Solutionbank M3
Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise A, Question 1

Question:

A particle P of mass 0.2 kg is moving on the x-axis. At time t seconds P is x metres from the origin O. The force acting on P has magnitude $2\cos t$ N and acts in the direction OP. When $t = 0$, P is at rest at O. Calculate

a the speed of P when $t = 2$,

b the speed of P when $t = 3$,

c the time when P first comes to instantaneous rest,

d the distance OP when $t = 2$,

e the distance OP when P first comes to instantaneous rest.

Solution:

a $F = ma$

$2\cos t = 0.2a$

$0.2\dfrac{dv}{dt} = 2\cos t$

Force is a function of time so use $a = \dfrac{dv}{dt}$.

$v = \dfrac{2}{0.2}\int \cos t \ dt$

Integrate to obtain an expression for v.

$v = 10\sin t + c$

Don't forget the constant.

$t = 0 \quad v = 0$

$0 = 0 + c \therefore c = 0$

$v = 10\sin t$

$t = 2 \quad v = 10\sin 2 = 9.092\ldots$

When $t = 2$ the speed of P is 9.09 m s^{-1} (3 s.f.)

b $t = 3 \quad v = 10\sin 3 = 1.411\ldots$

When $t = 3$ the speed of P is 1.41 m s^{-1} (3 s.f.)

c $v = 0 \quad 0 = 10\sin t$

P comes to rest when $v = 0$.

$\sin t = 0$

$t = 0, \pi, \ldots$

P first comes to rest when $t = \pi$.

Exact answers are best.

d $v = 10\sin t$

$\dfrac{dx}{dt} = 10\sin t$

$x = 10\int \sin t \ dt$

$x = -10\cos t + K$

Integrate to obtain an expression for x.

$t = 0, x = 0 \quad 0 = -10 + K \therefore K = 10$

$x = -10\cos t + 10$

$t = 2 \quad x = -10\cos t + 10 = 14.16\ldots$

When $t = 2$ $OP = 14.2$ m (3 s.f.)

e $t = \pi \quad x = -10\cos\pi + 10$

$= 10 + 10 = 20$

When P comes to rest $OP = 20$ m.

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Further dynamics
Exercise A, Question 2

Question:

A van of mass 1200 kg moves along a horizontal straight road. At time t seconds, the resultant force acting on the car has magnitude $\frac{60\,000}{(t+5)^2}$ N and acts in the direction of motion of the van. When $t = 0$, the van is at rest. The speed of the van approaches a limiting value V m s^{-1}. Find

a the value of V,
b the distance moved by the van in the first 4 seconds of its motion.

Solution:

a $F = ma$

$\frac{60\,000}{(t+5)^2} = 1200a$

$a = \frac{50}{(t+5)^2}$

$\frac{dv}{dt} = \frac{50}{(t+5)^2}$

Force is a function of time so use $a = \frac{dv}{dt}$.

$v = \int \frac{50}{(t+5)^2}\,dt$

Integrate to obtain an expression for v.

$v = -\frac{50}{(t+5)} + c$

$t = 0, v = 0 \quad \therefore\ 0 = -\frac{50}{5} + c$

$c = 10$

$v = -\frac{50}{t+5} + 10$

As $t \to \infty \quad -\frac{50}{t+5} \to 0$

$\therefore\ V = 10$

b $v = -\frac{50}{(t+5)} + 10$

$\frac{dx}{dt} = -\frac{50}{t+5} + 10$

$x = -50\ln(t+5) + 10t + K$

$t = 0,\ x = 0 \quad 0 = -50\ln 5 + K$

$K = 50\ln 5$

$\therefore\ x = -50\ln(t+5) + 10t + 50\ln 5$

$t = 4 \quad x = -50\ln 9 + 40 + 50\ln 5$

$x = 40 + 50\ln\frac{5}{9}$

$x = 10.61\ldots$

The van moves 10.6 m in the first 4 seconds (3 s.f.)

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Further dynamics
Exercise A, Question 3

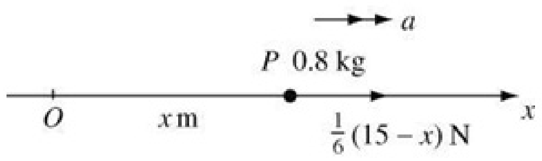
Question:

A particle P of mass 0.8 kg is moving along the x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x direction. At time t seconds, $OP = x$ metres and the velocity of P is $v\text{ m s}^{-1}$. The resultant force acting on P has magnitude $\frac{1}{6}(15 - x)\text{ N}$, and acts in the positive x direction. The maximum speed of P is 12 m s^{-1}.

a Explain why the maximum speed of P occurs when $x = 15$.

b Find the speed of P when $t = 0$.

Solution:



a Maximum speed $\Rightarrow$ acceleration zero

$\Rightarrow$ force is zero

$\therefore \frac{1}{6}(15-x)=0 \quad \therefore \ x=15$

b $F=ma$

$\frac{1}{6}(15-x)=0.8a$

$a=\frac{1}{4.8}(15-x)$

$v\frac{dv}{dx}=\frac{1}{4.8}(15-x)$ ← Force is a function of x so use $a=v\frac{dv}{dx}$.

$\int v\,dv=\frac{1}{4.8}\int(15-x)dx$ ← Separate the variables.

$\frac{1}{2}v^2=\frac{1}{4.8}\left(15x-\frac{1}{2}x^2\right)+c$

$x=15, v=12$ ← **a** tells you the initial conditions.

$\frac{1}{2}\times 12^2=\frac{1}{4.8}\left(15\times 15-\frac{1}{2}\times 15^2\right)+c$

$c=\frac{1}{2}\times 12^2-\frac{1}{4.8}\times\frac{1}{2}\times 15^2$

$c=48.5625$

$\frac{1}{2}v^2=\frac{1}{4.8}\left(15x-\frac{1}{2}x^2\right)+48.5625$

$t=0, x=0 \quad v^2=2\times 48.5625$ ← P is at 0 when $t=0$.

$v=9.855$

When $t=0$ P's speed is $9.86\ \text{m s}^{-1}$ (3 s.f.)

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Further dynamics
Exercise A, Question 4

Question:

A particle P of mass 0.75 kg is moving in a straight line. At time t seconds after it passes through a fixed point on the line, O, the distance OP is x metres and the force acting on P has magnitude $(2e^{-x}+2)$N and acts in the direction OP. Given that P passes through O with speed 5 m s^{-1}, calculate the speed of P when

a $x=3$,

b $x=7$.

Solution:

$$F = ma$$

$$(2e^{-x}+2) = 0.75\ddot{x}$$

$$0.75v\frac{dv}{dx} = 2e^{-x}+2$$

Force is a function of x so use $\ddot{x} = v\frac{dv}{dx}$.

$$0.75\int v\,dv = \int (2e^{-x}+2)\,dx$$

Separate the variables.

$$0.75\times\frac{1}{2}v^2 = -2e^{-x}+2x+c$$

$$x=0, v=5 \therefore 0.75\times\frac{1}{2}\times 5^2 = -2+c$$

$$c = 0.75\times\frac{1}{2}\times 5^2+2 = 11.375$$

$$\therefore 0.375v^2 = -2e^{-x}+2x+11.375$$

a $x=3$

$$v^2 = \frac{1}{0.375}(-2e^{-3}+6+11.375)$$

$$v = 6.787\ldots$$

When $x=3$ P's speed is 6.79 m s^{-1} (3 s.f.)

b $x=7$ $\quad v^2 = \frac{1}{0.375}(-2e^{-7}+14+11.375)$

$$v = 8.225\ldots$$

When $x=7$ P's speed is 8.23 m s^{-1} (3 s.f.)

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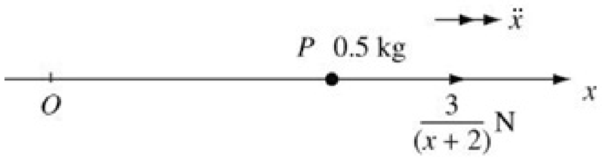
Further dynamics
Exercise A, Question 5

Question:

A particle P of mass 0.5 kg moves away from the origin O along the positive x-axis. When $OP = x$ metres the force acting on P has magnitude $\frac{3}{x+2}$ N and is directed away from O. When $x = 0$ the speed of P is 1.5 m s^{-1}. Find the value of x when the speed of P is 2 m s^{-1}.

Solution:



$F = ma$

Force is a function of x so use $\ddot{x} = v\frac{dv}{dx}$.

$$\frac{3}{x+2} = 0.5\ddot{x}$$

$$0.5v\frac{dv}{dx} = \frac{3}{x+2}$$

$$0.5\int v\,dv = 3\int \frac{1}{x+2}\,dx$$

$$0.5 \times \frac{1}{2}v^2 = 3\ln(x+2) + c$$

$$x = 0, v = 1.5$$

$$0.5 \times \frac{1}{2} \times 1.5^2 = 3\ln 2 + c$$

$$c = \frac{1.5^2}{4} - 3\ln 2$$

For the best final answer keep the exact value as long as possible.

$$\therefore \frac{1}{4}v^2 = 3\ln(x+2) + \frac{1.5^2}{4} - 3\ln 2$$

$$v = 2 \quad \frac{1}{4} \times 2^2 = 3\ln(x+2) + \frac{1.5^2}{4} - 3\ln 2$$

$$3\ln(x+2) = 1 - \frac{1.5^2}{4} + 3\ln 2$$

$$\ln(x+2) = 0.8389\ldots$$

$$x = e^{0.8389\ldots} - 2 = 0.3140\ldots$$

When P's speed is 2 m s^{-1}, $x = 0.314$ (3 s.f.)

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Further dynamics
Exercise A, Question 6

Question:

Calculate the magnitude of the impulse of a force of magnitude F N acting from time t_1 seconds to time t_2 seconds where

a $F = 3t^2 - \frac{1}{2}t \quad t_1 = 0, t_2 = 4,$

b $F = 2t + \frac{1}{3t-2} \quad t_1 = 1, t_2 = 2,$

c $F = 2\cos 4t \quad t_1 = 0, t_2 = \frac{\pi}{4},$

d $F = 3 + e^{-0.5t} \quad t_1 = 0, t_2 = 4.$

Solution:

a Impulse $= \int_0^4 \left(3t^2 - \frac{1}{2}t\right) dt$

$= \left[t^3 - \frac{1}{4}t^2\right]_0^4$

$= 64 - 4 - 0 = 60$

Impulse $= \int_{t_1}^{t_2} F\,dt$ or see end for an alternative method.

The magnitude of the impulse is 60 Ns.

b Impulse $= \int_1^2 \left(2t + \frac{1}{3t-2}\right) dt$

$= \left[t^2 + \frac{1}{3}\ln(3t-2)\right]_1^2$

$= 4 + \frac{1}{3}\ln(6-2) - \left(1 + \frac{1}{3}\ln 1\right)$

$= 3 + \frac{1}{3}\ln 4$

$= 3.462\ldots$

The magnitude of the impulse is 3.46 Ns (3 s.f.)

c Impulse $= \int_0^{\frac{\pi}{4}} 2\cos 4t\,dt = \left[\frac{2}{4}\sin 4t\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}[\sin\pi - \sin 0]$

$= 1$

The magnitude of the impulse is 1 Ns

d Impulse $= \int_0^4 (3 + e^{-0.5t})\,dt$

$= \left[3t - 2e^{-0.5t}\right]_0^4$

$= 12 - 2e^{-2} - (0 - 2\times 1)$

$= 14 - 2e^{-2}$

$= 13.72\ldots$

The magnitude of the impulse is 13.7 Ns (3 s.f.)

Alternative method for **a**

$F = ma$

Use $F = ma$ to find mv at each of the required times.

$3t^2 - \frac{1}{2}t = m\frac{dv}{dt}$

$t^3 - \frac{t^2}{4} + c = mv$

$t = 0 \quad mv_1 = c$

$t = 4 \quad 64 - 4 + c = mv_2$

impulse $= mv_2 - mv_1$

Impulse = change in momentum.

$= 60 + c - c = 60$

The magnitude of the impulse is 60 Ns.

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Further dynamics
Exercise A, Question 7

Question:

Calculate the work done by a force of magnitude F N directed along the x-axis which moves a particle from $x = x_1$ metres to $x = x_2$ metres where

a $F = 2x^{\frac{1}{2}} + \frac{1}{2}x^2 \quad x_1 = 1,\ x_2 = 4,$

b $F = 2\sin x + 3 \quad x_1 = 0,\ x_2 = \frac{\pi}{2},$

c $F = 3x^2 + e^{-2x} \quad x_1 = 1,\ x_2 = 3,$

d $F = \frac{3}{x} + \frac{2}{x-1} \quad x_1 = 2,\ x_2 = 4.$

Solution:

a work done $= \int_1^4 \left(2x^{\frac{1}{2}} + \frac{1}{2}x^2\right) dx$

$= \left[\frac{4}{3}x^{\frac{3}{2}} + \frac{1}{6}x^3\right]_1^4$

$= \frac{4}{3} \times 8 + \frac{1}{6} \times 64 - \left(\frac{4}{3} + \frac{1}{6}\right)$

$= 19\frac{5}{6}$ (or 19.83...)

The work done is $19\frac{5}{6}$ J (or 19.8 J (3 s.f.))

work done $= \int_{x_1}^{x_2} F \, ds$ or see for an alternative method.

b work done $= \int_0^{\frac{\pi}{2}} (2\sin x + 3) dx$

$= [-2\cos x + 3x]_0^{\frac{\pi}{2}}$

$= -2\cos\frac{\pi}{2} + \frac{3\pi}{2} - (-2\cos 0 + 0)$

$= \frac{3\pi}{2} + 2$

The work done is $\left(\frac{3\pi}{2} + 2\right)$ J or 6.71 J (3 s.f.)

Give the exact answer unless accuracy is specified.

c Work done $= \int_1^3 (3x^2 + e^{-2x}) dx$

$= \left[x^3 - \frac{1}{2}e^{-2x}\right]_1^3$

$= 27 - \frac{1}{2}e^{-6} - \left(1 - \frac{1}{2}e^{-2}\right)$

$= 26.06\ldots$

The work done is 26.1 J (3 s.f.)

d Work done $= \int_2^4 \left(\frac{3}{x} + \frac{2}{x-1}\right) dx$

$= [3\ln x + 2\ln(x-1)]_2^4$

$= 3\ln 4 + 2\ln 3 - (3\ln 2 + 2\ln 1)$

$= \ln 64 + \ln 9 - \ln 8 - 0$

$= \ln\left(\frac{64 \times 9}{8}\right) = \ln 72$

The work done is $\ln 72$ J or 4.28 J (3 s.f.)

Alternative method for **a**

$F = ma$

Use $F = ma$ with $a = v\frac{dv}{dx}$ and integrate to obtain $\frac{1}{2}mv^2$ for each value of x.

