Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

A stone was thrown with velocity 20 m s^{-1} at an angle of elevation of $30°$ from the top of a vertical cliff. The stone moved freely under gravity and reached the sea 5 s after it was thrown. Find

a the vertical height above the sea from which the stone was thrown,

b the horizontal distance covered by the stone from the instant when it was thrown until it reached the sea,

c the magnitude and direction of the velocity of the stone when it reached the sea.

Solution:

$$R(\rightarrow) \quad u_x = 20\cos 30^\circ = 10\sqrt{3}$$
$$R(\uparrow) \quad u_y = 20\sin 30^\circ = 10$$

The first step in most projectile questions is to resolve the velocity of projection horizontally and vertically.

a

$$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$$
$$= 10\times 5 - 4.9\times 5^2$$
$$= -72.5$$

The vertical motion is motion under constant acceleration of magnitude 9.8 m s^{-2}.

The vertical height above the sea from which the stone was thrown is 73 m (2 s.f.).

b

$$R(\rightarrow) \quad \text{distance} = \text{speed}\times\text{time}$$
$$= 10\sqrt{3}\times 5$$
$$= 50\sqrt{3} = 86.602\ldots$$

The horizontal motion is motion with constant speed. In this question, the horizontal component of the velocity is $10\sqrt{3} \text{ m s}^{-1}$ throughout. It never changes.

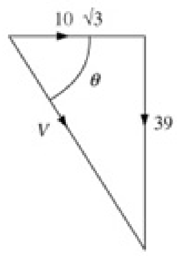
The horizontal distance covered by the stone is 87 m (2 s.f.).

c

$$R(\uparrow) \quad v^2 = u^2 + 2as$$
$$v_y^2 = 10^2 - 2\times 9.8\times(-72.5)$$
$$= 1521$$
$$v_y = \sqrt{1521} = -39$$

The stone is moving downwards as it reaches the sea, so the negative square root of 1521 is the appropriate root.

As the stone reaches the sea



$$V^2 = u_x^2 + v_y^2$$
$$= (10\sqrt{3})^2 + (-39)^2 = 1821$$
$$V = \sqrt{1821} = 42.673\ldots$$
$$\tan\theta = \frac{39}{10\sqrt{3}} \Rightarrow \theta \approx 66.05^\circ$$

Velocity is a vector quantity. Finding the magnitudes and directions of vectors is found in Chapter 6 of the M1 book.

The magnitude of the velocity of the stone as it reaches the sea is 43 m s^{-1} (2 s.f.), and the direction is 66°, (nearest degree), below the horizontal.

As the numerical value $g = 9.8$ has been used, you should give your answers to 2 significant figures. Answers cannot be more accurate than the data used to calculate them.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

A darts player throws darts at a dart board which hangs vertically. The motion of a dart is modelled as that of a particle moving freely under gravity. The darts move in a vertical plane which is perpendicular to the plane of the dart board. A dart is thrown horizontally with speed 12.6 m s^{-1}. It hits the board at a point which is 10 cm below the level from which it was thrown.

a Find the horizontal distance from the point where the dart was thrown to the dart board.

The darts player moves his position. He now throws a dart from a point which is at a horizontal distance of 2.5 m from the dart board. He throws the dart at an angle of elevation α to the horizontal where $\tan \alpha = \frac{7}{24}$. The dart hits the board at a point which is at the same level as the point from which it was thrown.

b Find the speed with which the dart was thrown.

Solution:

a The initial components of the velocity are

$$\text{R}(\rightarrow) \quad u_x = 12.6$$
$$\text{R}(\downarrow) \quad u_y = 0$$

As the dart is thrown horizontally, the vertical component of the initial velocity is zero.

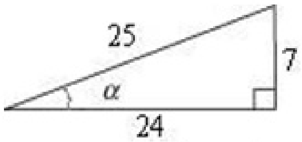
$$\text{R}(\downarrow) \quad s = ut + \tfrac{1}{2}at^2$$
$$0.1 = 0 + 4.9t^2$$
$$t^2 = \frac{0.1}{4.9} = \frac{1}{49} \Rightarrow t = \frac{1}{7}$$
$$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$= 12.6 \times \tfrac{1}{7} = 1.8$$

You usually work in metres, kilograms and seconds. Here, as the units of g are m s^{-2}, you need to convert 10 cm to 0.1 m before using the standard formula $s = ut + \frac{1}{2}at^2$ to find t.

