface area of a cone is given by the formula πrl where l is the length of the slant side. As $r = 3$ and $h = 4$ then $l = 5$ from Pythagoras-Theorem.]



Take moments about centre of base:

$$\therefore 15 \times \frac{4}{3} + 0 = 24\bar{x}$$

$$\therefore \bar{x} = \frac{20}{24} = \frac{5}{6}$$

So the centre of mass is at a distance $\frac{5}{6}$ cm above the base.

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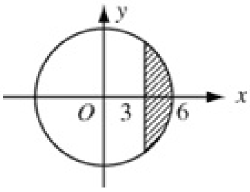
Statics of rigid bodies
Exercise B, Question 17

Question:

Use calculus to obtain your answer.
A cap of height 3 cm is cut from a uniform solid sphere of radius 6 cm. Using calculus, find the position of the centre of mass of the cap, giving the distance from the plane circular surface.

Solution:

This cap is the solid of revolution obtained when the arc of the circle with equation $x^2 + y^2 = 6^2, 3 \le x \le 6$, is rotated through $180°$ about the x-axis.



The centre of mass lies an the x-axis, from symmetry.

Use $$\bar{x} = \frac{\int_3^6 \pi y^2 x \, dx}{\int_3^6 \pi y^2 \, dx} = \frac{\int_3^6 (36 - x^2) x \, dx}{\int_3^6 (36 - x^2) \, dx}$$

$$= \frac{\int_3^6 36x - x^3 \, dx}{\int_3^6 36 - x^2 \, dx}$$

$$= \frac{\left[18x^2 - \frac{1}{4}x^4\right]_3^6}{\left[36x - \frac{1}{3}x^3\right]_3^6}$$

$$= \frac{\left(18 \times 6^2 - \frac{1}{4} \times 6^4\right) - \left(18 \times 3^2 - \frac{1}{4} \times 3^4\right)}{\left(36 \times 6 - \frac{1}{3} \times 6^3\right) - \left(36 \times 3 - \frac{1}{3} \times 3^3\right)}$$

$$= \frac{324 - 141.75}{144 - 99} = \frac{182.25}{45} = 4.05$$

$\therefore$ Distance from $x = 3$ is 1.05 cm.
$\therefore$ The distance of the centre of mass from the plane circular face is 1.05 cm.

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Statics of rigid bodies
Exercise B, Question 18

Question:

Use calculus to obtain your answer.
Show that the centre of mass of a cap of height h of a sphere of radius a is on its axis of symmetry at a distance $\frac{h(4a-h)}{4(3a-h)}$ from the circular base of the cap.

Solution:

The arc of the circle $x^2+y^2=a^2, a-h \le x \le a$ is rotated about the x-axis.

$$\bar{x}=\frac{\pi\int_{a-h}^{a}(a^2-x^2)x\,\mathrm{d}x}{\pi\int_{a-h}^{a}(a^2-x^2)\,\mathrm{d}x} = \frac{\left[\frac{1}{2}a^2x^2-\frac{1}{4}x^4\right]_{a-h}^{a}}{\left[a^2x-\frac{1}{3}x^3\right]_{a-h}^{a}}$$

$$=\frac{\frac{1}{4}a^4-\frac{1}{2}a^2(a-h)^2+\frac{1}{4}(a-h)^4}{\frac{2}{3}a^3-a^2(a-h)+\frac{1}{3}(a-h)^3}$$

$$=\frac{\frac{1}{4}(a^2-(a-h)^2)^2}{\frac{1}{3}(2a^2-3a^2(a-h)+(a-h)^3)}$$

$$=\frac{3}{4}\frac{(2ah-h^2)^2}{3h^2a-h^3}$$

$$=\frac{3}{4}\frac{(2a-h)^2}{(3a-h)}$$

$\therefore$ Distance of centre of mass from base of cap (i.e. $x=a-h$) is

$$\frac{3(2a-h)^2}{4(3a-h)}-(a-h)=\frac{3(2a-h)^2-4(3a-h)(a-h)}{4(3a-h)}$$

$$=\frac{4ah-h^2}{4(3a-h)}$$

i.e. required distance is $\frac{h(4a-h)}{4(3a-h)}$

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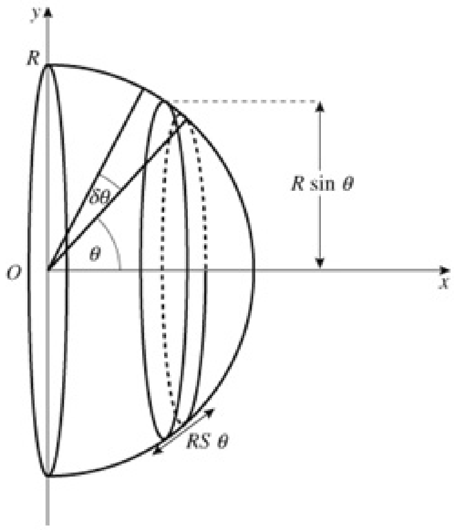
Statics of rigid bodies
Exercise B, Question 19

Question:

Use calculus to obtain your answer.
Using calculus, find the centre of mass of the uniform hemispherical shell with radius R.

Hint: Divide the shell into small elemental cylindrical rings, centred on the x-axis, with radius $R\sin\theta$, and height $R\delta\theta$, where θ is the angle between the radius R and the x-axis.



Solution:

The curved surface area of the elemental disc is $2\pi \ NPR \ \delta\theta = 2\pi R \sin\theta \times \delta\theta$

$\therefore$ its mass is $2\pi\rho R^2 \sin\theta\delta\theta$, where ρ is the mass per unit area.

$\therefore$ Since $\sum m_i \times \bar{x} = \sum m_i x_i$, where $x_i = R\cos\theta$

and $m_i = 2\pi \ \rho R^2 \sin\theta \ \delta\theta$ then

$$\bar{x} = \frac{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2 \sin\theta \times R\cos\theta \ \mathrm{d}\theta}{\int_0^{\frac{\pi}{2}} 2\pi\rho R^2 \sin\theta \ \mathrm{d}\theta}$$

$$= \frac{R\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \ \mathrm{d}\theta}{\int_0^{\frac{\pi}{2}} \sin\theta \ \mathrm{d}\theta}$$

$$= \frac{R \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta \ \mathrm{d}\theta}{\left[-\cos\theta\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{1}{4}R\frac{\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}}}{1}$$

$$= \frac{1}{4}R\left[1-(-1)\right]$$

$$= \frac{R}{2}$$

This proof is **not** expected to be known for the examination.

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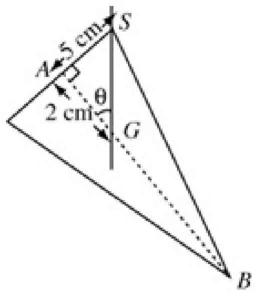
Statics of rigid bodies
Exercise C, Question 1

Question:

A uniform solid right circular cone is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 5 cm and the height of the cone is 8 cm, find the angle between the vertical and the axis of the cone when it is in equilibrium.

Solution:



The diagram shows the equilibrium position with the centre of mass G, vertically below the point of suspension S.

As $AG = \frac{1}{4}h$ for a cone

$\therefore AG = 2$ cm

Also the radius $AS = 5$ cm .

Let the angle between the vertical and the axis be θ.

Then from $\Delta ASG, \tan\theta = \frac{5}{2}$

$\therefore \theta = 68°$ (to the nearest degree)

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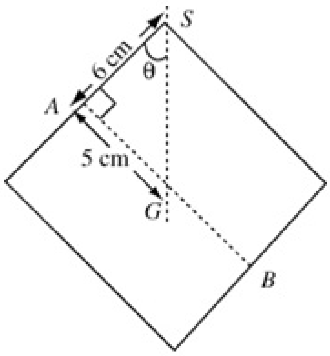
Statics of rigid bodies
Exercise C, Question 2

Question:

A uniform solid right circular cylinder is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 6 cm and the height of the cylinder is 10 cm, find the angle between the vertical and the circular base of the cylinder when it is in equilibrium.

Solution:



The diagram shows the equilibrium position with the centre of mass G below the point of suspension S.

As $AG = \frac{1}{2}h$ for a uniform cylinder

$\therefore AG = 5$ cm

Also the radius $AS = 6$ cm .

The angle between the vertical and the circular base of the cylinder is θ.

From $\Delta ASG, \tan\theta = \frac{5}{6}$

$\therefore \theta = 40^{\circ}$ (to the nearest degree)

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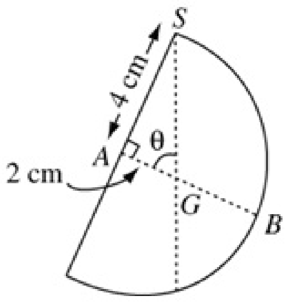
Statics of rigid bodies
Exercise C, Question 3

Question:

A uniform hemispherical shell is suspended by a string attached to a point on the rim of its base. Given that the radius of the base is 4 cm, find the angle between the vertical and the axis of the hemisphere when it is in equilibrium.

Solution:



The diagram shows the equilibrium position, with the centre of mass G below the point of suspension S.