$2x^{\frac{1}{2}} + \frac{1}{2}x^2 = mv\frac{dv}{dx}$

$m\int v\, dv = \int\left(2x^{\frac{1}{2}} + \frac{1}{2}x^2\right)dx$

$\frac{1}{2}mv^2 = \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{6}x^3 + c$

$x = 1 \quad \frac{1}{2}mv_1^2 = \frac{4}{3} + \frac{1}{6} + c = \frac{3}{2} + c$

$x = 4 \quad \frac{1}{2}mv_2^2 = \frac{4}{3} \times 8 + \frac{1}{6} \times 64 + c$

work done $= \frac{32}{3} + \frac{64}{6} - \frac{3}{2} = 19\frac{5}{6}$

work done = change in K.E.

The work done is $19\frac{5}{6}$ J.

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Further dynamics
Exercise A, Question 8

Question:

A particle P of mass 1.5 kg is moving in a straight line. The particle is initially at rest at a point O on the line. At time t seconds (where $t \geq 0$) the force acting on P has magnitude $(3t + 8)$N and acts in the direction OP. When $t = T$, P has speed 75 m s^{-1}.

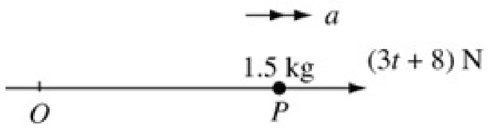
Calculate

a the magnitude of the impulse exerted by the force between the times $t = 1$ and $t = 4$,

b the speed of P when $t = 3$,

c the value of T.

Solution:



a Impulse $= \int_1^4 (3t+8)\,dt$

$= \left[\frac{3t^2}{2}+8t\right]_1^4$

$= (24+32)-\left(\frac{3}{2}+8\right)$

$= 46.5$

The impulse has magnitude 46.5 Ns.

> Impulse $= \int_{t_1}^{t_2} F\,dt$ or find velocities at $t=1$ and $t=4$ and use impulse = change in momentum.

b

$$F = ma$$
$$3t+8 = 1.5a$$
$$\frac{3}{2}\frac{dv}{dt} = 3t+8$$
$$\frac{3}{2}v = \int (3t+8)\,dt$$
$$\frac{3}{2}v = \frac{3t^2}{2}+8t+c$$
$$t=0\ v=0 \Rightarrow c=0$$
$$t=3\ \frac{3}{2}v = \frac{27}{2}+24$$
$$v = \frac{2}{3}\times\frac{75}{2}$$
$$v = 25$$

When $t=3$ the speed of P is 25 m s^{-1}.

c

$$\frac{3}{2}v = \frac{3t^2}{2}+8t$$
$$v=75,\ t=T$$
$$\frac{3}{2}\times 75 = \frac{3T^2}{2}+8T$$
$$3T^2+16T-225=0$$
$$T = \frac{-16\pm\sqrt{(16^2+4\times 3\times 225)}}{6}$$
$$= 6.394\ldots \text{ or } -11.72\ldots$$
$$T>0 \ \therefore\ T = 6.39\ (3\text{ s.f.})$$

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Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise A, Question 9

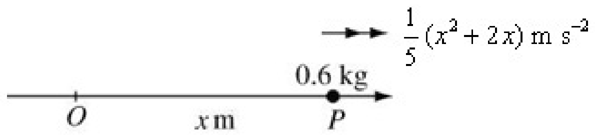
Question:

A particle of mass 0.6 kg moves in a straight line through a fixed point O. At time t seconds after passing through O the distance of P from O is x metres and the acceleration of P is $\frac{1}{5}(x^2+2x)$ m s^{-2}.

a Write down, in terms of x, an expression for the force acting on P.

b Calculate the work done by the force in moving P from $x=0$ to $x=4$.

Solution:



a $F = ma$

Use $F = ma$.

$F = 0.6 \times \frac{1}{5}(x^2+2x)$

$F = 0.12(x^2+2x)$

b Work done $= \int_0^4 F\,dx$

work done $= \int_{x_1}^{x_2} F\,dx$

$= \int_0^4 0.12(x^2+2x)dx$

$= 0.12\left[\frac{x^3}{3}+x^2\right]_0^4$

$= 0.12\left[\frac{64}{3}+16-0\right]$

$= 4.48$

The work done is 4.48 J.

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Further dynamics
Exercise B, Question 1

Question:

Above the Earth's surface, the magnitude of the force on a particle due to the Earth's gravitational force is inversely proportional to the square of the distance of the particle from the centre of the Earth. The acceleration due to gravity on the surface of the Earth is g and the Earth can be modelled as a sphere of radius R. A particle P of mass m is a distance $(x-R)$ (where $x \geq R$) above the surface of the Earth. Prove that the magnitude of the gravitational force acting on P is $\frac{mgR^2}{x^2}$.

Solution:

$F = \frac{k}{d^2}$ where $d =$ distance from centre

distance $(x-R)$ above surface

$\Rightarrow$ distance x from centre

$\therefore\ F = \frac{k}{x^2}$

On surface $F = mg, x = R$

$\therefore\ mg = \frac{k}{R^2}$

$k = mgR^2$

The magnitude of the gravitational force on a particle on the surface of the earth is the magnitude of the weight of the particle.

$\therefore$ Magnitude of the gravitational force is $\frac{mgR^2}{x^2}$.

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Further dynamics

Exercise B, Question 2

Question:

The Earth can be modelled as a sphere of radius R. At a distance x (where $x \geq R$) from the centre of the Earth the magnitude of the acceleration due to the Earth's gravitational force is A. On the surface of the Earth, the magnitude of the acceleration due to the Earth's gravitational force is g. Prove that $A = \frac{gR^2}{x^2}$.

Solution:

For a particle of mass m, distance x from the centre of the earth:

$F = ma$

$\frac{k}{x^2} = mA$ ← Use the inverse square law.

On the surface of the earth, $x = R, A = g$

$\therefore mg = \frac{k}{R^2}$

$k = mgR^2$

$\therefore mA = \frac{mgR^2}{x^2}$

$A = \frac{gR^2}{x^2}$

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Further dynamics
Exercise B, Question 3

Question:

A spacecraft S is fired vertically upwards from the surface of the Earth. When it is at a height R, where R is the radius of the Earth, above the surface of the Earth its speed is $\sqrt{gR}$. Model the spacecraft as a particle and the Earth as a sphere of radius R and find, in terms of g and R, the speed with which S was fired. (You may assume that air resistance can be ignored and that the rocket's engine is turned off immediately after the rocket fired.)

Solution:

$$F = ma$$

$$\frac{mgR^2}{x^2} = -m\ddot{x}$$

S is moving away from the earth, so the acceleration is in the direction of decreasing x.

where x is the distance of S from the centre of the Earth.

$$v\frac{dv}{dx} = -g\frac{R^2}{x^2}$$

$$\int v \, dv = -g\, R^2 \int \frac{1}{x^2}\, dx$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\frac{1}{2}v^2 = g\frac{R^2}{x} + C$$

$x = 2R \quad v = \sqrt{g\,R}$

$$\frac{1}{2}gR = \frac{gR^2}{2R} + C$$

$$C = 0$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x}$$

$x = R \quad \frac{1}{2}v^2 = \frac{gR^2}{R}$

$$v^2 = 2g\,R$$

$$v = \sqrt{2gR}$$

S was fired with speed $\sqrt{2gR}$.

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Further dynamics
Exercise B, Question 4

Question:

A rocket of mass m is fired vertically upwards from the surface of the Earth with initial speed U. The Earth is modelled as a sphere of radius R and the rocket as a particle. Find an expression for the speed of the rocket when it has travelled a distance X metres. (You may assume that air resistance can be ignored and that the rocket's engine is turned off immediately after the rocket is fired.)

Solution:

$F = ma$

$$\frac{mg R^2}{x^2} = -m\ddot{x}$$

The acceleration is in the direction of decreasing x.

where x is the distance of the rocket from the centre of the Earth.

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{g R^2}{x^2}$$

Use $\ddot{x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$ as the acceleration is a function of x.

$$\int v\,\mathrm{d}v = -g R^2 \int \frac{1}{x^2}\,\mathrm{d}x$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$x = R, \quad v = U$

On the Earth's surface.

$$\frac{1}{2}U^2 = g\frac{R^2}{R} + C$$

$$C = \frac{1}{2}U^2 - g R$$

$x = (X + R)$

After travelling a distance X, the rocket is a distance $(X + R)$ from the centre of the Earth.

$$\frac{1}{2}v^2 = \frac{gR^2}{(X+R)} + \frac{1}{2}U^2 - g R$$

$$v^2 = \frac{2g R^2 + U^2(X+R) - 2g R(X+R)}{(X+R)}$$

$$v = \sqrt{\left[\frac{U^2 X + U^2 R - 2g RX}{(X+R)}\right]}$$

When it has travelled X metres, the speed of the rocket is $\sqrt{\left[\frac{U^2 X + U^2 R - 2g RX}{(X+R)}\right]}$

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Further dynamics
Exercise B, Question 5

Question:

A particle is fired vertically upwards from the Earth's surface. The initial speed of the particle is u where $u^2 = 3gR$ and R is the radius of the Earth. Find, in terms of g and R, the speed of the particle when it is at a height $4R$ above the Earth's surface. (You may assume that air resistance can be ignored.)

Solution:

$$\ddot{x} = -\frac{gR^2}{x^2}$$

The acceleration is in the direction of decreasing x.

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v \, dv = -gR^2 \int \frac{1}{x^2} \, dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$$

$x = R \quad v^2 = 3gR$

$$\therefore \frac{1}{2} \times 3gR = \frac{gR^2}{R} + C$$

$$C = \frac{1}{2}gR$$

$$\therefore v^2 = \frac{2gR^2}{x} + gR$$

When $x = 5R$

At a height $4R$ above the Earth's surface, $x = 5R$.

$$v^2 = \frac{2gR^2}{5R} + gR$$

$$v^2 = \frac{7gR}{5}$$

$\therefore$ The speed at a height $4R$ above the Earth's surface is $\sqrt{\frac{7gR}{5}}$.

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Further dynamics
Exercise B, Question 6

Question:

A particle is moving in a straight line towards the centre of the Earth, which is assumed to be a sphere of radius R. The particle starts from rest when its distance from the centre of the Earth is $3R$. Find the speed of the particle as it hits the surface of the Earth. (You may assume that air resistance can be ignored.)

Solution:

$$m\ddot{x} = \frac{-mg R^2}{x^2}$$

The acceleration is in the direction of decreasing x.

$$v\frac{dv}{dx} = \frac{-gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v \, dv = -g R^2 \int \frac{1}{x^2} dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$x = 3R, \quad v = 0$

$$C = \frac{-gR^2}{3R} = \frac{-gR}{3}$$

$$\therefore \frac{1}{2}v^2 = \frac{gR^2}{x} - \frac{gR}{3}$$

$$x = R \quad \frac{1}{2}v^2 = \frac{gR^2}{R} - \frac{gR}{3}$$

$x = R$ on the surface of the Earth.

$$\frac{1}{2}v^2 = \frac{2}{3}g R$$

$$v^2 = \frac{4}{3}g R$$

$$v = 2\sqrt{\frac{gR}{3}}$$

The particle hits the surface of the Earth with speed $2\sqrt{\frac{gR}{3}}$.

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Further dynamics
Exercise C, Question 1

Question:

A particle P is moving in a straight line with simple harmonic motion. The amplitude of the oscillation is 0.5 m and P passes through the centre of the oscillation O with speed 2 m s^{-1}. Calculate

a the period of the oscillation,

b the speed of P when $OP = 0.2$ m.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$a = 0.5, \quad x = 0 \quad v = 2$

$$2^2 = \omega^2 \times 0.5^2$$

$$\omega = \frac{2}{0.5} = 4$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$$

The period is $\frac{\pi}{2}$ s.

b $x = 0.2 \text{ m} \quad v^2 = 4^2(0.5^2 - 0.2^2)$

$$v = 1.833\ldots$$

When $OP = 0.2$ m the speed of P is 1.83 m s^{-1} (3 s.f.)

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Further dynamics
Exercise C, Question 2

Question:

A particle P is moving in a straight line with simple harmonic motion. The period is $\frac{\pi}{3}$ s and P's maximum speed is 6 m s^{-1}. The centre of the oscillation is O. Calculate

a the amplitude of the motion,

b the speed of P 0.3s after passing through O.

Solution:

a $\text{period} = \frac{2\pi}{\omega} = \frac{\pi}{3}$

$\therefore \omega = 6$

$v^2 = \omega^2(a^2 - x^2)$

$6^2 = 6^2(a^2 - 0^2)$ ← Maximum speed occurs when $x = 0$.

$\therefore a = 1$

The amplitude is 1 m.

b $x = a \sin \omega t$ ← The speed 0.3 s after passing O is required.

$v = a\omega \cos \omega t$ ← Differentiate the line above to obtain v.

$t = 0.3 \text{ s} \quad v = 1 \times 6 \cos(6 \times 0.3)$

$v = 6 \cos 1.8$

$v = -1.363$

When $t = 0.3$, P has speed 1.36 m s^{-1} (3 s.f.) ← Speed is positive.

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Further dynamics
Exercise C, Question 3

Question:

A particle is moving in a straight line with simple harmonic motion. Its maximum speed is 10 m s^{-1} and its maximum acceleration is 20 m s^{-2}. Calculate

a the amplitude of the motion,
b the period of the motion.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$x = 0,\ v = 10 \text{ m s}^{-1} \quad 10^2 = \omega^2 a^2$ ①

Maximum speed occurs when $x = 0$.

$\ddot{x} = -\omega^2 x$

$x = -a, \ddot{x} = 20 \text{ m s}^{-2}$

$20 = +\omega^2 a$ ②

① ÷ ② $\dfrac{100}{20} = +\dfrac{\omega^2 a^2}{\omega^2 a}$

Maximum acceleration occurs when $x = -a$.

$a = 5$

The amplitude is 5 m.

b Using ① $10 = a\omega$

$10 = 5\omega$

$\omega = 2$

$\text{period} = \dfrac{2\pi}{\omega} = \pi$

The period is π s.

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Further dynamics

Exercise C, Question 4

Question:

A particle is moving in a straight line with simple harmonic motion. The period of the motion is $\frac{3\pi}{5}$ s and the amplitude is 0.4 m. Calculate the maximum speed of the particle.

Solution:

$$\text{period} = \frac{2\pi}{\omega} = \frac{3\pi}{5}$$

$$\omega = \frac{10}{3}$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = \left(\frac{10}{3}\right)^2 (0.4^2 - 0)$$

Maximum speed occurs when $x = 0$.

$$v = \frac{10}{3} \times 0.4 = \frac{4}{3}$$

The maximum speed is $\frac{4}{3}$ m s^{-1}.

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Further dynamics
Exercise C, Question 5

Question:

A particle is moving in a straight line with simple harmonic motion. Its maximum acceleration is 15 m s^{-2} and its maximum speed is 18 m s^{-1}. Calculate the speed of the particle when it is 2.5 m from the centre of the oscillation.