The horizontal distance from the point where the dart was thrown to the dart board is 1.8 m.

b $\tan\alpha = \frac{7}{24} \Rightarrow \sin\alpha = \frac{7}{25}, \cos\alpha = \frac{24}{25}$

As $7^2 + 24^2 = 25^2$, using



you can write down $\sin\alpha$ and $\cos\alpha$.

Let U m s^{-1} be the speed of projection.

$$\text{R}(\rightarrow) \quad u_x = U\cos\alpha = \frac{24U}{25}$$
$$\text{R}(\uparrow) \quad u_y = U\sin\alpha = \frac{7U}{25}$$
$$\text{R}(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$$
$$2.5 = \frac{24U}{25} \times t \Rightarrow t = \frac{62.5}{24U} \quad (1)$$
$$\text{R}(\uparrow) \quad s = ut + \tfrac{1}{2}at^2$$
$$0 = \frac{7U}{25} \times t - 4.9 \times t^2$$

As $t \neq 0$, dividing by t

$$0 = \frac{7U}{25} - 4.9 \times t$$
$$t = \frac{7U}{25 \times 4.9} = \frac{62.5}{24U}, \text{ from (1)}$$
$$U^2 = \frac{62.5 \times 25 \times 4.9}{7 \times 24} = 45.572\ldots$$
$$U = 6.750\ldots$$

The speed with which the dart was thrown is 6.8 m s^{-1} (2 s.f.).

The separate equations for the distances travelled horizontally and vertically give simultaneous equations in U and t from which both can be found. In this question, you are not asked for t but it is quite common to be asked for t and you could substitute your answer for U into (1) to find it.

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Review Exercise

Exercise A, Question 3

Question:

A particle is projected with velocity $(8\mathbf{i} + 10\mathbf{j})$ m s^{-1}, where **i** and **j** are unit vectors horizontally and vertically respectively, from a point O at the top of a cliff and moves freely under gravity.

Six seconds after projection, the particle strikes the sea at the point S. Calculate

a the horizontal distance between O and S,

b the vertical distance between O and S.

At time T seconds after projection, the particle is moving with velocity $(8\mathbf{i} - 14.5\mathbf{j})$ m s^{-1}.

c Find the value of T and the position vector, relative to O, of the particle at this instant.

Solution:

The initial components of the velocity are

R(→) $u_x = 8$
R(↑) $u_y = 10$

When the velocity of projection is given as a vector in terms of **i** and **j**, the usual first step of resolution is simpler. The horizontal component is 8 and the vertical 10.

a

R(→) distance = speed × time
$= 8 \times 6 = 48$

The horizontal distance between O and S is 48 m.

b

R(↑) $s = ut + \frac{1}{2}at^2$
$= 10 \times 6 - 4.9 \times 6^2 = -116.4$

The vertical distance between O and S is 120 m (2 s.f.).

c

R(↑) $v = u + at$
$-14.5 = 10 - 9.8T$
$T = \frac{24.5}{9.8} = \frac{245}{98} = \frac{5}{2} = 2\frac{1}{2}$

Considering the **j** components of the velocity, $v = -14.5$ and $u = 10$. Using $v = u + at$ with $a = -9.8$ then gives you the time.

R(→) distance = speed × time
$= 8 \times \frac{5}{2} = 20$

The **i** component of the velocity remains 8 throughout the motion.

20 is the **i** component of the position vector.

R(↑) $s = ut + \frac{1}{2}at^2$
$= 10 \times \frac{5}{2} - 4.9 \times \left(\frac{5}{2}\right)^2 = -\frac{45}{8}$

$-\frac{45}{8} = -5.625$ is the **j** component of the position vector.

The position vector of the particle after $2\frac{1}{2}$ seconds is $\left(20\mathbf{i} - \frac{45}{8}\mathbf{j}\right)$ m.