As $AG = \frac{1}{2}r$ for an hemispherical shell

$\therefore AG = 2$ cm

Also the radius $AS = 4$ cm

Let the angle between the vertical and the axis be θ.

Then from $\Delta ASG, \tan\theta = \frac{4}{2}$

$\therefore \theta = 63°$ (nearest degree)

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Statics of rigid bodies
Exercise C, Question 4

Question:

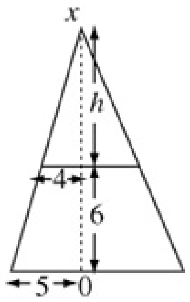
a Find the position of the centre of mass of the frustum of a right circular uniform solid cone, of end radii 4 cm and 5 cm and of height 6 cm. (Give your answer to 3 s.f.)

This frustum is now suspended by a string attached to a point on the rim of its smaller circular face.

b Find the angle between the vertical and the axis of the frustum when it is in equilibrium. (Give your answer to the nearest degree.)

Solution:

a



From similar triangles

$$\frac{h}{h+6} = \frac{4}{5}$$

$$\therefore 5h = 4h + 24$$

$$\text{i.e. } h = 24$$

Centre of mass lies at the axis of symmetry OX.

Shape	Mass	Mass ratios	Position of centre of mass i.e. distance from O
Large cone	$\frac{1}{3}\pi\rho \times 5^2 \times 30$	125	$\frac{30}{4} = 7.5$
Small cone	$\frac{1}{3}\pi\rho \times 4^2 \times 24$	64	$6 + \frac{24}{4} = 12$
Frustum	$\frac{250\pi}{3}\rho - 128\pi\rho$	61	$\bar{x}$

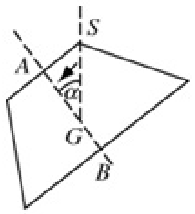
Take moments about O

$$125 \times 7.5 - 64 \times 12 = 61\bar{x}$$

$$\therefore 169.5 = 61\bar{x}$$

$$\therefore \bar{x} = 2.78 \text{ (3 s.f.) (or } \frac{339}{122})$$

b



In equilibrium the centre of mass G lies vertically below the point of suspension S.
Let the required angle be α.
AS is smaller radius = 4 cm

$$AG = 6 - 2.78 = 3.22 \text{ cm (3 s.f.)}$$

$$\tan\alpha = \frac{AS}{AG} = \frac{4}{3.22}$$

$$\therefore \alpha = 51° \text{ (to the nearest degree)}$$

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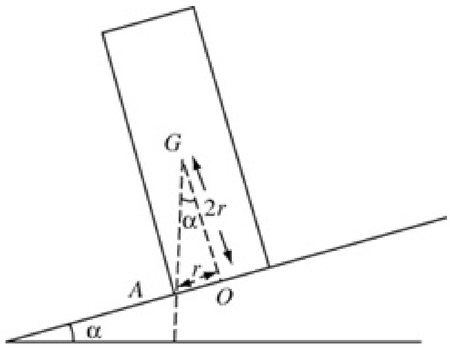
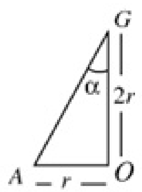
Statics of rigid bodies
Exercise C, Question 5

Question:

A uniform solid cylinder of radius r and height $4r$ rests in equilibrium with its base in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal. Show that equilibrium is maintained provided that $\tan\alpha \le k$ and find the value of k.

Solution:





The diagram shows the limiting case when the point vertically below the centre of mass G is on the edge of the area of contact.
In this position angle AGO is also equal to α, the angle of inclination of the plane to the horizontal.

From $\Delta AGO, \tan\alpha = \frac{r}{2r} = \frac{1}{2}$
$(\therefore \alpha = 26.6\ (3\text{ s.f.}))$

For any larger angle tilting will occur.

$\therefore$ Equilibrium is maintained provided
$\tan\alpha \le \frac{1}{2}$ $\left(\text{i.e. } k = \frac{1}{2}\right)$

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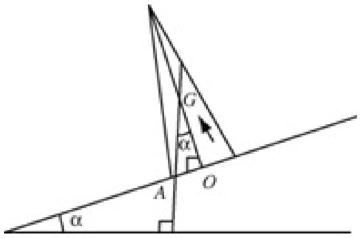
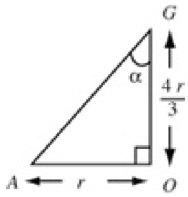
Statics of rigid bodies
Exercise C, Question 6

Question:

A uniform hollow cone of radius r and height $4r$ rests in equilibrium with its base in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal. Show that equilibrium is maintained provided that $\tan\alpha \le k$ and find the value of k.

Solution:





The diagram shows the limiting case where A, a point on the circumference of the circular base of the cone, is vertically below G – the centre of mass.

$A\hat{G}O = \alpha$

$AO = r$

$OG = \frac{1}{3} \times 4r = \frac{4r}{3}$

From $\Delta AGO, \tan\alpha = \frac{r}{\frac{4r}{3}} = \frac{3}{4} \quad (\alpha \approx 37°)$

For any larger angle tilting will occur.

Equilibrium is maintained provided

$\tan\alpha \le \frac{3}{4} \quad$ (i.e. $k = \frac{3}{4}$).

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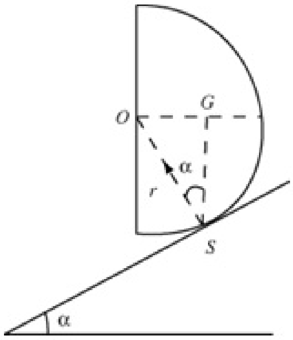
Statics of rigid bodies
Exercise C, Question 7

Question:

A uniform solid hemisphere of radius r rests in equilibrium with its curved surface in contact with a rough inclined plane, which is sufficiently rough to prevent sliding. The plane is inclined at an angle α to the horizontal, and the plane face of the hemisphere is in a vertical position. Find the value of α, giving your answer to the nearest degree.

Solution:



The diagram shows the centre of mass G above the point of contact S.

$G\hat{S}O = \alpha$

$SO = r$

and $OG = \frac{3}{8}r$.

From $\Delta GOS, \sin\alpha = \frac{GO}{OS} = \frac{\frac{3}{8}r}{r} = \frac{3}{8}$

$\therefore \alpha = 22°$ (nearest degree)

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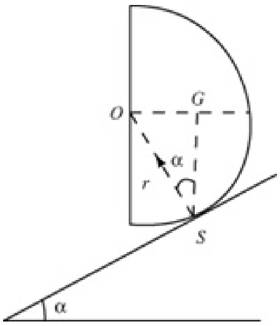
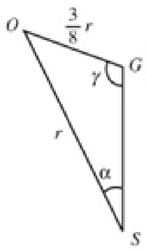
Statics of rigid bodies
Exercise C, Question 8

Question:

A solid uniform hemisphere rests in equilibrium with its curved surface in contact with a rough plane inclined at α to the horizontal where $\sin\alpha = \frac{3}{16}$.
Find the inclination of the axis of symmetry of the hemisphere to the vertical.

Solution:





You need a good diagram and use of the sine rule.

The diagram shows the equilibrium position with the centre of mass G vertically above the point of contact with the plane S. O is the centre of the plane face of the hemisphere.

Let the obtuse angle between the axis of symmetry and the vertical be γ. Let the radius of the hemisphere be r.

In ΔOGS, $O\hat{S}G = \alpha$

$OS = r$

and $OG = \frac{3}{8}r$

Using the sine rule

$$\frac{\sin\gamma}{r} = \frac{\sin\alpha}{\frac{3}{8}r} \quad \text{But } \sin\alpha = \frac{3}{16}$$

$$\therefore \sin\gamma = \frac{\frac{3}{16}r}{\frac{3}{8}r} = \frac{1}{2}$$

$\therefore \gamma = 150°$ and the acute angle between the axis of the hemisphere and the vertical is $30°$.

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Statics of rigid bodies
Exercise C, Question 9

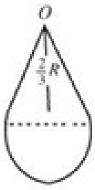
Question:

A solid object is made up of a right circular uniform solid cone joined to a uniform solid hemisphere so that the base of the cone coincides with the plane surface of the hemisphere. Their common radius is r and the height of the cone is $\frac{2}{3}r$.

a Find the position of the centre of mass of the composite object giving the distance of this centre of mass from the vertex of the cone.

b Show that the object will remain in equilibrium on a smooth horizontal plane, if it is placed with a curved surface of the cone in contact with the plane.

Solution:

a 

Let the vertex of the cone be O and let the mass per unit volume be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cone	$\frac{1}{3}\pi\rho r^2 \times \frac{2}{3}r$	2	$\frac{3}{4}\times\frac{2}{3}r = \frac{1}{2}r$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	6	$\frac{3}{8}r + \frac{2}{3}r = \frac{25}{24}r$
Composite body	$\frac{8}{9}\pi\rho r^3$	8	$\bar{x}$

Take moments about O:

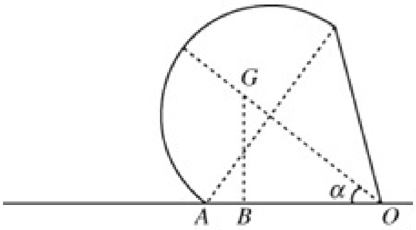
$$2\times\frac{1}{2}r + 6\times\frac{25}{24}r = 8\bar{x}$$

$$\therefore 8\bar{x} = r + \frac{25r}{4}$$

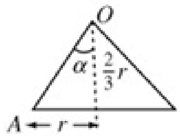
$$8\bar{x} = \frac{29r}{4}$$

$$\therefore \bar{x} = \frac{29r}{32}$$

b



α is the angle between the axis of symmetry and the slant side of the cone



For equilibrium, the point vertically below G, i.e. point B, must lie between O and A, where A is a point on the common face of the cone and hemisphere.