Solution:

$$\ddot{x} = -\omega^2 x$$

$\ddot{x} = 15\text{ m s}^{-2}$, $x = a$

$$15 = \omega^2 a \quad ①$$

$$v^2 = \omega^2(a^2 - x^2)$$

First find a and ω. (See question 3.)

$v = 18\text{ m s}^{-1}$, $x = 0$ $\quad 18^2 = \omega^2 a^2 \quad ②$

② ÷ ① $\quad \dfrac{18^2}{15} = \dfrac{\omega^2 a^2}{\omega^2 a}$

$$a = \frac{18^2}{15} = 21.6$$

Using ② $\quad a\omega = 18$

$$\omega = \frac{18}{21.6} = 0.8333\ldots$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = 0.833\ldots^2(21.6^2 - 2.5^2)$$

$$v = 17.87\ldots$$

The speed is 17.9 m s^{-1} (3 s.f.)

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Further dynamics
Exercise C, Question 6

Question:

A particle P is moving in a straight line with simple harmonic motion. The centre of the oscillation is O and the period is $\frac{\pi}{2}$ s. When $OP = 1.2$ m, P has speed 1.5 m s^{-1}.

a Find the amplitude of the motion.

At time t seconds the displacement of P from O is x metres. When $t = 0$, P is passing through O.

b Find an expression for x in terms of t.

Solution:

a period $= \frac{2\pi}{\omega} = \frac{\pi}{2}$ ← Use the period to find ω.

$\omega = 4$

$v^2 = \omega^2(a^2 - x^2)$ ← Then use $v^2 = \omega^2(a^2 - x^2)$ with $x = 1.2$ and $v = 1.5$ to find a.

$x = 1.2\text{ m}$ $\quad v = 1.5\text{ m s}^{-1}$

$1.5^2 = 4^2(a^2 - 1.2^2)$

$a^2 = \frac{1.5^2}{4^2} + 1.2^2$

$a = 1.257\ldots$

The amplitude is 1.26 m (3 s.f.).

b $x = a\sin\omega t$ ← Use $x = a\sin\omega t$ as $x = 0$ when $t = 0$.

$x = 1.26\sin 4t$

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Further dynamics
Exercise C, Question 7

Question:

A particle is moving in a straight line with simple harmonic motion. The particle performs 6 complete oscillations per second and passes through the centre of the oscillation, O, with speed 5 m s^{-1}. When P passes through the point A the magnitude of P's acceleration is 20 m s^{-1}. Calculate

a the amplitude of the motion,
b the distance OA.

Solution:

a $\text{period} = \frac{2\pi}{\omega} = \frac{1}{6}$

The period is the time for one complete oscillation.

$\omega = 12\pi$

$v^2 = \omega^2(a^2 - x^2)$

$5^2 = (12\pi)^2(a^2 - 0)$

$a = \frac{5}{12\pi} = 0.1326\ldots$

The amplitude is 0.133 m (3 s.f.).

b $\ddot{x} = -\omega^2 x$

You are told the magnitude of the acceleration at A.

$20 = |-12\pi^2|\, x$

$x = \frac{20}{12\pi^2}$

$x = 0.01407\ldots$

$OA = 0.0141 \text{ m}$ (3 s.f.)

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Further dynamics
Exercise C, Question 8

Question:

A particle P is moving on a straight line with simple harmonic motion between two points A and B. The mid-point of AB is O. When $OP = 0.6$ m, the speed of P is 3 m s^{-1} and when $OP = 0.2$ m the speed of P is 6 m s^{-1}. Find

a the distance AB,

b the period of the motion.

Solution:

a $v^2 = \omega^2(a^2 - x^2)$

$x = 0.6 \text{ m}, v = 3 \text{ m s}^{-1}$

$3^2 = \omega^2(a^2 - 0.6)^2$ ①

$x = 0.2 \text{ m}, v = 6 \text{ m s}^{-1}$

$6^2 = \omega^2(a^2 - 0.2)^2$ ②

② ÷ ① $\frac{6^2}{3^2} = \frac{\omega^2(a^2 - 0.2^2)}{\omega^2(a^2 - 0.6^2)}$

$4(a^2 - 0.6^2) = a^2 - 0.2^2$

$3a^2 = 4 \times 0.6^2 - 0.2^2$

$a^2 = \frac{4 \times 0.6^2 - 0.2^2}{3}$

$a = 0.6831\ldots$

The distance AB is 1.37 m (3 s.f.)

AB is twice the amplitude.

b Using ① $9 = \omega^2(0.6831^2 - 0.6^2)$

$\omega^2 = \frac{9}{(0.6831^2 - 0.6^2)}$

$\omega = 9.187$

$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{9.187} = 0.6838\ldots$

The period is 0.684s (3 s.f.).

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Further dynamics
Exercise C, Question 9

Question:

A particle is moving in a straight line with simple harmonic motion. When the particle is 1 m from the centre of the oscillation, O, its speed is 0.1 m s^{-1}. The period of the motion is 2π seconds. Calculate

a the maximum speed of the particle,

b the speed of the particle when it is 0.4 m from O.

Solution:

a

$$\text{period} = \frac{2\pi}{\omega} = 2\pi$$

$$\omega = 1$$

First find ω.

$$v^2 = \omega^2(a^2 - x^2)$$

Now use $v^2 = \omega^2(a^2 - x^2)$ to find a.

$$x = 1\text{ m}, v = 0.1\text{ m s}^{-1}$$

$$0.1^2 = 1^2(a^2 - 1^2)$$

$$a^2 = 0.1^2 + 1^2$$

$$a = 1.004\ldots$$

$$v_{\max} = \omega a$$

$$= 1 \times 1.004\ldots$$

Maximum speed occurs when $x = 0$.

The maximum speed is 1.00 m s^{-1} (3 s.f.).

b

$$v^2 = 1(1.004^2 - 0.4^2)$$

$$v = 0.9219\ldots$$

The speed is 0.922 m s^{-1} (3 s.f.).

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Further dynamics
Exercise C, Question 10

Question:

A piston of mass 1.2 kg is moving with simple harmonic motion inside a cylinder. The distance between the end points of the motion is 2.5 m and the piston is performing 30 complete oscillations per minute. Calculate the maximum value of the kinetic energy of the piston.

Solution:

$$a = \frac{2.5}{2} = 1.25$$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{60}{30} = 2$$

30 oscillations per minute $\Rightarrow$ 2s for 1 oscillation

$$\omega = \pi$$

$$v_{\max} = a\omega$$

$$= 1.25 \times \pi$$

$$\text{maximum K.E.} = \frac{1}{2} m v_{\max}^2$$

$$= \frac{1}{2} \times 1.2 \times 1.25^2 \times \pi^2$$

$$= 9.252\ldots$$

The maximum K.E. is 9.25 J (3 s.f.).

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Further dynamics
Exercise C, Question 11

Question:

A marker buoy is moving in a vertical line with simple harmonic motion. The buoy rises and falls through a distance of 0.8 m and takes 2 s for each complete oscillation.
Calculate

a the maximum speed of the buoy,

b the time taken for the buoy to fall a distance 0.6 m from its highest point.

Solution:

a $a = 0.8 \div 2 = 0.4\text{ m}$

The amplitude is half the distance between the highest and lowest points.

$\text{period} = \frac{2\pi}{\omega} = 2$

$\omega = \pi$

$v^2 = \omega^2(a^2 - x^2)$

$x = 0 \quad v = \omega a$

$v = \pi \times 0.4$

$v = 1.256\ldots$

The maximum speed is 1.26 m s^{-1} (3 s.f.).

b 0.6 m from highest point

$\Rightarrow x = -0.2\text{ m}$

The buoy is now below the centre.

$x = a\cos\omega t$

You want the time from the highest point.

$-0.2 = 0.4\cos\pi t$

$\cos\pi t = -0.5$

$t = \frac{1}{\pi}\cos^{-1}(-0.5)$

$t = \frac{1}{\pi} \times \left(\pi - \frac{\pi}{3}\right)$

$t = \frac{2}{3}$

The buoy takes $\frac{2}{3}$s to fall 0.6 m.

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Further dynamics
Exercise C, Question 12

Question:

Points O, A and B lie in that order in a straight line. A particle P is moving on the line with simple harmonic motion. The motion has period 2 s and amplitude 0.5 m. The point O is the centre of the oscillation, $OA = 0.2\text{ m}$ and $OB = 0.3\text{ m}$. Calculate the time taken by P to move directly from A to B.

Solution:

$$\text{period} = \frac{2\pi}{\omega} = 2$$

$$\therefore \omega = \pi$$

$$x = a \sin \omega t$$

$$x = 0.5 \sin \pi t$$

Use $x = a \sin \omega t$ to find the time to go from O to A and the time to go from O to B.

$x = 0.2\text{ m}$ $\quad 0.2 = 0.5 \sin \pi t_1$

$$\pi t_1 = \sin^{-1}\left(\frac{0.2}{0.5}\right) = \sin^{-1}\left(\frac{2}{5}\right)$$

$x = 0.3$ $\quad \pi t_2 = \sin^{-1}\left(\frac{3}{5}\right)$

$$\text{time } A \to B = t_2 - t_1$$

$$= \frac{1}{\pi}\left(\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{2}{5}\right)\right)$$

$$= 0.07384\ldots$$

The time to move directly from A to B is 0.0738 (3 s.f.).

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Further dynamics
Exercise C, Question 13

Question:

A particle P is moving along the x-axis. At time t seconds the displacement, x metres, of P from the origin O is given by $x = 4\sin 2t$.

a Prove that P is moving with simple harmonic motion.

b Write down the amplitude and period of the motion.

c Calculate the maximum speed of P.

d Calculate the least value of $t\,(t > 0)$ for which P's speed is 4 m s^{-1}.

e Calculate the least value of $t\,(t > 0)$ for which $x = 2$.

Solution:

a $x = 4\sin 2t$

$\dot{x} = 8\cos 2t$

$\ddot{x} = -16\sin 2t$

$\ddot{x} = -4(4\sin 2t)$

$\ddot{x} = -4x$

$\therefore$ S.H.M.

Differentiate the given equation twice.

b amplitude = 4 m

period $= \dfrac{2\pi}{2} = \pi$ s

Compare $x = 4\sin 2t$ with $x = a\sin\omega t$ to obtain a and ω.

c $v^2 = \omega^2(a^2 - x^2)$

$x = 0 \quad v^2 = 4(4^2 - 0)$

$v = 8$

The maximum speed is 8 m s^{-1}.

d $x = 4\sin 2t$

$\dot{x} = 8\cos 2t$

$\dot{x} = 4\text{ m s}^{-1} \quad 4 = 8\cos 2t$

$\cos 2t = 0.5$

$t = \dfrac{1}{2}\cos^{-1} 0.5$

$t = \dfrac{1}{2} \times \dfrac{\pi}{3}$

The least value of t is $\dfrac{\pi}{6}$.

From a.

e $x = 4\sin 2t$

$x = 2 \quad 2 = 4\sin 2t$

$\sin 2t = 0.5$

$t = \dfrac{1}{2}\sin^{-1} 0.5$

$t = \dfrac{1}{2} \times \dfrac{\pi}{6}$

The least value of t is $\dfrac{\pi}{12}$.

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Further dynamics
Exercise C, Question 14

Question:

A particle P is moving along the x-axis. At time t seconds the displacement, x metres, of P from the origin O is given by $x = 3\sin\left(4t + \frac{1}{2}\right)$.

a Prove that P is moving with simple harmonic motion.
b Write down the amplitude and period of the motion.
c Calculate the value of x when $t = 0$.
d Calculate the value of $t(t > 0)$ the first time P passes through O.

Solution:

a $x = 3\sin\left(4t + \frac{1}{2}\right)$

$\dot{x} = 12\cos\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -48\sin\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -16x$

$\therefore$ S.H.M.

b amplitude = 3 m

period $= \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ s

Compare with $x = a\sin(\omega t + \varepsilon)$ to obtain a and ω.

c $t = 0 \quad x = 3\sin\left(\frac{1}{2}\right)$

$= 1.438\ldots$

When $t = 0$, $x = 1.44$ (3 s.f.)

d $x = 0 \quad 0 = 3\sin\left(4t + \frac{1}{2}\right)$

$\sin\left(4t + \frac{1}{2}\right) = 0$

$4t + \frac{1}{2} = 0, \pi, \ldots$

$4t = \left(0 - \frac{1}{2}\right), \left(\pi - \frac{1}{2}\right), \ldots$

$t = -\frac{1}{8}$ (not applicable)

$t = \frac{1}{4}\left(\pi - \frac{1}{2}\right) = 0.6603\ldots$

The value of t is 0.660 (3 s.f.).

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Further dynamics
Exercise C, Question 15

Question:

On a certain day, low tide in a harbour is at 10 a.m. and the depth of the water is 5 m. High tide on the same day is at 4.15 p.m. and the water is then 15 m deep. A ship which needs a depth of water of 7 m needs to enter the harbour. Assuming that the water can be modelled as rising and falling with simple harmonic motion, calculate

a the earliest time, to the nearest minute, after 10 a.m. at which the ship can enter the harbour,

b the time by which the ship must leave.

Solution:

a amplitude $= \frac{(15-5)}{2} = 5\text{ m}$

The difference between high and low tides is twice the amplitude.

period $= (10\text{ am} \to 4.15\text{ pm}) \times 2$

$= 6.25 \times 2\text{ hr}$

$= 12.5\text{ hr}$

The time from low to high tide is half the period.

You can work through this question using hours as the unit for time.

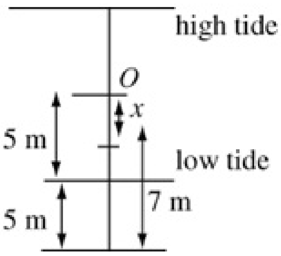
period $= \frac{2\pi}{\omega} = 12.5$

$\omega = \frac{2\pi}{12.5}$

$x = a\cos\omega t$

Start at low tide.

$x = 5\cos\left(\frac{2\pi}{12.5}t\right)$



$x = 3\text{ m}$

$3 = 5\cos\left(\frac{2\pi}{12.5}t\right)$

$\cos\left(\frac{2\pi}{12.5}t\right) = 0.6$

$t = \frac{12.5}{2\pi}\cos^{-1}0.6$

$t = 1.844\ldots$

The diagram shows that when the water is 7 m deep, $x = 3$.

Time after 10 am.

The ship can enter the harbour at 11.51 am. (nearest minute).

Remember to change the decimal part of 1.844 into minutes (0.844×60).

b The water is once more 7 m deep at $(12.5 - 1.844)$ hours after 10 am

$= 10.656$ hrs after 10 am

$= 10\text{ hr } 39.3\ldots\text{min}$.

$\therefore$ Ship must leave by 8.39 pm (nearest minute).

Use the symmetry of S.H.M. to find the time required.

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Further dynamics
Exercise C, Question 16

Question:

Points A, O and B lie in that order in a straight line. A particle P is moving on the line with simple harmonic motion with centre O. The period of the motion is 4 s and the amplitude is 0.75 m. The distance OA is 0.4 m and AB is 0.9 m. Calculate the time taken by P to move directly from B to A.