OA = slant side of cone $= \sqrt{r^2 + \left(\frac{2}{3}r\right)^2}$ (from Pythagoras)

$$= \frac{r}{3}\sqrt{9+4} = \frac{r\sqrt{13}}{3}$$

$$OB = OG\cos\alpha$$

$$= \frac{29r}{32} \times \frac{2}{\sqrt{13}}$$

$$= \frac{29r\sqrt{13}}{16 \times 13}$$

$$= \frac{29r\sqrt{13}}{208}$$

As $\frac{1}{3} > \frac{29}{208} \therefore OA > OB$ and equilibrium is maintained.

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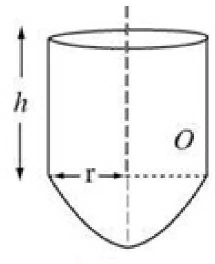
Statics of rigid bodies
Exercise C, Question 10

Question:

A uniform solid consists of a hemisphere of radius r and a right circular uniform cylinder of base radius r and height h fixed together so that their circular faces coincide. The solid can rest in equilibrium with any point of the curved surface of the hemisphere in contact with a horizontal plane. Find h in terms of r.

Solution:



Shape	Mass	Distance from O of centre of mass
Cylinder	$\pi\rho r^2 h$	$\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$-\frac{3}{8}r$
Composite body	$\pi\rho r^2\left(h+\frac{2}{3}r\right)$	$\bar{x}$

As the solid can rest in equilibrium with any point of the curved surface in contact with the horizontal plane, the centre of mass must be at the centre of the plane face of the hemisphere.

i.e. $\bar{x}=0$

$\therefore$ Taking moments about O

$$\rho\pi r^2 h\cdot\frac{h}{2}-\frac{2}{3}\pi\rho r^3\cdot\frac{3}{8}r=0$$

$$\frac{r^2h^2}{2}=\frac{1}{4}r^4$$

$$\therefore r^2=2h^2$$

$$\text{i.e. } h=\frac{r}{\sqrt{2}} \text{ or } \frac{r\sqrt{2}}{2}$$

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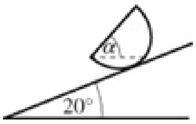
Statics of rigid bodies
Exercise C, Question 11

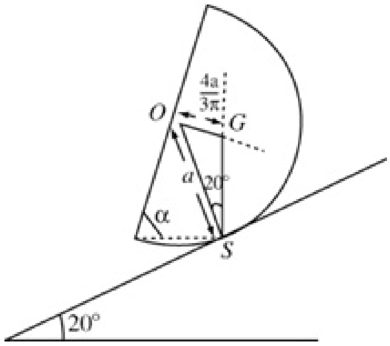
Question:

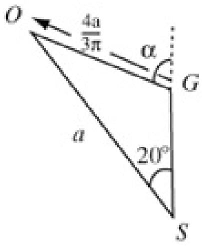
You may assume that the centre of mass of a uniform semi circular lamina of radius a is at a distance $\frac{4a}{3\pi}$ from the centre.



A uniform solid right circular cylinder is cut in half through its axis to form two prisms of semi-circular cross section. One of these is placed with its curved surface in contact with a rough inclined plane as shown in the figure. The inclined plane makes an angle of $20°$ with the horizontal. Show that when the prism is in equilibrium, its rectangular plane face makes an angle α with the horizontal, where α is approximately $54°$.

Solution:





Draw a clear diagram showing equilibrium with the centre of mass G above the point of contact S.

Identify the lengths of the sides and the angles in ΔOGS.

Use the sine rule in ΔOGS:

Then $\dfrac{\sin(180-\alpha)}{a} = \dfrac{\sin 20^\circ}{\frac{4a}{3\pi}}$

$$\frac{\sin(180-\alpha)}{a} = \frac{\sin 20^\circ}{\frac{4a}{3\pi}}$$

$$\therefore \sin\alpha = \frac{3\pi}{4}\times\sin 20$$

$$= 0.806$$

$$\therefore \alpha = 53.6^\circ \text{ (3 s.f.)}$$

$$\therefore \alpha \approx 54^\circ$$

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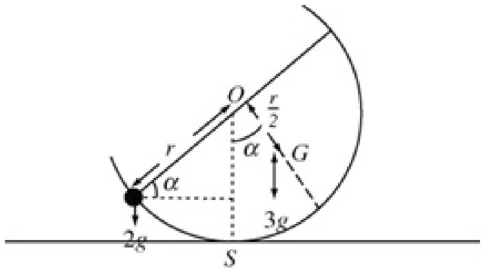
Statics of rigid bodies
Exercise C, Question 12

Question:

A hemispherical bowl, which may be modelled as a uniform hemispherical shell, has mass 3 kg. A weight of 2 kg is placed on the rim and the bowl rests in equilibrium on a smooth horizontal plane. The plane surface of the bowl makes an angle α with the horizontal. Show that $\tan\alpha = \frac{4}{3}$.

Solution:



Let the radius of the bowl be r.

The distance $OG = \frac{r}{2}$.

Take moments about point S – the point of contact with the plane:

$$3g \times \frac{r}{2}\sin\alpha = 2g \times r\cos\alpha$$

$$\therefore 3\sin\alpha = 4\cos\alpha$$

$$\therefore \tan\alpha = \frac{4}{3}, \text{ as required}$$

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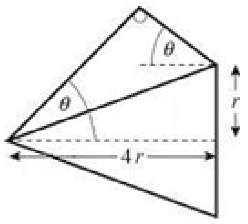
Statics of rigid bodies
Exercise C, Question 13

Question:

A uniform solid right circular cone has base radius r, height $4r$ and mass m. One end of a light inextensible string is attached to the vertex of the cone and the other end is attached to a point on the rim of the base. The string passes over a smooth peg and the cone rests in equilibrium with the axis horizontal, and with the strings equally inclined to the horizontal at an angle θ, as shown in the figure.

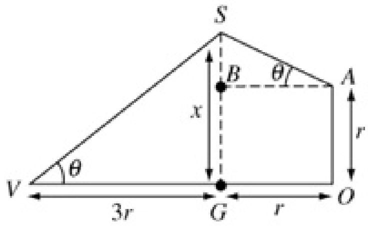


a Show that angle θ satisfies the equation $\tan\theta = \frac{1}{2}$.

b Find the tension in the string, giving your answer as an exact multiple of mg.

Solution:

a



In equilibrium the centre of mass G lies below the point of suspension S. Let distance $SG = x$.
O is the centre of the base of the cone and V is its vertex.
A and B are shown on the diagram.

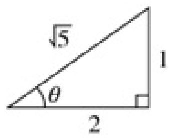
$\tan\theta = \frac{x}{3r}$ (from ΔVSG)

Also $\tan\theta = \frac{x-r}{r}$ (from ΔABS)

$$\therefore \frac{x}{3r} = \frac{x-r}{r}$$
$$\therefore x = 3x - 3r$$
$$\therefore 2x = 3r$$
$$\therefore x = \frac{3r}{2}$$
$$\therefore \tan\theta = \frac{1}{2}$$

b Resolve vertically for the forces acting an the cone:

$$2T\sin\theta = mg$$
$$\therefore T = \frac{mg}{2\sin\theta}$$



As $\tan\theta = \frac{1}{2}$, $\sin\theta = \frac{1}{\sqrt{5}}$ (from Pythagoras)

$$\therefore T = \frac{\sqrt{5}\,mg}{2}$$

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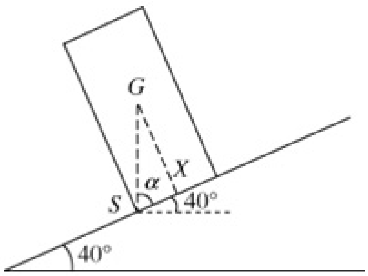
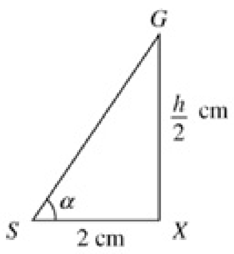
Statics of rigid bodies
Exercise D, Question 1

Question:

A uniform solid right circular cylinder with base diameter 4 cm stands on a rough plane inclined at $40°$ to the horizontal. What is the maximum height that such a cylinder can have without toppling over?

Solution:





Let the maximum height be h cm. The cylinder is about to topple and so its centre of mass G is directly above the point S on the circumference of the base. X is the mid-point of the base.

As $\alpha + 40° = 90°, \alpha = 50°$.