Solution:

$$\text{period} = \frac{2\pi}{\omega} = 4$$

$$\omega = \frac{\pi}{2}$$

$$x = a\sin\omega t$$

$$x = 0.75\sin\frac{\pi}{2}t$$

Find the time taken from O to B (using $x = 0.5\text{ m}$) and from O to the point where $x = 0.4\text{ m}$.

$x = 0.5\text{ m}$ $\quad 0.5 = 0.75\sin\frac{\pi}{2}t$

$$\sin\frac{\pi t}{2} = \frac{0.5}{0.75}$$

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.5}{0.75}\right)$$

$x = 0.4\text{ m}$ $\quad t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.4}{0.75}\right)$

Time $B \rightarrow A$

$$= \frac{2}{\pi}\left[\sin^{-1}\left(\frac{0.5}{0.75}\right) + \sin^{-1}\left(\frac{0.4}{0.75}\right)\right]$$

$$= 0.8226\ldots$$

Adding these times will give the time to go directly from B to A due to the symmetry of S.H.M.

P takes 0.823s to travel directly from B to A (3 s.f.).

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Further dynamics
Exercise D, Question 1

Question:

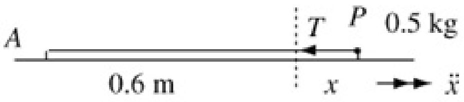
A particle P of mass 0.5 kg is attached to one end of a light elastic spring of natural length 0.6 m and modulus of elasticity 60 N. The other end of the spring is fixed to a point A on the smooth horizontal surface on which P rests. The particle is held at rest with $AP = 0.9$ m and then released.

a Show that P moves with simple harmonic motion.

b Find the period and amplitude of the motion.

c Calculate the maximum speed of P.

Solution:



$\lambda = 60$ N

a $F = ma$

$-T = 0.5\ddot{x}$

Hooke's law: $T = \frac{\lambda x}{l}$

$T = \frac{60x}{0.6} = 100x$

$-100x = 0.5\ddot{x}$

$\ddot{x} = -\frac{100}{0.5}x$

$\ddot{x} = -200x$

$\therefore$ S.H.M.

The equation of motion must reduce to the form $\ddot{x} = -\omega^2 x$.

b $\omega^2 = 200 \quad \omega = \sqrt{200} = 10\sqrt{2}$

period $= \frac{2\pi}{\omega} = \frac{2\pi}{10\sqrt{2}} = \frac{\pi}{10}\sqrt{2}$

$\therefore$ period is $\frac{\pi}{10}\sqrt{2}$s (or 0.444s (3 s.f.))

amplitude $= 0.9 - 0.6 = 0.3$

$\therefore$ amplitude is 0.3 m

The amplitude is the same as the initial extension.

c $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a = 10\sqrt{2} \times 0.3$

$= 3\sqrt{2}$

Use $x = 0$ for the maximum speed.

The maximum speed is $3\sqrt{2}$ m s^{-1} or 4.24 m s^{-1} (3 s.f.)

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Further dynamics
Exercise D, Question 2

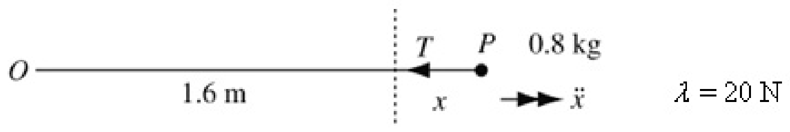
Question:

A particle P of mass 0.8 kg is attached to one end of a light elastic string of natural length 1.6 m and modulus of elasticity 20 N. The other end of the string is fixed to a point O on the smooth horizontal surface on which P rests. The particle is held at rest with $OP = 2.6$ m and then released.

a Show that, while the string is taut, P moves with simple harmonic motion.

b Calculate the time from the instant of release until P returns to its starting point for the first time.

Solution:



a $F = ma$

$-T = 0.8\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{20}{1.6}x$$

$$-\frac{20}{1.6}x = 0.8\ddot{x}$$

$$\ddot{x} = -\frac{20x}{1.6 \times 0.8} = -\frac{10x}{0.8^2}$$

$\therefore$ S.H.M.

b $\omega = \frac{\sqrt{10}}{0.8}$

$\therefore$ period $= \frac{2\pi}{\omega} = 2\pi \times \frac{0.8}{\sqrt{10}} = \frac{1.6\pi}{\sqrt{10}}$

amplitude $= 2.6 - 1.6 = 1$ m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a = 1 \times \frac{\sqrt{10}}{0.8}$$

total distance at this speed $= 4 \times 1.6$

$= 6.4$ m

The oscillation is split into 2 parts which are twice the natural length apart

$$\text{time} = 6.4 \times \frac{0.8}{\sqrt{10}}$$

$$\therefore \text{ total time} = 6.4 \times \frac{0.8}{\sqrt{10}} + \frac{1.6\pi}{\sqrt{10}} = 3.208\ldots$$

For the middle section the particle moves at a constant speed (= the maximum speed of the S.H.M.)

the total time is 3.21 s (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise D, Question 3

Question:

A particle P of mass 0.4 kg is attached to one end of a light elastic string of modulus of elasticity 24 N and natural length 1.2 m. The other end of the string is fixed to a point A on the smooth horizontal table on which P rests. Initially P is at rest with $AP = 1$ m. The particle receives an impulse of magnitude 1.8 N s in the direction AP.

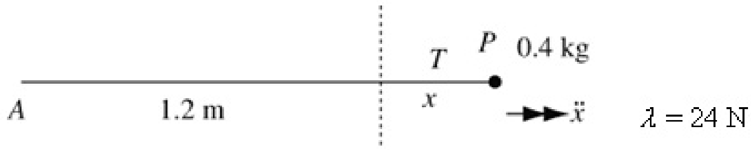
a Show that, while the string is taut, P moves with simple harmonic motion.

b Calculate the time that elapses between the moment P receives the impulse and the next time the string becomes slack.

The particle comes to instantaneous rest for the first time at the point B.

c Calculate the distance AB.

Solution:



a $F = ma$

$-T = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{\lambda x}{l}$

$T = \frac{24x}{1.2} = 20x$

$\therefore -20x = 0.4\ddot{x}$

$\ddot{x} = -\frac{20}{0.4}x$

$\ddot{x} = -50x$

$\therefore$ S.H.M.

b For the impact $I = mv - mu$

$1.8 = 0.4v$

$v = \frac{1.8}{0.4} = 4.5$

This is the speed of P while the string is slack. It is also the maximum speed for the S.H.M.

$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{5\sqrt{2}}$

The required time includes half a period.

$\therefore$ time for half an oscillation $= \frac{\pi}{5\sqrt{2}}$ s

time at constant speed

$= \frac{0.2}{4.5} = \frac{2}{45}$ s

P travels 0.2 m before the string becomes taut.

total time $= \frac{\pi}{5\sqrt{2}} + \frac{2}{45} = 0.4887\ldots$

time is 0.489 s (3 s.f.)

c $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = 4.5$ m s^{-1}

$\therefore 4.5 = a\omega$

ω and the maximum speed are known so the amplitude can be found.

$a = \frac{4.5}{5\sqrt{2}}$

$AB = 1.2 + \frac{4.5}{5\sqrt{2}}$

$= 1.836$

AB is the natural length of the string plus the amplitude of the S.H.M.

Distance AB is 1.84 m (3 s.f.)

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Further dynamics

Exercise D, Question 4

Question:

A particle P of mass 0.8 kg is attached to one end of a light elastic spring of natural length 1.2 m and modulus of elasticity 80 N. The other end of the spring is fixed to a point O on the smooth horizontal surface on which P rests. The particle is held at rest with $OP = 0.6$ m and then released.

a Show that P moves with simple harmonic motion.

b Find the period and amplitude of the motion.

c Calculate the maximum speed of P.

Solution:

$\lambda = 80$ N

a $F = ma$

$-T = 0.8\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{80x}{1.2}$$

$$0.8\ddot{x} = -\frac{80}{1.2}x$$

$$\ddot{x} = -\frac{100}{1.2}x$$

$\therefore$ SHM

b $\omega = \sqrt{\frac{100}{1.2}} = \frac{10}{\sqrt{1.2}}$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{10}\sqrt{1.2}$$

$$= 0.6882\ldots$$

period is 0.688 s (3 s.f.)

amplitude $= 1.2 - 0.6 = 0.6$ m

c $v^2 = \omega^2(a^2 - x^2)$

$$v_{max} = \omega a$$

$$= \frac{10}{\sqrt{1.2}} \times 0.6$$

$$= 5.477\ldots$$

The max speed is 5.48 m s^{-1} (3 s.f.)

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Further dynamics
Exercise D, Question 5

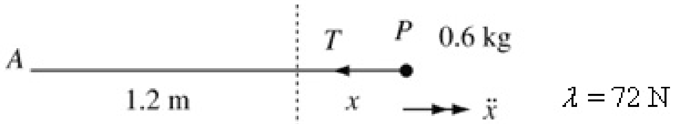
Question:

A particle P of mass 0.6 kg is attached to one end of a light elastic spring of modulus of elasticity 72 N and natural length 1.2 m. The other end of the spring is fixed to a point A on the smooth horizontal table on which P rests. Initially P is at rest with $AP = 1.2$ m. The particle receives an impulse of magnitude 3 N s in the direction AP.
Given that t seconds after the impulse the displacement of P from its initial position is x metres

a find an equation for x in terms of t,

b calculate the maximum magnitude of the acceleration of P.

Solution:



a $F = ma$

$-T = 0.6\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{72x}{1.2} = 60x$

$\therefore -60x = 0.6\ddot{x}$

$\ddot{x} = -\frac{60}{0.6}x$

$\ddot{x} = -100x$

$\therefore \omega^2 = 100$, $\omega = 10$

Use $F = ma$ and Hooke's Law to obtain the value of ω.

For the impact: $I = mv - mu$

$3 = 0.6v - 0$

$v = \frac{3}{0.6} = 5$

Use impulse = change of momentum to obtain the maximum speed.

$\therefore$ maximum speed is 5 m s^{-1}

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$5 = 10a$

$a = \frac{5}{10} = 0.5$

Now the amplitude can be obtained.

$x = a \sin \omega t$

$\therefore x = 0.5 \sin 10t$

P is at the centre of the oscillation when $t = 0$.

b $\ddot{x} = -100x$

$|\ddot{x}| = 100|x|$

$|\ddot{x}|_{max} = 100 \times 0.5 = 50$

The amplitude gives the maximum value of $|x|$.

The maximum magnitude of the acceleration is 50 m s^{-2}.

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Further dynamics
Exercise D, Question 6

Question:

A particle of mass 0.9 kg rests on a smooth horizontal surface attached to one end of a light elastic string of natural length 1.5 m and modulus of elasticity 24 N. The other end of the string is attached to a point on the surface. The particle is pulled so that the string measures 2 m and released from rest.

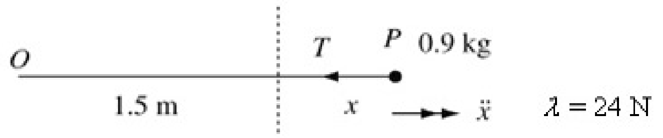
a State the amplitude of the resulting oscillation.

b Calculate the speed of the particle when the string becomes slack.

Before the string becomes taut again the particle hits a vertical surface which is at right angles to the particle's direction of motion. The coefficient of restitution between the particle and the vertical surface is $\frac{3}{5}$.

c Calculate **i** the period and **ii** the amplitude of the oscillation which takes place when the string becomes taut once more.

Solution:



a amplitude $= (2 - 1.5)\text{ m} = 0.5\text{ m}$

b energy: K.E. gained $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.9v^2$

E.P.E. lost $= \frac{\lambda x^2}{2l} = \frac{24 \times 0.5^2}{2 \times 1.5}$

$$\frac{1}{2} \times 0.9v^2 = 24 \times \frac{0.5^2}{2 \times 1.5}$$

$$v^2 = \frac{2 \times 24 \times 0.5^2}{0.9 \times 2 \times 1.5}$$

$$v = 2.108\ldots$$

The speed is 2.11 m s^{-1} (3 s.f.).

b can be solved by using conservation of energy or by S.H.M. methods, finding the maximum speed for the oscillation.

c Impact with the wall:

Newton's law of impact: $eu = v$

$$\therefore v = \frac{3}{5} \times 2.108\ldots$$

$$= 1.264\ldots$$

$\therefore$ maximum speed for the new oscillation is 1.264 m s^{-1}

$F = ma$

$-T = 0.9\ddot{x}$

S.H.M. methods essential for this part.

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{24}{1.5}x = 16x$$

$\therefore -16x = 0.9\ddot{x}$

$$\ddot{x} = -\frac{16}{0.9}x$$

$$\therefore \omega = \frac{4}{\sqrt{0.9}}$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\frac{\sqrt{0.9}}{4} = 1.490\ldots$$

The period is 1.49s (3 s.f.).

$v^2 = \omega^2(a^2 - x^2)$

$v_{\max} = \omega a$

Now ω is known you can find the amplitude using $v^2 = \omega^2(a^2 - x^2)$ with the maximum speed.

$$1.264 = \frac{4}{\sqrt{0.9}}a$$

$$a = 1.264 \times \frac{\sqrt{0.9}}{4}$$

$$a = 0.2997$$

The amplitude is 0.300 m (3 s.f.)

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Further dynamics
Exercise D, Question 7

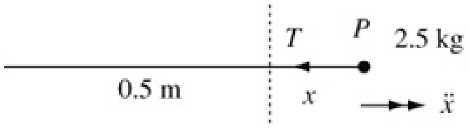
Question:

A smooth cylinder is fixed with its axis horizontal. A piston of mass 2.5 kg is inside the cylinder, attached to one end of the cylinder by a spring of modulus of elasticity 400 N and natural length 50 cm. The piston is held at rest in the cylinder with the spring compressed to a length of 42 cm. The piston is then released. The spring can be modelled as a light elastic spring and the piston can be modelled as a particle.

a Find the period of the resulting oscillations.

b Find the maximum value of the kinetic energy of the piston.

Solution:



$\lambda = 400$ N

a $F = ma$

$-T = 2.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{400x}{0.5} = 800x$$

$$-800x = 2.5\ddot{x}$$

$$\ddot{x} = -\frac{800}{2.5}x$$

$$\ddot{x} = -320x$$

$$\omega = \sqrt{320}$$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{320}} = 0.3512\ldots$$

The period is 0.351 s (3 s.f.)

b amplitude $= (50 - 42)$cm

$= 0.08$ m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

$$= \sqrt{320} \times 0.08$$

$$\text{maximum K.E} = \frac{1}{2} \times 2.5 \times (\sqrt{320} \times 0.08)^2$$

$$= 2.56$$

The maximum K.E. is 2.56 J.