In $\Delta GSX, SX = 2$ cm (radius)

$$GX = \frac{h}{2} \quad \text{(position of centre of mass)}$$

$$\therefore \tan 50° = \frac{\frac{h}{2}}{2}$$

$$\therefore h = 4 \tan 50°$$

$$\therefore h = 4.77 \text{ cm (3 s.f.)}$$

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Statics of rigid bodies
Exercise D, Question 2

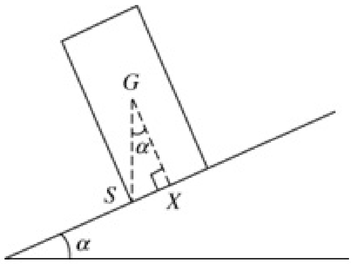
Question:

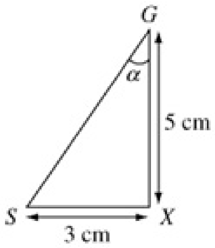
A uniform solid right circular cylinder with base radius 3 cm and height 10 cm is placed with its circular plane base on a rough plane. The plane is gradually tilted.

a Find the angle which the plane makes with the horizontal if the cylinder topples over before it slides.

b What can you deduce about the value of the coefficient of friction?

Solution:





a When the cylinder is about to topple, G is vertically above point S. X is the mid-point of the base.

Let α be the angle which the plane makes with the horizontal.

In triangle GSX, $S\hat{G}X = \alpha$

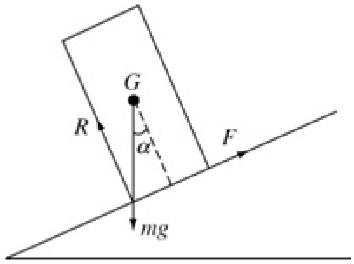
$GX = \frac{1}{2} \times 10 \text{ cm} = 5 \text{ cm}$ (position of centre of mass)

$SX = 3 \text{ cm}$ (radius)

$\therefore \tan\alpha = \frac{3}{5}$

i.e. $\alpha = 31°$ (to the nearest degree)

b



R(↗)

$F - mg\sin\alpha = 0$

$\therefore F = Mg\sin\alpha$

R(↖)

$R - Mg\cos\alpha = 0$

$\therefore R = Mg\cos\alpha$

As $F \le \mu R, Mg\sin\alpha \le Mg\cos\alpha \times \mu$

$\therefore \mu \ge \tan\alpha$

i.e. $\mu \ge \frac{3}{5}$

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Statics of rigid bodies
Exercise D, Question 3

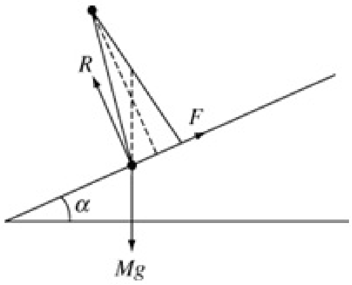
Question:

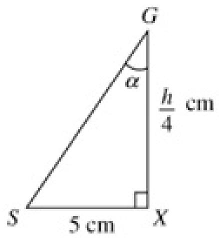
A uniform solid right circular cone with base radius 5 cm and height h cm is placed with its circular plane base on a rough plane. The coefficient of friction is $\frac{\sqrt{3}}{3}$. The plane is gradually tilted.

a Find the angle which the plane makes with the horizontal if the cone is about to slide and topple at the same time.

b Calculate the value of the height of the cone, h cm.

Solution:





a When the cone is about to slide $F = \mu R$

$$\text{i.e.} F = \frac{\sqrt{3}}{3} R \qquad ①$$

R(↗)
Then $F - mg\sin\alpha = 0 \quad \therefore F = Mg\sin\alpha$ ②
R(↖)
Then $R - Mg\cos\alpha = 0 \quad \therefore R = Mg\cos\alpha$ ③

Substituting F and R into equation ①

Then $Mg\sin\alpha = \frac{\sqrt{3}}{3} Mg\cos\alpha$

$$\therefore \tan\alpha = \frac{\sqrt{3}}{3}$$
$$\therefore \alpha = 30°$$

b From ΔGSX, where G is the centre of mass of the cone, X the centre of its base and S a point an the circumference of the base about which topping is about to occur:

$$\tan\alpha = \frac{5}{\frac{h}{4}} = \frac{20}{h}$$

$$\therefore h = \frac{20}{\tan\alpha} = 20 \div \frac{\sqrt{3}}{3} = 20\sqrt{3} = 35 \text{ cm (2 s.f.)}$$

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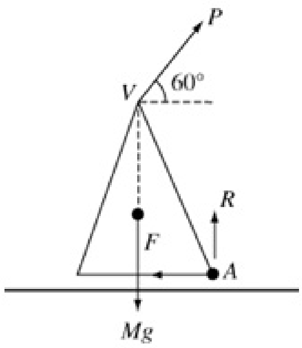
Statics of rigid bodies
Exercise D, Question 4

Question:

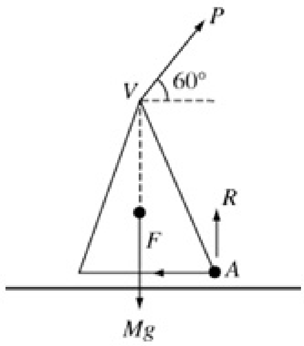
A uniform solid right circular cone of mass M with base radius r and height $2r$ is placed with its circular plane base on a rough horizontal plane. A force P is applied to the vertex V of the cone at an angle of $60°$ above the horizontal as shown in the figure. The cone begins to topple and to slide at the same time.

a Find the magnitude of the force P in terms of M.

b Calculate the value of the coefficient of friction.



Solution:



Let the point about which toppling occurs be A.

Take moments about point A.

When toppling is about to occur, R and F act through point A.

So $P\cos 60 \times 2r + P\sin 60 \times r = Mg \times r$

$$\therefore Pr + \frac{P\sqrt{3}}{2}r = Mgr$$

$$\therefore P\left(1 + \frac{\sqrt{3}}{2}\right) = Mg$$

So $P = \dfrac{2Mg}{2+\sqrt{3}}$

R(→)

$P\cos 60° - F = 0$

$$\therefore F = \frac{Mg}{2+\sqrt{3}}$$

R(↑)

$P\sin 60° + R - Mg = 0$

$\therefore R = Mg - \dfrac{Mg\sqrt{3}}{2+\sqrt{3}} = \dfrac{2Mg}{2+\sqrt{3}}$

As the cone is on the point of slipping, $F = \mu R$

$\therefore \mu = F \div R = \dfrac{1}{2}$

i.e. μ, the coefficient of friction, $= \dfrac{1}{2}$

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Statics of rigid bodies
Exercise D, Question 5

Question:

A frustum of a right circular solid cone has two plane circular end faces with radii r and $2r$ respectively. The distance between the end faces is $2r$.

a Show that the centre of mass of the frustum is at a distance $\frac{11r}{14}$ from the larger circular face.

b Find whether this solid can rest without toppling on a rough plane, inclined to the horizontal at an angle of $40°$, if the face in contact with the inclined plane is

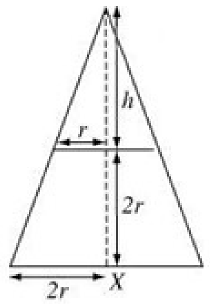
i the large circular end,

ii the small circular end.

c In order to answer part **b** you assumed that slipping did not occur. What does this imply about the coefficient of friction μ?

Solution:

a



Let the height of the small cone shown be h.
Using similar triangles

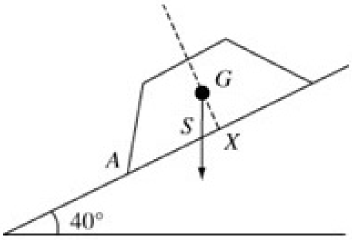
$$\frac{h}{h+2r} = \frac{r}{2r}$$
$$\therefore 2h = h + 2r$$
$$\therefore h = 2r$$

Shape	Mass	Ratio of masses	Distance of centre of mass from X
Large cone	$\rho\frac{1}{3}\pi(2r)^2(4r)$	8	r
Small cone	$\rho\frac{1}{3}\pi r^2 \times 2r$	1	$2r + \frac{2r}{4} = \frac{5r}{2}$
Frustum	$\rho\frac{1}{3}\pi \times 14r^3$	7	$\bar{x}$

Take moments about X:

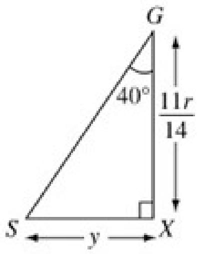
$$8r - \frac{5r}{2} = 7\bar{x}$$
$$\therefore \bar{x} = \frac{11r}{14}$$

b **i**



Let G be the position of the centre of mass.
Let S be the point an the plane vertically below G.
Let X be the centre of the circular face with radius $2r$ and A be the point about which tilting would occur.

If $SX < AX$ then the solid rests in equilibrium without toppling



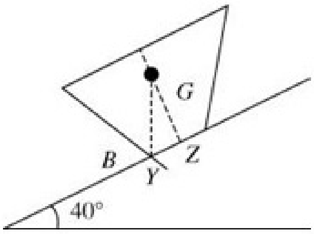
Let $SX = y$.

Then $\tan 40° = \dfrac{y}{\frac{11r}{14}}$

$\therefore y = \dfrac{11r}{14}\tan 40° = 0.66r \quad (2 \text{ s.f.})$

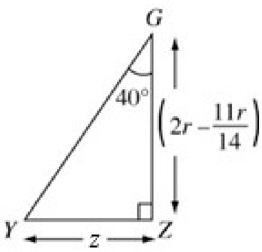
As $SX = 0.66r$ and $AX = 2r$
$SX < AX$ and the solid rests without toppling.

ii



This time Y is vertically below G. Z is the centre of the circular face and B is the point about which toppling would occur.

If $YZ > BZ$ then toppling occurs.



Let $YZ = z$

Then $\tan 40° = \dfrac{z}{\frac{17r}{14}}$

$\therefore z = \dfrac{17r}{14}\tan 40° = 1.02r$

As $YZ = 1.02r$ and $BZ = r$
$YZ > BZ$ and toppling would occur.

c As the angle of slope is $40°$ limiting friction would imply $\mu = \tan 40°$.
No slipping implies $\mu \geq 0.839$ (3 s.f.)

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Statics of rigid bodies
Exercise D, Question 6

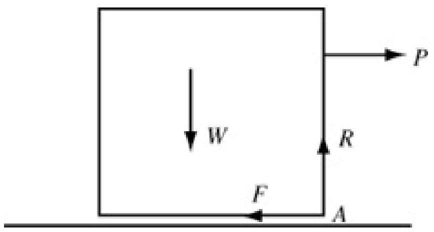
Question:

A uniform cube with edges of length $6a$ and weight W stands on a rough horizontal plane. The coefficient of friction is μ. A gradually increasing force P is applied at right angles to a vertical face of the cube at a point which is a distance a above the centre of that face.

a Show that equilibrium will be broken by sliding or toppling depending on whether $\mu < \frac{3}{4}$ or $\mu > \frac{3}{4}$.

b If $\mu = \frac{1}{4}$, and the cube is about to slip, find the distance from the point where the normal reaction acts, to the nearest vertical face of the cube.

Solution:



a Consider the cube in equilibrium, on the point of toppling, so R acts through the corner A.

$R(\rightarrow): P - F = 0 \therefore F = P$

$R(\uparrow): R - W = 0 \therefore R = W$

$\circlearrowleft M(A): P \times 4a = W \times 3a$

$\therefore P = \frac{3}{4}W$

If equilibrium is broken by toppling $P = \frac{3}{4}W$, so $F = \frac{3}{4}W$

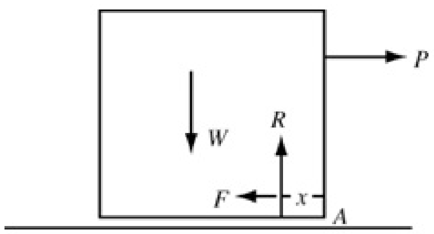
But $F < \mu R$

$\therefore \frac{3}{4}W < \mu W$ so $\mu > \frac{3}{4}$ is the condition for toppling.

If however, $\mu < \frac{3}{4}$ then the cube will be on the point of slipping when $F = \mu R$

i.e. when $P = \mu W$ the cube will start to slip.

b



Let R act at a point x from A.

$R(\rightarrow) P - F = 0 \therefore P = F$

$R(\uparrow) R - W = 0 \therefore R = W$

When the cube is about to slip: $F = \mu R$

$$\therefore P = \frac{1}{4}W$$

$\circlearrowleft M(A): P \times 4a + Rx = W \times 3a$ (substitute for P)

$\therefore \frac{1}{4}W \times 4a + Rx = W \times 3a$ (substitute for R)

$\therefore Wx = W \times 2a$

i.e. $x = 2a$

The required distance is $2a$.

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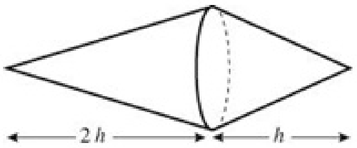
Statics of rigid bodies
Exercise D, Question 7

Question:

A spindle is formed by joining two solid right circular cones so that their circular bases coincide. The cones have the same base radius and have the same uniform density. The heights of the two cones are h and $2h$ as shown in the figure.



a Find the distance of the centre of mass of the spindle from the vertex of the larger cone.

The spindle is placed on horizontal ground with the sloping surface of the smaller cone in contact with the ground. It rests in equilibrium but is on the point of toppling.

b Show that the radius of the common base of the two cones is $\frac{1}{2}h$.

Solution:

a Let the point V be at the vertex of the large cone. The centre of mass lies an the axis of symmetry. Let the radius of the bases of the cones be r.

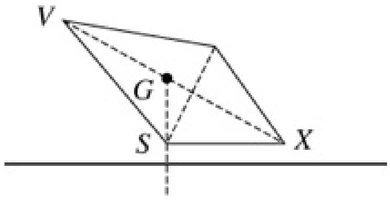
Shape	Mass	Mass ratios	Distance of centre of mass from V
Large cone	$\frac{1}{3}\pi\rho r^2 2h$	2	$\frac{3}{4}\times 2h$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	1	$2h+\frac{1}{4}h$
Spindle	$\frac{1}{3}\pi\rho r^2 3h$	3	$\bar{x}$

$$\circlearrowleft \mathrm{M}(V): 2\times\frac{3}{4}\times 2h + 1\times\left(2h+\frac{1}{4}h\right) = 3\bar{x}$$

$$\therefore 3h + \frac{9h}{4} = 3\bar{x}$$

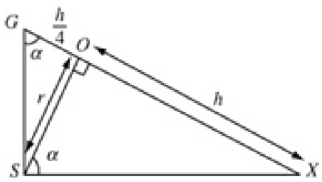
$$\text{i.e. } \bar{x} = \frac{7}{4}h$$

b



When the spindle is on the point of toppling the centre of mass G is vertically above point S, on the rim of the common face.

Let the vertex of the small cone be X and the centre of the bases of the cones be O.



In the figure let $X\hat{S}O = \alpha$,

then $S\hat{G}O = \alpha$ also

$$GO = 2h - \left(\frac{7h}{4}\right)$$

$$= \frac{h}{4}$$

$OS = r$ and $OX = h$

From ΔGOS, $\tan\alpha = \dfrac{r}{\frac{1}{4}h}$

and from ΔSOX, $\tan\alpha = \dfrac{h}{r}$

$$\therefore \frac{h}{r} = \frac{4r}{h}$$

$$\text{i.e. } h^2 = 4r^2$$

$$\text{or } r = \frac{1}{2}h$$

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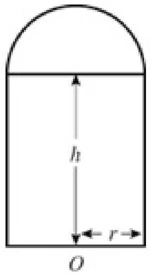
Statics of rigid bodies
Exercise D, Question 8

Question:

A uniform solid cylinder of base radius r and height h has the same density as a uniform solid hemisphere of radius r. The plane face of the hemisphere is joined to a plane face of the cylinder to form the composite solid S shown. The point O is the centre of the plane base of S.

a Show that the distance from O to the centre of mass of S is $\dfrac{6h^2+8hr+3r^2}{4(3h+2r)}$



The solid is placed on a rough plane which is inclined at an angle α to the horizontal. The plane base of S is in contact with the inclined plane.

b Given that $h = 3r$ and that S is on the point of toppling, find α to the nearest degree.

c Given that the solid did not slip before it toppled, find the range of possible values for the coefficient of friction. **[E] [adapted]**

Solution:

a

Shape	Mass	Mass ratios	Distance of centre of mass from O
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$h+\frac{3}{8}r$
Cylinder	$\pi\rho r^2 h$	$3h$	$\frac{h}{2}$
Composite solid	$\pi\rho r^2\left(\frac{2}{3}r+h\right)$	$2r+3h$	$\bar{x}$

$$\circlearrowleft M: 2r\left(h+\frac{3}{8}r\right)+3h\times\frac{h}{2} = (2r+3h)\bar{x}$$

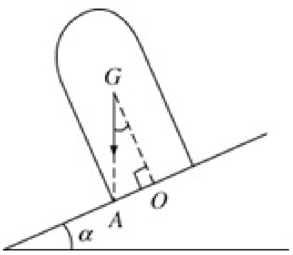
$$\therefore 2rh+\frac{3}{4}r^2+\frac{3}{2}h^2 = (2r+3h)\bar{x}$$

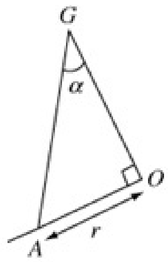
Multiply both sides by 4

$$8rh+3r^2+6h^2 = 4(2r+3h)\bar{x}$$

$$\therefore \bar{x} = \frac{6h^2+8hr+3r^2}{4(3h+2r)}$$

b





When the solid is on the point of toppling the centre of mass G is vertically above point A as shown.