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Further dynamics
Exercise D, Question 8

Question:

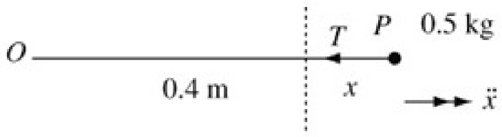
A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 0.4 m and modulus of elasticity 30 N. The other end of the string is attached to a point on the smooth horizontal surface on which P rests. The particle is pulled until the string measures 0.6 m and then released from rest.

a Calculate the speed of P when the string becomes slack for the first time.

When P has travelled a distance 0.3 m from the point of release the surface becomes rough. The coefficient of friction between P and the surface is 0.25. The particle comes to rest T seconds after it was released.

b Find the value of T.

Solution:



a $F = ma$

$-T = 0.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{30}{0.4}x = 75x$

$\therefore\ 0.5\ddot{x} = -75x$

$\ddot{x} = -\frac{75}{0.5}x$

$\ddot{x} = -150x$

$\therefore\ \omega = \sqrt{150}$

amplitude $= 0.6 - 0.4 = 0.2$ m

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = a\omega$

$= \sqrt{150} \times 0.2$

$= 2.449\ldots$

When the string becomes slack P's speed is $2.45\ \text{m s}^{-1}$ (3 s.f.).

> **a** can be done by conservation of energy but the period of the oscillation is needed for **b**.

b period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{150}}$

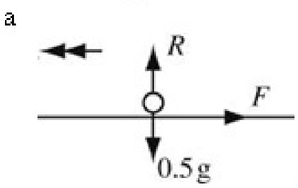
> The first part of the motion is $\frac{1}{4}$ of an oscillation.

On the smooth floor:

time $= \frac{0.1}{2.449}$

> For the first 0.2 m the string is taut.

On the rough floor:



$\mu = 0.25$

$$-F = 0.5a$$
$$F = \mu R = 0.25 \times 0.5g$$
$$\therefore\ 0.5a = -0.25 \times 0.5g$$
$$a = -0.25g$$

Find the acceleration.

$$v = u + at$$
$$0 = 2.449 - 0.25gt$$
$$t = \frac{2.449}{0.25 \times 9.8}$$

Use $v = u + at$ with $u = 2.449$ and $a = -0.25\,g$ to find the time taken to come to rest.

$$\text{total time} = \frac{1}{4} \times \frac{2\pi}{\sqrt{150}} + \frac{0.1}{2.449} + \frac{2.449}{0.25 \times 9.8}$$
$$= 1.168\ldots$$
$$\therefore\ T = 1.17\ (3\text{ s.f.})$$

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Further dynamics
Exercise D, Question 9

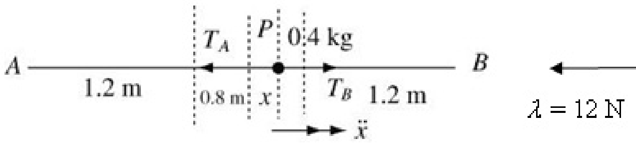
Question:

A particle P of mass 0.4 kg is attached to two identical light elastic springs each of natural length 1.2 m and modulus of elasticity 12 N. The free ends of the strings are attached to points A and B which are 4 m apart on a smooth horizontal surface. The point C lies between A and B with $AC = 1.4$ m and $CB = 2.6$ m. The particle is held at C and released from rest.

a Show that P moves with simple harmonic motion.

b Calculate the maximum value of the kinetic energy of P.

Solution:



The centre of the oscillation is at the mid-point of AB.

a $F = ma$

$T_B - T_A = 0.4\ddot{x}$

The tensions in the two parts of the string are different.

Hooke's Law: $T = \dfrac{\lambda x}{l}$

AP: extension $= (0.8 + x)$

$\therefore\ T_A = \dfrac{12(0.8+x)}{1.2} = 10(0.8+x)$

BP: extension $= (0.8 - x)$

$\therefore\ T_B = \dfrac{12(0.8-x)}{1.2} = 10(0.8-x)$

$\therefore\ 10(0.8-x) - 10(0.8+x) = 0.4\ddot{x}$

$-20x = 0.4\ddot{x}$

$\ddot{x} = -\dfrac{20}{0.4}x = -50x$

$\therefore$ P moves with S.H.M.

b $\omega^2 = 50$

amplitude $= 0.6\ m$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max}^2 = \omega^2 a^2$

$= 50 \times 0.6^2$

maximum K.E. $= \dfrac{1}{2} m v_{max}^2$

$= \dfrac{1}{2} \times 0.4 \times 50 \times 0.6^2$

$= 3.6$

The maximum K.E. is 3.6 J.

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Further dynamics
Exercise D, Question 10

Question:

A particle P of mass m is attached to two identical light strings of natural length l and modulus of elasticity $3mg$. The free ends of the strings are attached to fixed points A and B which are $5l$ apart on a smooth horizontal surface. The particle is held at the point C, where $AC = l$ and A, B and C lie on a straight line, and is then released from rest.

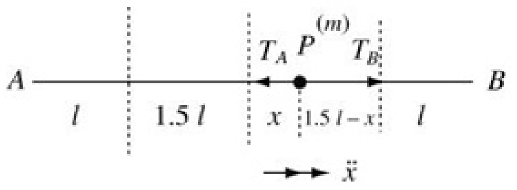
a Show that P moves with simple harmonic motion.

b Find the period of the motion.

c Write down the amplitude of the motion.

d Find the speed of P when $AP = 3l$.

Solution:



$\lambda = 3\,mg$

The centre of the oscillation is at the mid-point of AB.

a $F = ma$

$T_B - T_A = m\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

extension $= 1.5l + x$

AP: $T_A = \dfrac{3mg(1.5l + x)}{l}$

PB: extension $= 1.5l - x$

$T_B = \dfrac{3mg(1.5l - x)}{l}$

$\therefore \dfrac{3mg(1.5l - x)}{l} - \dfrac{3mg(1.5l + x)}{l} = m\ddot{x}$

$-\dfrac{6mgx}{l} = m\ddot{x}$

$\ddot{x} = -\dfrac{6g}{l}x$

$\therefore$ S.H.M.

b $\omega^2 = \dfrac{6g}{l}$ $\omega^2 = \sqrt{\dfrac{6g}{l}}$

period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{6g}}$

c Amplitude $= 1.5l$

d $v^2 = \omega^2(a^2 - x^2)$

$AP = 3l \Rightarrow x = \dfrac{l}{2}$

$\therefore v^2 = \dfrac{6g}{l}\left(\left(\dfrac{3l}{2}\right)^2 - \left(\dfrac{l}{2}\right)^2\right)$

$v^2 = \dfrac{6g}{l}\left(\dfrac{9l^2}{4} - \dfrac{l^2}{4}\right)$

$v^2 = \dfrac{6g}{l} \times \dfrac{8l^2}{4}$

$v^2 = 12gl$

When $AP = 3l$, P's speed is $\sqrt{12gl}$ (or $2\sqrt{3gl}$).

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Further dynamics

Exercise D, Question 11

Question:

A light elastic string has natural length 2.5 m and modulus of elasticity 15 N. A particle P of mass 0.5 kg is attached to the string at the point K where K divides the unstretched string in the ratio 2 : 3. The ends of the string are then attached to the points A and B which are 5 m apart on a smooth horizontal surface. The particle is then pulled aside and held at rest in contact with the surface at the point C where $AC = 3\text{ m}$ and ACB is a straight line. The particle is then released from rest.

a Show that P moves with simple harmonic motion of period $\frac{\pi}{5}\sqrt{2}$.

b Find the amplitude of the motion.

Solution:

a When P is in equilibrium:

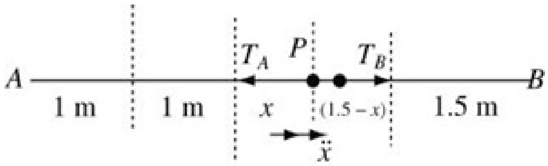
$AP = \frac{2}{5} \times 5 = 2$ m

$BP = 3$ m

Natural lengths: $AP = 1$ m

$BP = 1.5$ m

Use the ratio condition to obtain the necessary lengths for the two parts of the string.



$$F = ma$$

$$T_B - T_A = 0.5\ddot{x}$$

Hooke's Law: $T = \frac{\lambda x}{l}$

AP: extension $= 1 + x$

$$T_A = \frac{15(1+x)}{1}$$

BP: extension $= 1.5 - x$

$$T_B = \frac{15(1.5-x)}{1.5} = 10(1.5 - x)$$

$$\therefore\ 10(1.5-x) - 15(1+x) = 0.5\ddot{x}$$

$$-25x = 0.5\ddot{x}$$

$$\ddot{x} = -50x$$

$\therefore$ S.H.M.

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{50}} = \frac{2\pi}{5\sqrt{2}} = \frac{\pi}{5}\sqrt{2}$$

b Amplitude $= (3-2)\text{m} = 1$ m .

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Further dynamics
Exercise E, Question 1

Question:

A particle P of mass 0.75 kg is hanging in equilibrium attached to one end of a light elastic spring of natural length 1.5 m and modulus of elasticity 80 N. The other end of the spring is attached to a fixed point A vertically above P.

a Calculate the length of the spring.

The particle is pulled downwards and held at a point B which is vertically below A. The particle is then released from rest.

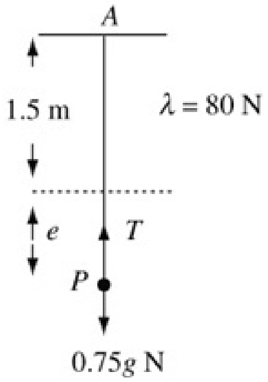
b Show that P moves with simple harmonic motion.

c Calculate the period of the oscillations.

The particle passes through its equilibrium position with speed 2.5 m s^{-1}.

d Calculate the amplitude of the oscillations.

Solution:



a In equilibrium:

$R(\uparrow) T = 0.75\text{g}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

> Resolve and use Hooke's Law with the equilibrium extension (e).

$$T = \frac{80e}{1.5}$$

$$\therefore\ 0.75\text{g} = \frac{80e}{1.5}$$

$$e = 0.75 \times 9.8 \times \frac{1.5}{80}$$

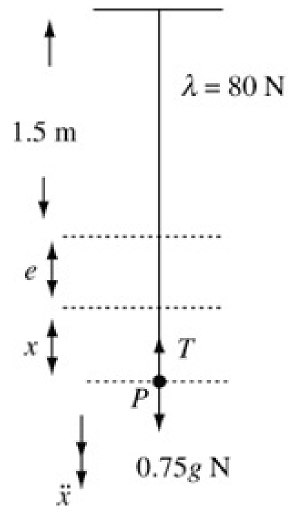
$$= 0.1378\ldots$$

$$e + l = 1.637\ldots$$

> The total length of the spring is required.

The length of the spring is 1.64 m (3 s.f.)

b



$$F = ma$$

$$0.75\text{g} - T = 0.75\ddot{x}$$

> x is measured from the equilibrium level, and $\ddot{x}$ is in the direction of increasing x.

Hooke's Law:

$$T = \frac{80(x+e)}{1.5}$$

$$\therefore\ 0.75\text{g} - \frac{80(x+e)}{1.5} = 0.75\ddot{x}$$

> To avoid decimals use $(e + x)$ for the extension
>
> where $\dfrac{80e}{1.5} = 0.75\text{g}$
>
> (from **a**)

from **a** $0.75\,\text{g} = \dfrac{80\,\text{e}}{1.5}$

$$\therefore\ 0.75\,\ddot{x} = -\frac{80}{1.5}x$$

$$\ddot{x} = -\frac{80}{1.5 \times 0.75}x$$

> of the form $\ddot{x} = -\omega^2 x$

$\therefore$ S.H.M.

c

$$\omega^2 = \frac{80}{1.5 \times 0.75}$$

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1.5 \times 0.75}{80}}$$

$$= 0.7450\ldots$$

The period is 0.745s (3 s.f.)

d $v^2 = \omega^2(a^2 - x^2)$

$2.5^2 = \frac{80}{1.5 \times 0.75} a^2$

$a^2 = \frac{2.5^2 \times 1.5 \times 0.75}{80}$

$a = 0.2964\ldots$

The amplitude is 0.296 m (3 s.f.)

$x = 0$ at the equilibrium level.

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Further dynamics
Exercise E, Question 2

Question:

A particle P of mass 0.5 kg is attached to the free end of a light elastic spring of natural length 0.5 m and modulus of elasticity 50 N. The other end of the spring is attached to a fixed point A and P hangs in equilibrium vertically below A.

a Calculate the extension of the spring.

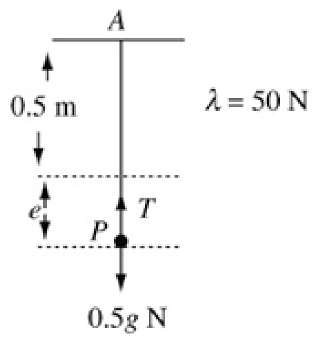
The particle is now pulled vertically down a further 0.2 m and released from rest.

b Calculate the period of the resulting oscillations.

c Calculate the maximum speed of the particle.

Solution:

a



$R(\uparrow)\ T = 0.5g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{50e}{0.5}$$

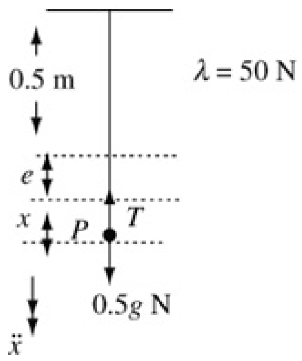
$$T = 100e$$

$$\therefore 100e = 0.5g$$

$$e = \frac{0.5 \times 9.8}{100} = 0.049$$

The extension is 0.049 m (or 4.9 cm)

b



$$F = ma$$

$$0.5g - T = 0.5\ddot{x}$$

Use $F = ma$ and Hooke's Law to find ω.

Hooke's Law: $T = \frac{\lambda x}{l}$

$$T = \frac{50(e+x)}{0.5}$$

$$T = 100(e+x)$$

$$\therefore 0.5g - 100(e+x) = 0.5\ddot{x}$$

from **a** $100e = 0.5g$

$$\therefore -100x = 0.5\ddot{x}$$

$$\ddot{x} = -200x$$

$$\omega^2 = 200$$

Compare previous line with $\ddot{x} = -\omega^2 x$.

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{200}} = \frac{2\pi}{10\sqrt{2}} = \frac{\pi}{10}\sqrt{2}$$

The period is $\frac{\pi}{10}\sqrt{2}$ s (or 0.444s (3 s.f.)).

c amplitude = 0.2 m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

$$= \sqrt{200} \times 0.2$$

$$= 2\sqrt{2}$$

The maximum speed occurs at the equilibrium level (i.e. when $x = 0$).

The maximum speed is $2\sqrt{2}$ m s^{-1} (or 2.83 m s^{-1} (3 s.f.)).