In ΔGOA,

$\angle AGO = \alpha$

$OA = r$

and $OG = \frac{6(3r)^2+8(3r^2)+3r^2}{4(9r+2r)}$

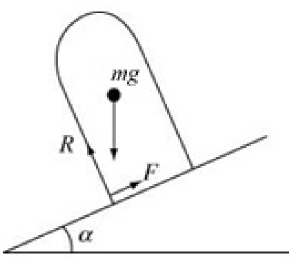
(i.e. $\bar{x}$ with $h = 3r$)

$$\therefore OG = \frac{81r^2}{44r} = \frac{81r}{44}$$

$$\therefore \tan\alpha = \frac{r}{\frac{81}{44}r} = \frac{44}{81}$$

$\therefore \alpha = 29°$ (nearest degree)

c



R(↗) $F - mg\sin\alpha = 0 \therefore F = mg\sin\alpha$

R(↖) $R - mg\cos\alpha = 0 \therefore R = mg\cos\alpha$

The solid does not slip

$\therefore F \le \mu R$

i.e., $mg\sin\alpha \le \mu mg\cos\alpha$

$\therefore \mu \ge \tan\alpha$

i.e: $\mu > \frac{44}{81}$ if the solid did not slip before it toppled.

[If $\mu = \frac{44}{81}$ it slips and topples at the same time.]

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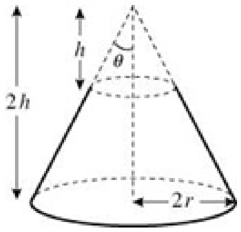
Statics of rigid bodies
Exercise D, Question 9

Question:

A uniform solid paperweight is in the shape of a frustum of a cone. It is formed by removing a right circular cone of height h from a right circular cone of height $2h$ and base radius $2r$.



a Show that the centre of mass of the paperweight lies at a height of $\frac{11}{28}h$ from its base.

When placed with its curved surface on a horizontal plane, the paperweight is on the point of toppling.

b Find θ, the semi-vertical angle of the cone, to the nearest degree.

Solution:

a Let the mass per unit volume be ρ.

Shape	Mass	Mass ratio	Position of centre of mass – distance from O
Large cone	$\frac{1}{3}\pi\rho(2r^2)2h$	8	$\frac{2h}{4}$
Small cone	$\frac{1}{3}\pi\rho r^2 h$	1	$h+\frac{h}{4}$
Frustum	$\frac{1}{3}\pi\rho(8r^2h-r^2h)$	7	$\bar{x}$

The centre of the base is the point O.
The radius of the small cone is obtained by sinular triangles.

$$\circlearrowleft MO: 8\times\frac{2h}{4}-1\times\frac{5h}{4}=7\bar{x}$$

$$\therefore \frac{11h}{4} = 7\bar{x}$$

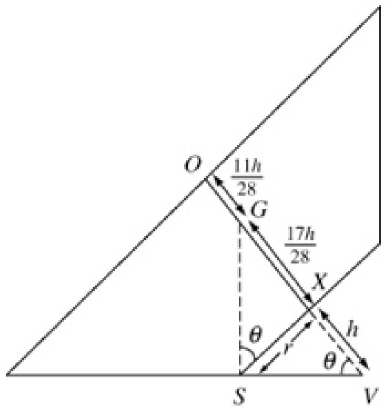
$$\text{i.e. } \bar{x} = \frac{11}{28}h$$

b As $OG=\frac{11h}{28}, GX = h-\frac{11h}{28}$

$$=\frac{17h}{28}$$

From ΔS GXS and VXS shown:

$$\tan\theta = \frac{\frac{17h}{28}}{r} \text{ and } \tan\theta=\frac{r}{h}$$



Eliminating r, $h\tan\theta = \frac{\frac{17h}{28}}{\tan\theta}$

$$h\tan\theta = \frac{\frac{17h}{28}}{\tan\theta}$$

$$\therefore \tan^2\theta = \frac{17}{28}$$

$$\therefore \theta = 38^\circ \text{ (nearest degree)}$$

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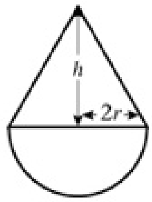
Statics of rigid bodies
Exercise D, Question 10

Question:

A child's toy is made from joining a right circular uniform solid cone, radius r and height h, to a uniform solid hemisphere of the same material and radius r. They are joined so that their plane faces coincide as shown in the figure.



a Show that the distance of the centre of mass of the toy from the base of the cone is $\left|\frac{h^2 - 3r^2}{4(2r + h)}\right|$.

The toy is placed with its hemisphere in contact with a horizontal plane and with its axis vertical. It is slightly displaced and released from rest.

b Given that the plane is sufficiently rough to prevent slipping, explain clearly, with reasons, what will happen in each of the following cases:

i $h > r\sqrt{3}$

ii $h < r\sqrt{3}$

iii $h = r\sqrt{3}$. **[E]**

Solution:

a Let the mass per unit volume of the solids be ρ. Let O be the centre of the plane circular faces which coincide.

Shape	Mass	Ratio of masses	Distance of centres of mass from O
Cone	$\frac{1}{3}\pi\rho r^2 h$	h	$\frac{h}{4}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$2r$	$\frac{-3r}{8}$
Toy	$\frac{1}{3}\pi\rho(r^2h+2r^3)$	$h+2r$	$\bar{x}$

$$\circlearrowleft O\ (h+2r)\bar{x} = h\times\frac{h}{4}+2r\left(\frac{-3r}{8}\right)$$
$$= \frac{h^2}{4}-\frac{3r^2}{4}$$
$$\therefore \bar{x} = \frac{(h^2-3r^2)}{4(h+2r)}$$

b **i** If $h > r\sqrt{3}$ then $\bar{x} > 0$ so the centre of mass is in the cone – the cone will fall over.

ii If $h < r\sqrt{3}$ then $\bar{x} < 0$ so the centre of mass is in the hemisphere, the toy will return to vertical position

iii If $h = r\sqrt{3}$, then $\bar{x} = 0$ so the centre of mass is on the join at point O. The toy will stay still in equilibrium.

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Statics of rigid bodies
Exercise E, Question 1

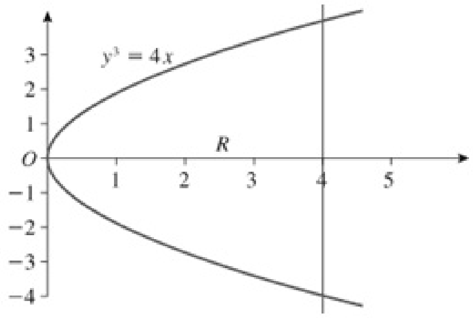
Question:

The curve shows a sketch of the region R bounded by the curve with equation $y^2 = 4x$ and the line with equation $x = 4$. The unit of length on both the axes is the centimetre. The region R is rotated through π radians about the x-axis to form a solid S.

a Show that the volume of the solid S is 32π cm^3.

b Given that the solid is uniform, find the distance of the centre of mass of S from O.

[E]



Solution:

a $V = \int \pi y^2 \, dx = \pi \int_0^4 4x \, dx$

$= \pi \left[2x^2 \right]_0^4$

$= 32\pi$

b $M\bar{x} = \rho \int x\pi y^2 \, dx = \rho\pi \int_0^4 4x^2 \, dx$

$= \rho\pi \left[\frac{4}{3}x^3 \right]_0^4$

$= \frac{256}{3}\rho\pi$

$\therefore 32\pi\rho\bar{x} = \frac{256}{3}\pi\rho$

$\therefore \bar{x} = \frac{8}{3}$

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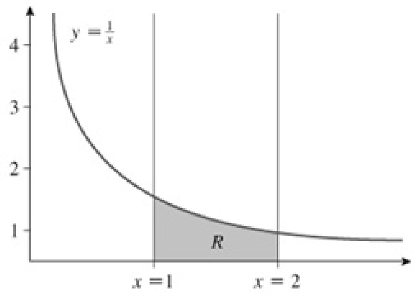
Statics of rigid bodies
Exercise E, Question 2

Question:



The region R is bounded by the curve with equation $y = \frac{1}{x}$, the lines $x = 1, x = 2$ and the x-axis, as shown in the figure. The unit of length on both the axes is 1 m. A solid plinth is made rotating R through 2π radians about the x-axis.

a Show that the volume of the plinth is $\frac{\pi}{2}\text{m}^3$.

b Find the distance of the centre of mass of the plinth from its larger plane face, giving your answer in cm to the nearest cm.

Solution:

a $V = \int \pi y^2 \, dx = \pi \int_1^2 \frac{1}{x^2} \, dx$

$$= \pi \left[\frac{-1}{x} \right]_1^2$$

$$= \pi \left[\frac{-1}{2} + 1 \right]$$

$$\text{Volume} = \frac{\pi}{2}\text{m}^3$$

b $M\bar{x} = \rho \int x \pi y^2 \, dx = \rho\pi \int_1^2 x \times \frac{1}{x^2} \, dx$

$$= \rho\pi \int_1^2 \frac{1}{x} \, dx$$

$$= \rho\pi \left[\ln x\right]_1^2$$

$$= \rho\pi \ln 2$$

$$\therefore \frac{\pi}{2}\rho\bar{x} = \rho\pi \ln 2$$

$$\therefore \bar{x} = 2\ln 2$$

So the distance of the centre of mass from the plane face $x = 1$ is $2\ln 2 - 1 = 0.386$ m (3 s.f.)
i.e. 39 cm to the nearest cm.