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Edexcel AS and A Level Modular Mathematics

Further dynamics
Exercise E, Question 3

Question:

A particle P of mass 2 kg is hanging in equilibrium attached to the free end of a light elastic spring of natural length 1.5 m and modulus of elasticity λ N. The other end of the spring is fixed to a point A vertically above P. The particle receives an impulse of magnitude 3 Ns in the direction AP.

a Find the speed of P immediately after the impact.

b Show that P moves with simple harmonic motion.

The period of the oscillations is $\frac{\pi}{2}$ s.

c Find the value of λ.

d Find the amplitude of the oscillations.

Solution:

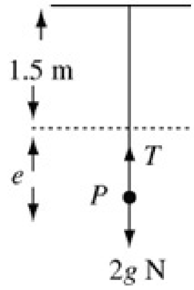
a For the impact: $I = mv - mu$

$3 = 2v$

$v = 1.5$

The speed immediately after the impact is 1.5 m s^{-1}.

b



In equilibrium

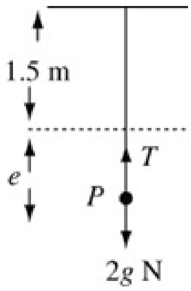
$R(\uparrow) \quad T = 2g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda e}{1.5}$

$\dfrac{\lambda e}{1.5} = 2g$

λ is unknown so will remain in this expression.



When oscillating:

$F = ma$

$2g - T = 2\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda(e + x)}{1.5}$

$\therefore\ 2g - \dfrac{\lambda(e + x)}{1.5} = 2\ddot{x}$

From above: $\dfrac{\lambda e}{1.5} = 2g$

$\therefore\ -\dfrac{\lambda x}{1.5} = 2\ddot{x}$

$\ddot{x} = -\dfrac{\lambda}{3}x$

as $\lambda > 0$, this is S.H.M.

c $\text{period} = \dfrac{2\pi}{\omega} = \dfrac{\pi}{2}$

$\therefore\ \omega = 4$

From $\ddot{x} = -\dfrac{\lambda}{3}x,\ \omega^2 = \dfrac{\lambda}{3}$

$\therefore\ \dfrac{\lambda}{3} = 16$

$\lambda = 48$

d maximum speed $= 1.5 \text{ m s}^{-1}$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$1.5 = 4a$

$a = \dfrac{1.5}{4} = 0.375$

Maximum speed occurs when $x = 0$.

The amplitude is 0.375 m.

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Further dynamics
Exercise E, Question 4

Question:

A light elastic spring has one end A fixed and hangs vertically with a particle P of mass 0.6 kg attached to its free end. Initially P is hanging freely in equilibrium. The particle is then pulled vertically downwards and released from rest.

a Show that P moves with simple harmonic motion.

The period of the motion is $\frac{\pi}{5}$ s and the maximum and minimum distances of P below A are 1.2 m and 0.8 m respectively. Calculate

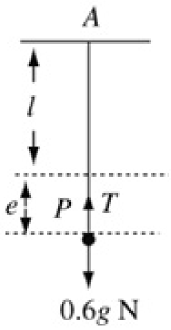
b the amplitude of the oscillation,

c the maximum speed of P,

d the maximum magnitude of the acceleration of P.

Solution:

a



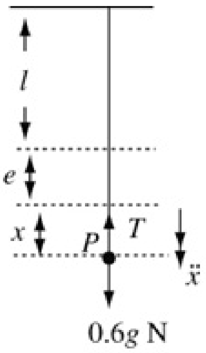
In equilibrium:

$R(\uparrow) T = 0.6g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda e}{l}$

$\therefore \dfrac{\lambda e}{l} = 0.6g$



For the oscillation:

$F = ma$

$0.6g - T = 0.6\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda(e+x)}{l}$

$\therefore 0.6g - \dfrac{\lambda(e+x)}{l} = 0.6\ddot{x}$

from above $\dfrac{\lambda e}{l} = 0.6g$

$\therefore 0.6\ddot{x} = -\dfrac{\lambda x}{l}$

$\ddot{x} = -\dfrac{\lambda x}{0.6l}$

As l and λ are both positive this is S.H.M.

The equation of motion must reduce to the form $\ddot{x} = -\omega^2 x$ but ω^2 can be expressed algebraically.

b amplitude $= \dfrac{1}{2}(1.2 - 0.8)$

$= 0.2$

The amplitude is 0.2 m.

The difference between the maximum and minimum distances below A is twice the amplitude c.

c period $= \dfrac{2\pi}{\omega} = \dfrac{\pi}{5}$

$\therefore \omega = 10$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$= 10 \times 0.2 = 2$

The maximum speed is 2 m s^{-1}.

d $\ddot{x} = -\omega^2 x$

$\ddot{x} = -100x$

Take maximum value of x.

$\therefore$ maximum magnitude of the acceleration $= 100 \times 0.2 \text{ m s}^{-2}$

$= 20 \text{ m s}^{-2}$

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Further dynamics
Exercise E, Question 5

Question:

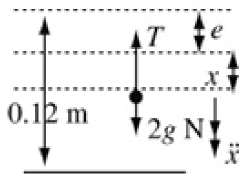
A piston of mass 2 kg moves inside a smooth cylinder which is fixed with its axis vertical. The piston is attached to the base of the cylinder by a spring of natural length 12 cm and modulus of elasticity 500 N. The piston is released from rest at a point where the spring is compressed to a length of 8 cm. Assuming that the spring can be modelled as a light elastic spring and the piston as a particle, calculate

a the period of the resulting oscillations,

b the maximum speed of the piston.

Solution:

a



$\lambda = 500$ N

In equilibrium:

$R(\uparrow)\ T = 2g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{500e}{l}$

$\therefore\ 2g = \frac{500e}{0.12}$ ← Change cm to m.

For the oscillations:

$F = ma$

$2g - T = 2\ddot{x}$

Hooke's Law $T = \frac{\lambda x}{l}$

$T = \frac{500(e+x)}{0.12}$

$\therefore\ 2g - \frac{500(e+x)}{0.12} = 2\ddot{x}$

From above: $\frac{500e}{0.12} = 2g$

$\therefore\ -\frac{500x}{0.12} = 2\ddot{x}$

$\ddot{x} = -\frac{250}{0.12}x$

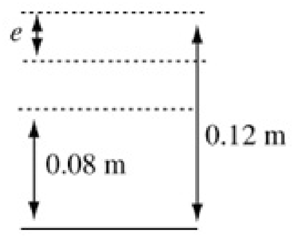
$\omega^2 = \frac{250}{0.12}$ ← Compare line above with $\ddot{x} = -\omega^2 x$.

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.12}{250}}$

$= 0.1376\ldots$

The period is 0.138 s (3 s.f.)

b $e = \frac{2g \times 0.12}{500}$



$\text{amplitude} = 0.04 - e$

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = a\omega$ ← Maximum speed occurs when $x = 0$.

$= \sqrt{\frac{250}{0.12}} \times (0.04 - e)$

$= \sqrt{\frac{250}{0.12}} \times \left(0.04 - \frac{2g \times 0.12}{500}\right)$

$= 1.611\ldots$

The maximum speed is $1.61\ \text{m s}^{-1}$ (3 s.f.).

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Further dynamics
Exercise E, Question 6

Question:

A light elastic string of natural length 40 cm has one end A attached to a fixed point. A particle P of mass 0.4 kg is attached to the free end of the string and hangs freely in equilibrium vertically below A. The distance AP is 45 cm.

a Find the modulus of elasticity of the string.

The particle is now pulled vertically downwards until AP measures 52 cm and then released from rest.

b Show that, while the string is taut, P moves with simple harmonic motion.

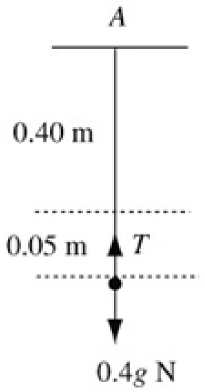
c Find the period and amplitude of the motion.

d Find the greatest speed of P during the motion.

e Find the time taken by P to rise 11 cm from the point of release.

Solution:

a



In equilibrium:

$R(\uparrow)\, T = 0.4g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda 0.05}{0.4}$

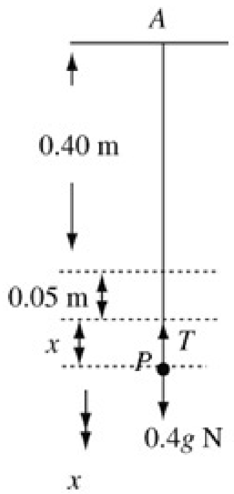
$\therefore\ \lambda \times \dfrac{0.05}{0.4} = 0.4g$

$\lambda = \dfrac{0.4^2}{0.05} g$

$\lambda = 3.2g = 31.36$

The modulus of elasticity is 31.4 N (3 s.f.)

b



For oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{31.36(x + 0.05)}{0.4}$

$0.4g - \dfrac{31.36(x+0.05)}{0.4} = 0.4\ddot{x}$

$\ddot{x} = -\dfrac{31.36}{0.4^2} x$

$\therefore$ S.H.M.

c From $\ddot{x} = -\dfrac{31.36}{0.4^2} x$

$\omega = \dfrac{\sqrt{31.36}}{0.4}$

$\text{period} = \dfrac{2\pi}{\omega} = 2\pi \times \dfrac{0.4}{\sqrt{31.36}} = 0.4487\ldots$

The period is 0.449 s.

amplitude $= 52 - 45 = 7$(cm)

The amplitude is 0.07 m.

d $v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

$= \dfrac{\sqrt{31.36}}{0.4} \times 0.07$

$= 0.98$

The maximum speed is $0.98\ \text{m s}^{-1}$.

e 11 cm from the lowest point

$\Rightarrow AP = 41\text{ cm.}$

$\therefore\ x = -4\text{ cm} = -0.04\text{ m}$

$$x = a\cos\omega t$$

P starts from an end point.

$$-0.04 = 0.07\cos\omega t$$

$$\omega t = \cos^{-1}\left(-\frac{0.04}{0.07}\right) = \cos^{-1}\left(-\frac{4}{7}\right)$$

$$t = \frac{1}{\omega}\cos^{-1}\left(-\frac{4}{7}\right) = \frac{0.4}{\sqrt{31.36}}\cos^{-1}\left(-\frac{4}{7}\right)$$

$$= 0.1556\ldots$$

P takes 0.156s to rise 11 cm (3 s.f.).

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Further dynamics
Exercise E, Question 7

Question:

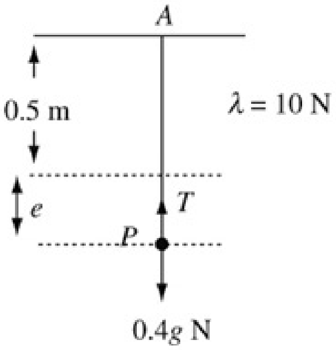
A particle P of mass 0.4 kg is attached to one end of a light elastic string of natural length 0.5 m and modulus of elasticity 10 N. The other end of the string is attached to a fixed point A and P is initially hanging freely in equilibrium vertically below A. The particle is then pulled vertically downwards a further 0.2 m and released from rest.

a Calculate the time from release until the string becomes slack for the first time.

b Calculate the time between the string first becoming slack and the next time it becomes taut.

Solution:

a



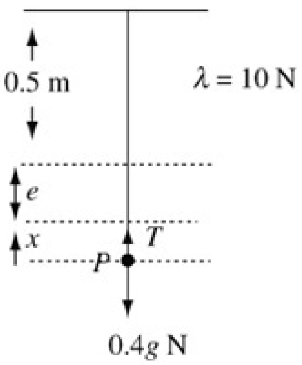
In equilibrium:

$R(\uparrow) T = 0.4g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{10e}{0.5} = 20e$

$\therefore\ 20e = 0.4g$



For the oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{10(e + x)}{0.5}$

$\therefore\ 0.4g - \frac{10(e + x)}{0.5} = 0.4\ddot{x}$

From above $0.4g = \frac{10e}{0.5}$

$\therefore\ -\frac{10x}{0.5} = 0.4\ddot{x}$

$\ddot{x} = -\frac{20x}{0.4} = -50x$

$\therefore$ S.H.M. with $\omega^2 = 50$

amplitude $= 0.2$ m

$x = a\cos\omega t$

$x = 0.2\cos\sqrt{50}t$

String becomes slack when $x = -e$

$-\frac{0.4g}{20} = 0.2\cos\sqrt{50}t$

$\cos\sqrt{50}t = -\frac{2g}{20} = -\frac{g}{10} = -0.98$

$\sqrt{50}t = \cos^{-1}(-0.98)$

$t = \frac{1}{\sqrt{50}}\cos^{-1}(-0.98)$

$t = 0.4159$

The string becomes slack after 0.416s (3 s.f.)

b $v^2 = \omega^2(a^2 - x^2)$

$x = -e = -\frac{0.4}{20}g$

$v^2 = 50\left(0.2^2 - \left(\frac{0.4}{20}g\right)^2\right)$

$v^2 = 0.0792$

$v = u + at$

$\sqrt{0.0792} = -\sqrt{0.0792} + 9.8t$

$t = \frac{2\sqrt{0.0792}}{9.8} = 0.05743\ldots$

The string is slack for 0.0574 s (3 s.f.)

Find the speed when the string becomes slack.

The particle moves freely under gravity while the string is slack.

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Further dynamics
Exercise E, Question 8

Question:

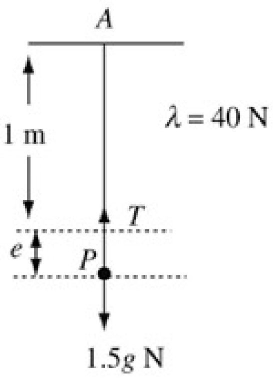
A particle P of mass 1.5 kg is hanging freely attached to one end of a light elastic string of natural length 1 m and modulus of elasticity 40 N. The other end of the string is attached to a fixed point A on a ceiling. The particle is pulled vertically downwards until AP is 1.8 m and released from rest. When P has risen a distance 0.4 m the string is cut.

a Calculate the greatest height P reaches above its equilibrium position.

b Calculate the time taken from release to reach that greatest height.

Solution:

a



In equilibrium:

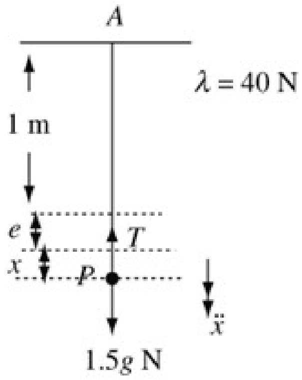
$R(\uparrow) T = 1.5g$

a can be done by using conservation of energy but **b** needs S.H.M. So S.H.M. has been used for both parts.