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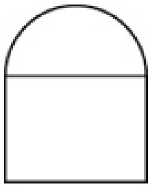
Statics of rigid bodies
Exercise E, Question 3

Question:

The figure shows a uniform solid standing on horizontal ground. The solid consists of a uniform solid right circular cylinder, of diameter 80 cm and height 40 cm, joined to a uniform solid hemisphere of the same density. The circular base of the hemisphere coincides with the upper circular end of the cylinder and has the same diameter as that of the cylinder. Find the distance of the centre of mass of the solid from the ground.



Solution:

Let the density of the solids be ρ . Let O be the centre of the circular base of the solid.

Shape	Mass	Ratio of masses	Distance of centre of mass from O
Cylinder	$\pi \times 40^2 \times 40\rho$	1	20 cm
Hemisphere	$\frac{2}{3}\pi\rho \times 40^3$	$\frac{2}{3}$	$\left(40 + \frac{3}{8} \times 40\right)$ cm
Solid	$\pi\rho \times 40^3\left(1 + \frac{2}{3}\right)$	$\frac{5}{3}$	$\bar{x}$

$$\circlearrowleft \text{M}(O) \quad \frac{5}{3}\bar{x} = 1 \times 20 + \frac{2}{3} \times \left(40 + \frac{3}{8} \times 40\right)$$
$$= 20 + \frac{110}{3}$$
$$\therefore \bar{x} = \frac{170}{5}$$
$$= 34$$

$\therefore$ The centre of mass of the solid is at a height of 34 cm above the ground.

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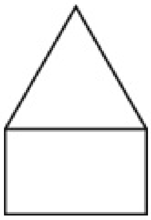
Statics of rigid bodies
Exercise E, Question 4

Question:

A simple wooden model of a rocket is made by taking a uniform cylinder, of radius r and height $3r$, and carving away part of the top two thirds to form a uniform cone of height $2r$ as shown in the figure. Find the distance of the centre of mass of the model from its plane face.



Solution:

Let the mass per unit volume be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from plane face
Cylinder	$\pi\rho r^2 \times r$	1	$\frac{r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times 2r$	$\frac{2}{3}$	$r+\frac{2r}{4}$
Model	$\pi\rho r^2 \times 1\frac{2}{3}r$	$1\frac{2}{3}$	$\bar{x}$

Note that the cylindrical base of this rocket has height r.

$$\circlearrowleft \text{M (plane face)}: 1\frac{2}{3}\bar{x} = 1\times\frac{r}{2}+\frac{2}{3}\times\left(r+\frac{2r}{4}\right)$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{r}{2}+\frac{2r}{3}+\frac{1}{3}r$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{3r}{2}$$

$$\therefore \bar{x} = \frac{9r}{10}$$

$\therefore$ The centre of mass is at a distance $\frac{9r}{10}$ from the plane face.

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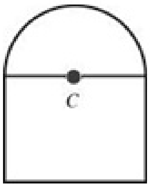
Statics of rigid bodies
Exercise E, Question 5

Question:

The figure shows a cross section containing the axis of symmetry of a uniform body consisting of a solid right circular cylinder of base radius r and height kr surmounted by a solid hemisphere of radius r. Given that the centre of mass of the body is at the centre C of the common face of the cylinder and the hemisphere, find the value of k, giving your answer to 2 significant figures.
Explain briefly why the body remains at rest when it is placed with any part of its hemispherical surface in contact with a horizontal plane.

[E]



Solution:

Let the density of the solid be ρ.

Shape	Mass	Mass ratio	Distance of C of mass from C
Cylinder	$\pi\rho r^2 \times kr$	k	$-\frac{kr}{2}$
Hemisphere	$\frac{2}{3}\pi\rho r^3$	$\frac{2}{3}$	$\frac{3}{8}r$
Composite body	$\pi\rho r^3\left(k+\frac{2}{3}\right)$	$k+\frac{2}{3}$	0

$$\circlearrowleft \text{M(about } C\text{)}: k \times \left(-\frac{kr}{2}\right) + \frac{2}{3} \times \frac{3}{8}r = 0$$

$$\therefore \frac{k^2 r}{2} = \frac{r}{4}$$

$$\therefore k^2 = \frac{1}{2} \Rightarrow k = \frac{1}{\sqrt{2}} = 0.71 \text{ (2 s.f.)}$$

The centre of mass of the body is at C which is always directly above the contact point.

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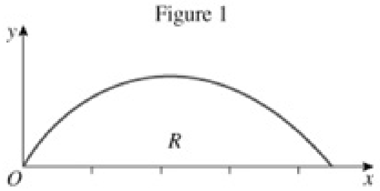
Statics of rigid bodies
Exercise E, Question 6

Question:

A uniform lamina occupies the region R bounded by the x-axis and the curve with equation $y=\frac{1}{4}x(4-x)0 \le x \le 4$, as shown in Figure 1.

a Show by integration that the y-coordinate of the centre of mass of the lamina is $\frac{2}{5}$.

Figure 1



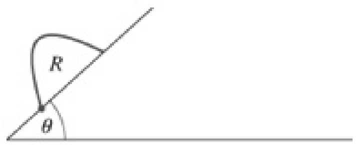


Figure 2

A uniform prism P has cross section R. The prism is placed with its rectangular face on a slope inclined at an angle θ to the horizontal. The cross section R lies in a vertical plane as shown in Figure 2. The surfaces are sufficiently rough to prevent P from sliding.

b Find the angle θ, for which P is about to topple.

Solution:

a $\bar{y} = \dfrac{\rho \int \frac{1}{2} y^2 \, dx}{\rho \int y \, dx} = \dfrac{\frac{1}{2}\int_0^4 \frac{x^2}{16}(16 - 8x + x^2)\, dx}{\frac{1}{4}\int_0^4 4x - x^2 \, dx}$

$= \dfrac{\frac{1}{2}\int_0^4 x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4 \, dx}{\frac{1}{4}\left[2x^2 - \frac{1}{3}x^3\right]_0^4}$

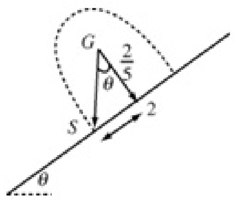
$= 2\dfrac{\left[\frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{80}x^5\right]_0^4}{32 - \frac{64}{3}}$

$= \frac{6}{32}\left[\frac{64}{3} - 32 + \frac{64}{5}\right]$

$= \frac{6}{32} \times \frac{32}{15}$

$= \frac{6}{15} = \frac{2}{5}$

b From symmetry the x-coordinate of the centre of mass is 2.
When P is about to topple the centre of mass G is directly above the lower edge of the prism S.



$\therefore \tan\theta = \dfrac{2}{\frac{2}{5}} = 5$

$\therefore \theta = 79°$ (nearest degree)

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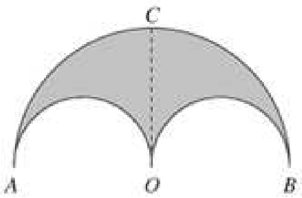
Statics of rigid bodies
Exercise E, Question 7

Question:

A uniform semi-circular lamina has radius $2a$ and the mid-point of the bounding diameter AB is O.

a Using integration, show that the centre of mass of the lamina is at a distance $\frac{8a}{3\pi}$ from O.



The two semi-circular laminas, each of radius a and with AO and OB as diameters, are cut away from the original lamina to leave the lamina $AOBC$ shown in the diagram, where OC is perpendicular to AB.

b Show that the centre of mass of the lamina $AOBC$ is at a distance $\frac{4a}{\pi}$ from O.

The lamina $AOBC$ is of mass M and a particle of mass M is attached to the lamina at B to form a composite body.

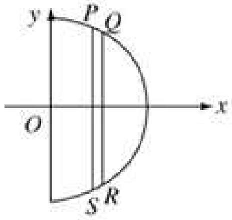
c State the distance of the centre of mass of the body from OC and from OB.

The body is smoothly hinged at A to a fixed point and rests in equilibrium in a vertical plane.

d Calculate, to the nearest degree, the acute angle between AB and the horizontal.

Solution:

a



Take the diameter as the y-axis and the mid-point of the diameter as the origin.

Then $M\bar{x} = \rho \int 2yx \, dx$ where

$M = \frac{1}{2}\rho\pi(2a)^2$ and where $x^2 + y^2 = (2a)^2$

$$\therefore 2\rho\pi a^2 \bar{x} = \rho \int_0^{2a} 2x\sqrt{4a^2 - x^2}\, dx$$

$$= \frac{-2\rho}{3}\left[(4a^2 - x^2)^{\frac{3}{2}}\right]_0^{2a}$$

$$\therefore 2\rho\pi a^2 \bar{x} = \frac{2\rho}{3} \times 8a^3$$

$$\therefore \bar{x} = \frac{16}{3}a^3 \div 2\pi a^2$$

$$= \frac{8a}{3\pi}$$

b

Shape	Mass	Mass ratios	Centre of mass (distance from AB)
Large semi-circle	$2\pi\rho a^2$	4	$\frac{8a}{3\pi}$
Semi-circle diameter AD	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Semi-circle diameter OB	$\frac{1}{2}\pi\rho a^2$	1	$\frac{4a}{3\pi}$
Remainder	$\pi\rho a^2$	2	$\bar{x}$

$$\circlearrowleft MO: 4 \times \frac{8a}{3\pi} - 1 \times \frac{4a}{3\pi} - 1 \times \frac{4a}{3\pi} = 2\bar{x}$$

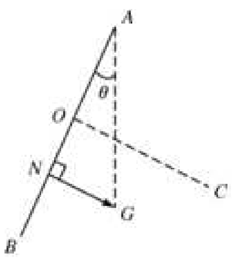
$$\therefore \frac{24a}{3\pi} = 2\bar{x}$$

$$\therefore \bar{x} = \frac{4a}{\pi}$$

c The distance from OC is a

The distance from OB is $\frac{2a}{\pi}$

d



Let N be the foot of the perpendicular from G onto AB.
In the diagram θ is the angle between AB and the vertical.
From ΔANG

$$\tan\theta = \frac{NG}{AN} = \frac{\frac{2a}{\pi}}{2a + a}$$

$$= \frac{2}{3\pi}$$

$\therefore \theta = 12°$ (to the nearest degree)

e The angle between AB and the horizontal is $90 - 12 = 78°$ (to the nearest degree)

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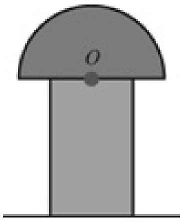
Statics of rigid bodies
Exercise E, Question 8

Question:



A uniform wooden 'mushroom', used in a game, is made by joining a solid cylinder to a solid hemisphere. They are joined symmetrically, such that the centre O of the plane face of the hemisphere coincides with the centre of one of the ends of the cylinder. The diagram shows the cross section through a plane of symmetry of the mushroom, as it stands on a horizontal table.
The radius of the cylinder is r, the radius of the hemisphere is $3r$, and the centre of mass of the mushroom is at the point O.

a Show that the height of the cylinder is $r\sqrt{\frac{81}{2}}$.

The table top, which is rough enough to prevent the mushroom from sliding, is slowly tilted until the mushroom is about to topple.

b Find, to the nearest degree, the angle with the horizontal through which the table top has been tilted.

Solution:

a

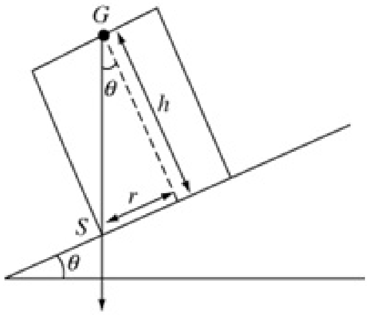
Shape	Mass	Mass ratio	Distance of centre of mass from O
Cylinder	$\pi\rho r^2 h$	h	$-\frac{h}{2}$
Hemisphere	$\frac{2}{3}\pi\rho(3r)^3$	$18r$	$\frac{3}{8}(3r)$
Mushroom	$\pi\rho r^2(h+18r)$	$h+18r$	0

$$\circlearrowleft \text{M}(O)\; -h\times\frac{h}{2}+18r\times\frac{3}{8}\times 3r = 0$$

$$\therefore \frac{h^2}{2} = \frac{81r^2}{4}$$

$$\therefore h = r\sqrt{\frac{81}{2}}$$

b



When the mushroom is about to topple GS is vertical.

From the diagram $\tan\theta = \frac{r}{h}$

$$= \sqrt{\frac{2}{81}}$$

$\therefore \theta = 9°$ (nearest degree)

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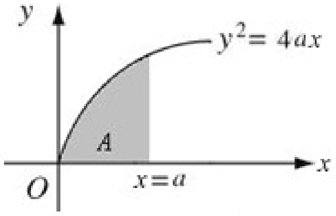
Statics of rigid bodies
Exercise E, Question 9

Question:

Figure 1 shows a finite region A which is bounded by the curve with equation $y^2 = 4ax$, the line $x = a$ and the x-axis.
A uniform solid S_1 is formed by rotating A through 2π radians about the x-axis.



a Show that the volume of S_1 is $2\pi a^3$.

b Show that the centre of mass of S_1 is a distance $\frac{2a}{3}$ from the origin O.

Figure 2

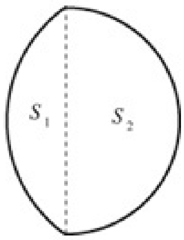


Figure 2 shows a cross section of a uniform solid S which has been obtained by attaching the plane base of solid S_1 to the plane base of a uniform hemisphere S_2 of the base radius $2a$.

c Given that the densities of solids S_1 and S_2 are ρ_1 and ρ_2 respectively, find the ratio $\rho_1 : \rho_2$ which ensures that the centre of mass of S lies in the common plane face of S_1 and S_2.

d Given that $\rho_1 : \rho_2 = 6$, explain why the solid S may rest in equilibrium with any point of the curved surface of the hemisphere in contact with a horizontal plane.

Solution:

a $V = \pi \int y^2 \, dx$

$= \pi \int_0^a 4ax \, dx$

$= \pi \left[2ax^2 \right]_0^a$

$= 2\pi a^3$

b $\bar{x} = \dfrac{\pi \int xy^2 \, dx}{\pi \int y^2 \, dx}$

$= \dfrac{\pi \int_0^a 4ax^2 \, dx}{2\pi a^3}$

$= \pi \dfrac{\left[\dfrac{4ax^3}{3} \right]_0^a}{2\pi a^3}$

$= \dfrac{\frac{4}{3}\pi a^4}{2\pi a^3}$

$= \dfrac{2}{3} a$

c

Shape	Mass	Mass ratios	Distance of centre of mass from X
S_1	$2\pi\rho a^3$	ρ_1	$-\dfrac{a}{3}$
S_2	$\dfrac{2}{3}\pi\rho_2(2a)^3$	$\dfrac{8}{3}\rho_2$	$\dfrac{3}{8}(2a)$
Combined solid	$2\pi a^3(\rho_1 + \dfrac{8}{3}\rho_2)$	$\rho_1 + \dfrac{8}{3}\rho_2$	0

The centre of mass of S_1 is at a distance $\left(a - \dfrac{2a}{3} \right)$ from its plane face.

X is the centre of the common plane base.

↺ M(X):

$-\rho_1 \times \dfrac{a}{3} + \dfrac{8}{3}\rho_2 \times \dfrac{6a}{8} = 0$

$\therefore \dfrac{1}{3}\rho_1 = 2\rho_2$

$\therefore \rho_1 = 6\rho_2$

$\rho_1 : \rho_2 = 6 : 1$

d Given that $\rho_1 : \rho_2 = 6 : 1$, then as centre of mass is at centre of hemisphere this will always be above the point of contact with the plane when a point of the curved surface area of the hemisphere is in contact with a horizontal plane.
(Tangent – radius property)

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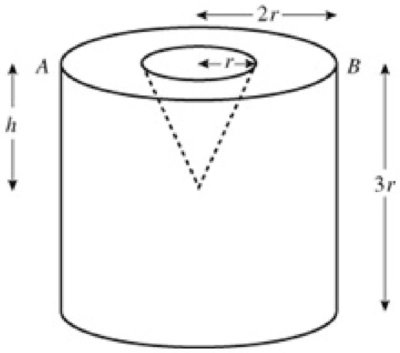
Statics of rigid bodies
Exercise E, Question 10

Question:

A mould for a right circular cone, base radius r and height h, is produced by making a conical hole in a uniform cylindrical block, base radius $2r$ and height $3r$. The axis of symmetry of the conical hole coincides with that of the cylinder, and AB is a diameter of the top of the cylinder, as shown in the figure.

a Show that the distance from AB of the centre of mass of the mould is $\frac{216r^2 - h^2}{4(36r - h)}$



The mould is suspended from the point A, and hangs freely in equilibrium.

b In the case $h = 2r$, calculate, to the nearest degree, the angle between AB and the downward vertical.

Solution:

a

Shape	Mass	Mass ratio	Distance of centre of mass from AB
Cylinder	$\pi\rho(2r)^2\times 3r$	$12r$	$\frac{3r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2\times h$	$\frac{1}{3}h$	$\frac{1}{4}h$
Remainder	$\pi\rho(12r^3-\frac{1}{3}r^3h)$	$12r-\frac{1}{3}h$	$\bar{x}$

$$\circlearrowleft:\left(12r-\frac{1}{3}h\right)\bar{x} = 12r\times\frac{3r}{2}-\frac{1}{3}h\times\frac{1}{4}h$$

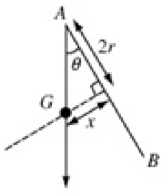
$$\therefore\left(12r-\frac{1}{3}h\right)\bar{x} = 18r^2-\frac{1}{12}h^2$$

$$\therefore \bar{x} = \frac{18r^2-\frac{1}{12}h^2}{12r-\frac{1}{3}h}$$

Multiply numerator and denominator by 12

$$\therefore \bar{x} = \frac{216r^2-h^2}{4(36r-h)}$$

b



From the diagram

$$\tan\theta = \frac{\bar{x}}{2r}$$

As $h=2r$, $\bar{x} = \frac{216r^2-(2r)^2}{4(36r-2r)} = \frac{212r^2}{136r} = \frac{53}{34}r$

$$\therefore \tan\theta = \frac{53}{68}$$

$\therefore \theta = 38°$ (nearest degree)