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$1.5g = \dfrac{40e}{1}$

$e = \dfrac{1.5g}{40} = 0.3675 \text{ m}$



For the oscillation:

$F = ma$

$1.5g - T = 1.5\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$= \dfrac{40(x+e)}{1}$

$\therefore 1.5g - 40(x+e) = 1.5\ddot{x}$

From above $1.5g = 40e$

$\therefore 1.5\ddot{x} = -40x$

$\ddot{x} = -\dfrac{80}{3}x$

$\omega = \sqrt{\dfrac{80}{3}}$

amplitude $= 0.8 - 0.3675 = 0.4325 \text{ m}$

$v^2 = \omega^2(a^2 - x^2)$

When the string is cut: $x = 0.4325 - 0.4$

$= 0.0325$

and $v^2 = \dfrac{80}{3}(0.4325^2 - 0.0325^2)$

$= 4.96$

Find the speed when the string is cut.

motion under gravity:

$v^2 = u^2 + 2as$

Use motion under gravity.

$0 = 4.96 - 2 \times 9.8s$

$s = \dfrac{4.96}{2 \times 9.8} = 0.2530\ldots$

height above equilibrium position

$= 0.2530 - 0.0325 = 0.2205$

Height is 0.221 m.

b For S.H.M. $x = a\cos\omega t$ ← Particle starts from an end-point.

$$x = 0.4325\cos\sqrt{\frac{80}{3}}t$$

$x = 0.0325$ $\quad 0.0325 = 0.4325\cos\sqrt{\frac{80}{3}}t$

$$\cos\sqrt{\frac{80}{3}}t = \frac{0.0325}{0.4325}$$

$$t = \sqrt{\frac{3}{80}}\cos^{-1}\left(\frac{0.0325}{0.4325}\right)$$

$$= 0.2896$$

Motion under gravity:

$v = u + at$

$O = \sqrt{4.96} - 9.8t$

$t = \dfrac{\sqrt{4.96}}{9.8}$

$\text{total time} = 0.2896\ldots + \dfrac{\sqrt{4.96}}{9.8} = 0.5168\ldots$

The time taken to reach the highest point is 0.517s (3 s.f.)

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Further dynamics
Exercise E, Question 9

Question:

A particle P of mass 1.5 kg is attached to the mid-point of a light elastic string of natural length 1.2 m and modulus of elasticity 15 N. The ends of the string are fixed to the points A and B where A is vertically above B and $AB = 2.8$ m.

a Given that P is in equilibrium calculate the length AP.

The particle is now pulled downwards a distance 0.15 m from its equilibrium position and released from rest.

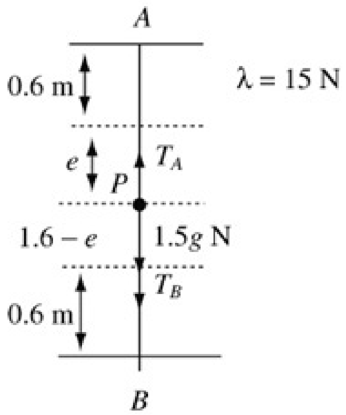
b Prove that P moves with simple harmonic motion.

T seconds after being released P is 0.1 m above its equilibrium position.

c Find the value of T.

Solution:

a



In equilibrium

$R(\uparrow)\ T_A = 1.5g + T_B$

Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T_A = \frac{15e}{0.6} = 25e$$

$$T_B = \frac{15(1.6-e)}{0.6} = 40 - 25e$$

You must consider P to be attached to two strings. The tensions in the two parts will be different.

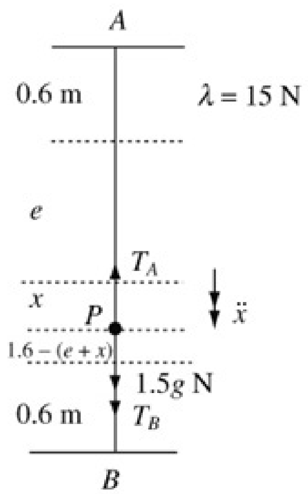
$$\therefore\ 25e = 1.5g + 40 - 25e$$

$$50e = 1.5g + 40$$

$$e = \frac{1}{50}(1.5g + 40) = 1.094$$

In equilibrium, $AP = 1.69$ m (3 s.f.)

b



For the oscillations:

$$T_A = \frac{15(e+x)}{0.6}$$

$$T_B = \frac{15(1.6-(e+x))}{0.6}$$

$F = ma$

$$1.5g + \frac{15(1.6-(e+x))}{0.6} - \frac{15(e+x)}{0.6} = 1.5\ddot{x}$$

$$1.5g + 40 - 25(e+x) - 25(e+x) = 1.5\ddot{x}$$

$$1.5g + 40 - 50e - 50x = 1.5\ddot{x}$$

from **a** $50e = 1.5g + 40$

$$\therefore\ 1.5\ddot{x} = -50x$$

$$\ddot{x} = -\frac{50}{1.5}x = -\frac{100}{3}x$$

$\therefore$ P moves with S.H.M.

c amplitude $= 0.15\,\text{m}$

$x = a\cos\omega t = 0.15\cos\left(\frac{10}{\sqrt{3}}\right)t$

When $x = -0.1$

The equilibrium position is the centre of the oscillation.

$$-0.1 = 0.15\cos\left(\frac{10}{\sqrt{3}}T\right)$$

$$\cos\left(\frac{10}{\sqrt{3}}T\right) = -\frac{0.1}{0.15}$$

$$T = \frac{\sqrt{3}}{10}\cos^{-1}\left(-\frac{0.1}{0.15}\right)$$

$$= 0.3984\ldots$$

$$\therefore\ T = 0.398 \quad (3\text{ s.f.})$$

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Further dynamics
Exercise E, Question 10

Question:

A rock climber of mass 70 kg is attached to one end of a rope. He falls from a ledge which is 8 m vertically below the point to which the other end of the rope is fixed. The climber falls vertically without hitting the rock face. Assuming that the climber can be modelled as a particle and the rope as a light elastic string of natural length 16 m and modulus of elasticity 40 000 N, calculate

a the climber's speed at the instant when the rope becomes taut,

b the maximum distance of the climber below the ledge,

c the time from falling from the ledge to reaching his lowest point.

Solution:

a Until rope is taut:

$v^2 = u^2 + 2as$

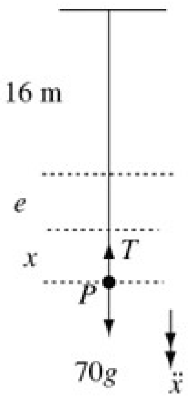
$v^2 = 0 + 2 \times 9.8 \times 8$

$v = 12.52\ldots$

When the rope becomes taut the climber's speed is 12.5 m s^{-1} (3 s.f.)

Climber falling freely under gravity.

b



At the equilibrium level:

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{40\,000e}{16}$

$R(\uparrow)\ T = 70g$

$\therefore\ 70g = \dfrac{40\,000e}{16}$

$e = \dfrac{16 \times 70g}{40\,000} = \dfrac{7g}{250}$

b can be solved by using conservation of energy. However **c** involves time, so S.H.M. methods are needed. It is more efficient to use S.H.M. for both parts.

For the oscillation:

$F = ma$

$70g - T = 70\ddot{x}$

Hooke's Law: $T = \dfrac{40\,000(x+e)}{16}$

$70g - \dfrac{40\,000(x+e)}{16} = 70\ddot{x}$

$\ddot{x} = -\dfrac{4000}{16 \times 7}x = -\dfrac{250}{7}x$

From **a**: $70g = \dfrac{40\,000e}{16}$

$\omega^2 = \dfrac{250}{7}$

$v^2 = \omega^2(a^2 - x^2)$

$156.8 = \dfrac{250}{7}\left(a^2 - \left(\dfrac{7g}{250}\right)^2\right)$

Use the result from part **a**, ie the speed when $x = e\left(= \dfrac{7g}{250}\right)$.

$a^2 = \dfrac{156.8 \times 7}{250} + \left(\dfrac{7g}{250}\right)^2$

$a^2 = 4.4656\ldots$

$a = 2.113\ldots$

Total distance $= 2.113 + e + 8$

$= 2.113 + \dfrac{7g}{250} + 8$

$= 10.38\ldots$

The total distance fallen is 10.4 m (3 s.f.).

The amplitude is the greatest distance below the equilibrium level.

c $x = a\cos\omega t$

$x = 2.113\cos\sqrt{\frac{250}{7}}t$

When $x = e$

$\frac{7g}{250} = 2.113\cos\sqrt{\frac{250}{7}}t$

$t = \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7}{250}}$

Time while the rope is taut:

$= \frac{2\pi}{2}\sqrt{\frac{7}{250}} - \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

$= 0.2846\ldots$

While moving under gravity:

$s = ut + \frac{1}{2}at^2$

$8 = \frac{1}{2}\times 9.8t^2$

$t^2 = \frac{16}{9.8}$

total time $= \frac{4}{\sqrt{9.8}} + 0.2846\ldots$

$= 1.562\ldots$

The total time is 1.56 s (3 s.f.).

Because of the symmetry of S.H.M. there are several methods available for c.

This method assumes the oscillation is complete and finds the time from the highest point $(x = a)$ to the equilibrium level $(x = e)$. This time will be subtracted from half the period. So it does not matter that this part of the oscillation does not exist.

Time from highest point to lowest point of a complete oscillation is half the period. Subtract the time for the missing part (before the rope is taut) to obtain the time while the rope is taut.

The time before the rope becomes taut is also needed.

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Further dynamics
Exercise F, Question 1

Question:

A particle P is moving with simple harmonic motion between two points A and B which are 0.4 m apart on a horizontal line. The mid-point of AB is O. At time $t = 0$, P passes through O, moving towards A, with speed u m s^{-1}. The next time P passes through O is when $t = 2.5$ s.

a Find the value of u.

b Find the speed of P when $t = 3$ s.

c Find the distance of P from A when $t = 3$ s.

Solution:

a amplitude $= 0.4 \div 2 = 0.2$ m

period $= 2 \times 2.5 = 5$s

$$\therefore \frac{2\pi}{\omega} = 5 \quad \omega = \frac{2\pi}{5}$$

> Find a and ω from the given information.

$$v^2 = \omega^2(a^2 - x^2)$$

$$u^2 = \left(\frac{2\pi}{5}\right)^2 (0.2^2 - 0)$$

$$u = \frac{2\pi}{5} \times 0.2 = \frac{4\pi}{50} \text{ (or } 0.2513\ldots)$$

$$\therefore u = \frac{4\pi}{50} \text{ (or } 0.251 \text{ (3 s.f.))}$$

> Now use $v^2 = \omega^2(a^2 - x^2)$ to find u.

b

$$x = a \sin \omega t$$

$$x = 0.2 \sin\left(\frac{2\pi}{5} t\right)$$

> P is at the centre of oscillation when $t = 0$.

$$\dot{x} = \frac{2\pi}{5} \times 0.2 \cos\left(\frac{2\pi}{5} t\right)$$

> Differentiate x with respect to t to find $\dot{x}$.

$$t = 3 \quad \dot{x} = \frac{0.4\pi}{5} \cos \frac{6\pi}{5} = -0.2033$$

When $t = 3$ P's speed is 0.203 m^{-1} (3 s.f.).

> Speed is positive.

c

$$x = a \sin \omega t$$

$$x = 0.2 \sin\left(\frac{2\pi}{5} t\right)$$

$$t = 3 \quad x = 0.2 \sin\left(\frac{6\pi}{5}\right) = -0.1175\ldots$$

$\therefore$ Distance from A is $0.2 + 0.1175\ldots$

$= 0.318$ m (3 s.f.)

> P is moving towards A when $t = 0$, so x is negative between O and B.

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Further dynamics
Exercise F, Question 2

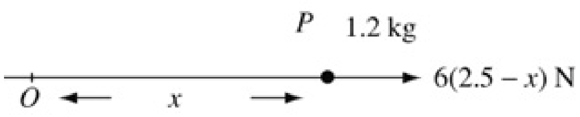
Question:

A particle P of mass 1.2 kg moves along the x-axis. At time $t = 0$, P passes through the origin O, moving in the positive x-direction. At time t seconds, the velocity of P is v m s^{-1} and $OP = x$ metres. The resultant force acting on P has magnitude $6(2.5 - x)$ N and acts in the positive x-direction. The maximum speed of P is 8 m s^{-1}.

a Write down the value of x when the speed of P is 8 m s^{-1}.

b Find an expression for v^2 in terms of x.

Solution:



a $x = 2.5$

The acceleration (and therefore the resultant force) are zero when the speed is maximum.

b

$$F = ma$$

$$6(2.5 - x) = 1.2a$$

$$6(2.5 - x) = 1.2v\frac{dv}{dx}$$

When the force is a function of x use $a = v\frac{dv}{dx}$.

$$v\frac{dv}{dx} = 5(2.5 - x)$$

$$\int v\,dv = \int 5(2.5 - x)dx$$

Separate the variables.

$$\frac{1}{2}v^2 = 5\left(2.5x - \frac{x^2}{2}\right) + C$$

Integrate. Don't forget the constant!

$x = 2.5 \quad v = 8$

a gives the initial conditions.

$$\frac{1}{2} \times 8^2 = 5\left(2.5 \times 2.5 - \frac{2.5^2}{2}\right) + C$$

$$C = 32 - 5 \times \frac{2.5^2}{2} = 16.375$$

$$v^2 = 10\left(2.5x - \frac{x^2}{2}\right) + 2 \times 16.375$$

$$v^2 = 25x - 5x^2 + 32.75$$

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Further dynamics
Exercise F, Question 3

Question:

A particle P of mass 0.6 kg moves along the positive x-axis under the action of a single force which is directed towards the origin O and has magnitude $\frac{k}{(x+2)^2}$ N where $OP = x$ metres and k is a constant. Initially P is moving away from O. At $x = 2$ the speed of P is 8 m s^{-1} and at $x = 10$ the speed of P is 2 m s^{-1}.

a Find the value of k.

The particle first comes to instantaneous rest at the point B.

b Find the distance OB.

Solution:

a $F = ma$

$-\frac{k}{(x+2)^2} = 0.6a$

The force is a function of x so use $a = v\frac{dv}{dx}$.

$0.6v\frac{dv}{dx} = -\frac{k}{(x+2)^2}$

$0.6\int v\,dv = -\int \frac{k}{(x+2)^2}\,dx$

Separate the variables and integrate.

$0.3v^2 = \frac{k}{(x+2)} + c$

$x = 2, v = 8 \quad 0.3\times 8^2 = \frac{k}{4} + c$

$x = 10, v = 2 \quad 0.3\times 2^2 = \frac{k}{12} + c$

Use the given information to obtain a pair of simultaneous equations in k and c.

Subtract: $0.3(8^2 - 2^2) = \frac{k}{4} - \frac{k}{12}$

$0.3\times 60 = \frac{k}{6}$

$k = 0.3\times 60\times 6 = 108$

Solve to find k.

b From above $0.3\times 4 = \frac{k}{12} + c$

$c = 1.2 - \frac{108}{12} = -7.8$

$\therefore\ 0.3v^2 = \frac{108}{(x+2)} - 7.8$

Find c to complete the expression for v^2.

$v = 0 \quad 0 = \frac{108}{x+2} - 7.8$

$7.8(x+2) = 108$

$x = \frac{108}{7.8} - 2 = 11.84\ldots$

The distance OB is 11.8 m (3 s.f.).

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Further dynamics
Exercise F, Question 4

Question:

A particle P moves along the x-axis in such a way that at time t seconds its distance x metres from the origin O is given by $x = 3\sin\left(\frac{\pi t}{4}\right)$.

a Prove that P moves with simple harmonic motion.
b Write down the amplitude and the period of the motion.
c Find the maximum speed of P.
The points A and B are on the same side of O with $OA = 1.2\text{ m}$ and $OB = 2\text{ m}$.
d Find the time taken by P to travel directly from A to B.

Solution:

a $x = 3\sin\left(\frac{\pi}{4}t\right)$

$\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

$\ddot{x} = -3\left(\frac{\pi}{4}\right)^2 \sin\left(\frac{\pi}{4}t\right)$

> Differentiate $x = 3\sin\left(\frac{\pi}{4}t\right)$ twice.

$\ddot{x} = -\left(\frac{\pi}{4}\right)^2 x$

> Obtain an equation of the form $\ddot{x} = -\omega^2 x$.

$\therefore$ S.H.M.

b amplitude $= 3\,m$

period $= \frac{2\pi}{\omega} = 2\pi \times \frac{4}{\pi} = 8s$

c From **a** $\dot{x} = \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)$

> Or use $v_{max} = a\omega$.

$\Rightarrow$ maximum speed $= \frac{3\pi}{4}$ m s^{-1}

(or 2.36 m s^{-1} (3 s.f.))

d

$x = 3\sin\left(\frac{\pi}{4}t\right)$

At A, $x = 1.2$ $\quad 1.2 = 3\sin\left(\frac{\pi}{4}t_a\right)$

$t_a = \frac{4}{\pi}\sin^{-1}\left(\frac{1.2}{3}\right)$

At B, $x = 2$ $\quad t_b = \frac{4}{\pi}\sin^{-1}\left(\frac{2}{3}\right)$

Time $A \to B = \frac{4}{\pi}\left[\sin^{-1}\left(\frac{2}{3}\right) - \sin^{-1}\left(\frac{1.2}{3}\right)\right]$

$= 0.4051$

> Leave calculator work as late as possible.

The time to go directly from A to B is 0.405 s (3 s.f.)

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Further dynamics
Exercise F, Question 5

Question:

A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 1.2 m and modulus of elasticity λ N. The other end of the string is attached to a fixed point A. The particle is hanging in equilibrium at the point O, which is 1.4 m vertically below A.

a Find the value of λ.

The particle is now displaced to a point B, 1.75 m vertically below A, and released from rest.

b Prove that while the string is taut P moves with simple harmonic motion.

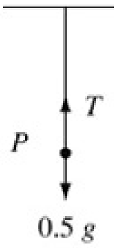
c Find the period of the simple harmonic motion.

d Calculate the speed of P at the first instant when the string becomes slack.

e Find the greatest height reached by P above O.

Solution:

a



In equilibrium:

$R(\uparrow)\, T = 0.5g$

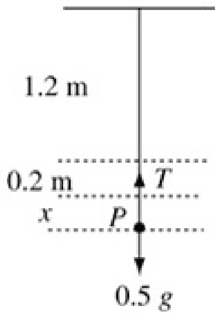
Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$0.5g = \frac{\lambda \times 0.2}{1.2}$$

$$\lambda = 0.5g \times \frac{1.2}{0.2}$$

$$\therefore \lambda = 3g \quad (\text{or } 29.4)$$

b



For oscillations:

$$F = ma$$

$$0.5g - T = 0.5\,\ddot{x}$$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{3g(0.2 + x)}{1.2}$$

$$\therefore\ 0.5g - \frac{3g(0.2 + x)}{1.2} = 0.5\,\ddot{x}$$

$$\ddot{x} = -\frac{3g}{0.5 \times 1.2}x = -5gx$$

$\therefore$ S.H.M.

Of form $\ddot{x} = \omega^2 x$.

c $\omega^2 = 5g$

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5g}} = 0.8975\ldots$$

The period is 0.898 s (3 s.f.).

From $\ddot{x} = -5gx$.

d String becomes slack when $x = -0.2$ m.

amplitude $= 0.35$ m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = 5g(0.35^2 - 0.2^2)$$

$$v = 2.010\ldots$$

The speed is $2.01\ \text{m s}^{-1}$ (3 s.f.).

Use the exact value for ω^2.

e $v^2 = u^2 + 2as$

$$0 = 2.010^2 - 2 \times 9.8s$$

$$s = \frac{2.010^2}{2 \times 9.8} = 0.2061\ldots$$

Once the string is slack the particle moves freely under gravity.

Distance above $O = 0.2 + 0.2061\ldots$

$= 0.406$ m (3 s.f.)

The particle is 0.2 m above O when the string becomes slack.

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Further dynamics
Exercise F, Question 6

Question:

A spacecraft S of mass m is moving in a straight line towards the centre of the Earth. When the distance of S from the centre of the Earth is x metres, the force exerted by the Earth on S has magnitude $\frac{k}{x^2}$, where k is a constant, and is directed towards the centre of the Earth.

a By modelling the Erth as a sphere of radius R and S as a particle, show that $k = mgR^2$.

The spacecraft starts from rest when $x = 5R$.

b Assuming that air resistance can be ignored find the speed of S as it crashes onto the Earth's surface.

Solution:

a $F = \frac{k}{x^2}$

when $x = R, F = mg$

$\therefore \frac{k}{R^2} = mg, \quad k = mg\,R^2$

When $x = R$, S is on the surface of the Earth and the force exerted by the Earth on S is mg.

b $\text{Force} = -\frac{mg\,R^2}{x^2}$

$F = ma$

$-\frac{mg\,R^2}{x^2} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$

The force is in the direction of *decreasing* x.

The force is a function of x so use $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$.

$-\int \frac{gR^2}{x^2}\,\mathrm{d}x = \int v\,\mathrm{d}v$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$

$x = 5R, v = 0 \quad c = \frac{-gR^2}{5R}$

$\therefore v^2 = 2g\frac{R^2}{x} - \frac{2g\,R^2}{5R}$

When $x = R$ $v^2 = 2g\frac{R^2}{R} - \frac{2g\,R^2}{5R}$

$v^2 = \frac{8Rg}{5}$

The speed of the spacecraft is

$\sqrt{\left(\frac{8\,Rg}{5}\right)}$ or $2\sqrt{\left(\frac{2\,Rg}{5}\right)}$

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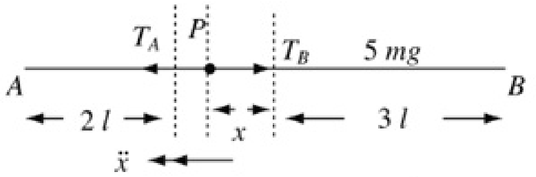
Further dynamics
Exercise F, Question 7

Question:

A particle P of mass m is attached to the mid-point of a light elastic string of natural length $4l$ and modulus of elasticity $5mg$. One end of the string is attached to a fixed point A and the other end to a fixed point B, where A and B lie on a smooth horizontal surface and $AB = 6l$. The particle is held at the point C where A, C and B are collinear and $AC = \frac{9l}{4}$, and released from rest.

a Prove that P moves with simple harmonic motion.
Find, in terms of g and l,
b the period of the motion,
c the maximum speed of P.

Solution:



$\lambda = 5\,mg$

a Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T_A = \frac{5mg(l-x)}{2l}$$

$$T_B = \frac{5mg(l+x)}{2l}$$

Consider the particle to be attached to two strings, AP and PB, both with natural length $2l$ and modulus $5\,mg$.

$$F = ma$$

$$T_A - T_B = m\ddot{x}$$

$$\frac{5mg(l-x)}{2l} - \frac{5mg(l+x)}{2l} = m\ddot{x}$$

$$-\frac{5mgx}{l} = m\ddot{x}$$

$$\ddot{x} = -\frac{5gx}{l}$$

$\therefore$ S.H.M.

b $\omega^2 = \dfrac{5g}{l}$ period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{l}{5g}}$

The period is $2\pi\sqrt{\left(\dfrac{l}{5g}\right)}$

c amplitude $= \dfrac{3l}{4}$

Find the amplitude from the given information.

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a = \sqrt{\frac{5g}{l}} \times \frac{3l}{4} = \frac{3}{4}\sqrt{5gl}$$

Maximum speed when $x = 0$.

The maximum speed is $\dfrac{3}{4}\sqrt{5gl}$.

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Further dynamics
Exercise F, Question 8

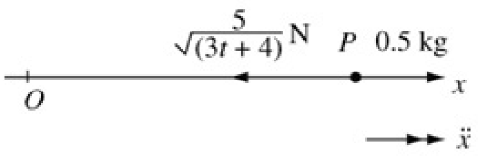
Question:

A particle P of mass 0.5 kg is moving along the x-axis, in the positive x-direction. At time t seconds (where $t > 0$) the resultant force acting on P has magnitude $\frac{5}{\sqrt{(3t+4)}}$ N and is directed towards the origin O. When $t = 0$, P is moving through O with speed 12 m s^{-1}.

a Find an expression for the velocity of P at time t seconds.
b Find the distance of P from O when P is instantaneously at rest.

Solution:

a



$$F = ma$$

$$-\frac{5}{\sqrt{(3t+4)}} = 0.5\,\ddot{x}$$

$$\ddot{x} = -10(3t+4)^{-\frac{1}{2}}$$

$$\dot{x} = -\frac{10}{\frac{1}{2}\times 3}(3t+4)^{\frac{1}{2}} + c$$

$$t = 0 \quad \dot{x} = 12 \quad 12 = -\frac{20}{3}\sqrt{4} + c$$

$$c = 12 + \frac{40}{3} = \frac{76}{3}$$

$$\therefore\ \dot{x} = -\frac{20}{3}(3t+4)^{\frac{1}{2}} + \frac{76}{3}$$

b $x = -\frac{20}{3\times\frac{3}{2}\times 3}(3t+4)^{\frac{3}{2}} + \frac{76}{3}t + A$ ← Integrate line above.

$$t = \ \ x = 0 \therefore A = \frac{40}{27}\times 4^{\frac{3}{2}} = \frac{320}{27}$$

P at rest $\Rightarrow \frac{76}{3} = \frac{20}{3}(3t+4)^{\frac{1}{2}}$ ← Using result from a.

$$\left(\frac{76}{20}\right)^2 = 3t+4$$

$$t = \frac{1}{3}\left[\left(\frac{76}{20}\right)^2 - 4\right]$$

$$t = 3.48$$

When $t = 3.48$ ← $3t+4 = \left(\frac{76}{20}\right)^2$, so use the exact value here.

$$x = -\frac{40}{27}\left(\frac{76}{20}\right)^3 + \frac{76}{3}\times 3.48 + \frac{320}{27}$$

$$x = 18.72$$

P is 18.7 m from O (3 s.f.)

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Further dynamics
Exercise F, Question 9

Question:

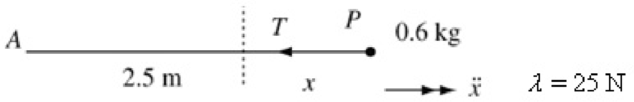
A particle P of mass 0.6 kg is attached to one end of a light elastic spring of natural length 2.5 m and modulus of elasticity 25 N. The other end of the spring is attached to a fixed point A on the smooth horizontal table on which P lies. The particle is held at the point B where $AB = 4$ m and released from rest.

a Prove that P moves with simple harmonic motion.

b Find the period and amplitude of the motion.

c Find the time taken for P to move 2 m from B.

Solution:



a $F = ma$

$-T = 0.6\,\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{25}{2.5}x = 10x$$

$$\therefore\ 0.6\,\ddot{x} = -10x$$

$$\ddot{x} = -\frac{10}{0.6}x$$

$\therefore$ S.H.M.

b $\omega^2 = \dfrac{10}{0.6}$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.6}{10}} = 1.539\ldots$$

$$\text{period} = 1.54\ \text{s (3 s.f.)}$$

$$\text{amplitude} = (4 - 2.5)\text{m} = 1.5\ \text{m}$$

c $x = a\cos\omega t$ ← B is an end-point.

$$x = 1.5\cos\left(\sqrt{\frac{10}{0.6}}t\right)$$

$x = -0.5\text{m}\quad -0.5 = 1.5\cos\left(\sqrt{\dfrac{10}{0.6}}t\right)$ ← D is on the other side of the centre from O so x is negative.

$$t = \sqrt{\frac{0.6}{10}}\cos^{-1}\left(-\frac{0.5}{1.5}\right) = 0.4680\ldots$$

P takes 0.468s to move 2 m from B (3 s.f.).

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Further dynamics
Exercise F, Question 10

Question:

A particle P of mass 0.4 kg is attached to the mid-point of a light elastic string of natural length 1.2 m and modulus of elasticity 2.5 N. The ends of the string are attached to points A and B on a smooth horizontal table where $AB = 2$ m . The particle P is released from rest at the point C on the table, where A, C and B lie in a straight line and $AC = 0.7$ m .

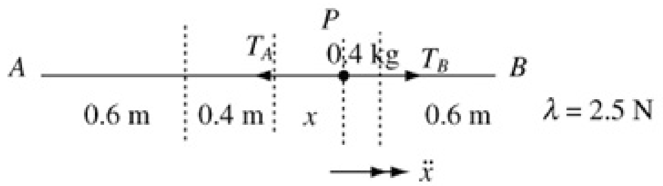
a Show that P moves with simple harmonic motion.

b Find the period of the motion.

The point D lies between A and B and $AD = 0.85$ m .

c Find the time taken by P to reach D for the first time.

Solution:



a $F = ma$

$T_B - T_A = 0.4\,\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T_A = \dfrac{2.5(0.4 + x)}{0.6}$

$T_B = \dfrac{2.5(0.4 - x)}{0.6}$

$\therefore \dfrac{2.5(0.4 - x)}{0.6} - \dfrac{2.5(0.4 + x)}{0.6} = 0.4\,\ddot{x}$

$-2 \times \dfrac{2.5x}{0.6} = 0.4\,\ddot{x}$

$\ddot{x} = -\dfrac{2 \times 2.5}{0.6 \times 0.4}x$

$\therefore$ S.H.M

Consider P to be attached to two strings, each of natural length 0.6 m and modulus 2.5 N.

b $\omega^2 = \dfrac{2 \times 2.5}{0.6 \times 0.4} = \dfrac{5}{0.24}$

period $= \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{0.24}{5}} = 1.376\ldots$

The period is 1.38 s (3 s.f.)

c $x = a\cos\omega t$

For time from B (an end-point)

At D, $x = 1 - 0.85 = 0.15$

$0.15 = 0.3\cos\left(\sqrt{\dfrac{5}{0.24}}t\right)$

D and C are on the same side of the centre, so x is positive.

$\sqrt{\dfrac{5}{0.24}}t = \cos^{-1} 0.5$

$t = \sqrt{\dfrac{0.24}{5}}\cos^{-1} 0.5$

$t = 0.2294\ldots$

P takes 0.229s (3 s.f.) to reach D.