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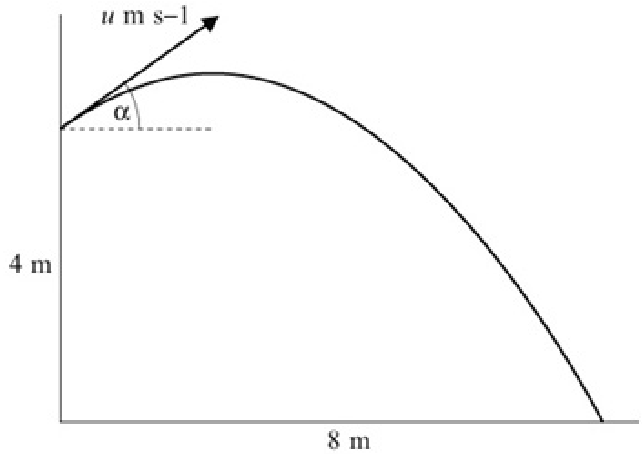
Review Exercise
Exercise A, Question 4

Question:



A ball is thrown from a point 4 m above horizontal ground. The ball is projected at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. The ball hits the ground at a point which is a horizontal distance 8 m from its point of projection, as shown in the figure above. The initial speed of the ball is u m s^{-1} and the time of flight is T seconds.

a Prove that $uT = 10$.

b Find the value of u.

As the ball hits the ground, its direction of motion makes an angle ϕ with the horizontal.

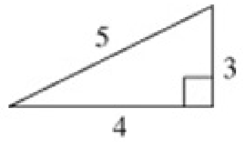
c Find $\tan\phi$.

Solution:

$\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

$R(\rightarrow) \quad u_x \quad = \quad u\cos\alpha = \frac{4}{5}u$

$R(\uparrow) \quad u_y \quad = \quad u\sin\alpha = \frac{3}{5}u$



This diagram shows that if $\tan\alpha = \frac{3}{4}$, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

a

$R(\rightarrow) \quad \text{distance} = \text{speed} \times \text{time}$

$8 = \frac{4}{5}u \times T$

$uT = 8 \times \frac{5}{4} = 10,$

as required

The horizontal component of the velocity remains unchanged throughout the question.

The separate equations for the distances travelled horizontally and vertically give simultaneous equations in u and T.

b

$R(\uparrow) \quad s = ut + \frac{1}{2}at^2$

$-4 = \frac{3}{5}uT - 4.9T^2$

$-4 = \frac{3}{5} \times 10 - 4.9T^2$

$4.9T^2 = 6 + 4 = 10$

$T^2 = \frac{10}{4.9} = \frac{100}{49} \Rightarrow T = \frac{10}{7}$

The ball descends 4 m before hitting the ground. So, if the upwards direction is taken as positive, $s = -4$.

From part **a**

$uT = 10$

$u \times \frac{10}{7} = 10 \Rightarrow u = 7$

c At the point where the ball hits the ground

$R(\uparrow) \quad v = u + at$

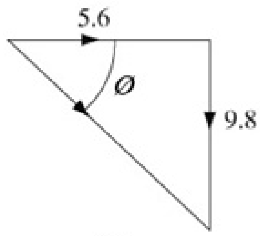
$v = v_y, u = 7\sin\alpha = 7 \times \frac{3}{5} = \frac{21}{5}, t = \frac{10}{7}, a = -9.8$

$v_y = \frac{21}{5} - 9.8 \times \frac{10}{7} = -9.8$

$u_x = 7\cos\alpha = 7 \times \frac{4}{5} = 5.6$

The vertical component of the velocity as the ball hits the ground has to be found, but you can do this in several ways. When you have a choice, $v = u + at$ is usually the simplest formula to use.

As the ball hits the ground



$\tan\varphi = \frac{9.8}{5.6} = 1.75 \ \varphi$

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Review Exercise
Exercise A, Question 5

Question:

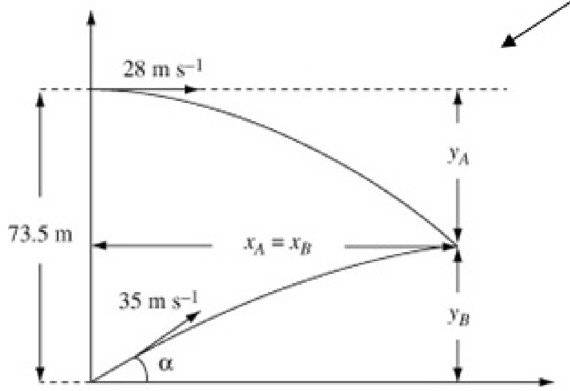
A vertical cliff is 73.5 m high. Two stones A and B are projected simultaneously.

Stone A is projected horizontally from the top of a cliff with speed 28 m s^{-1}. Stone B is projected from the bottom of the cliff with speed 35 m s^{-1} at an angle α above the horizontal. The stones move freely under gravity in the same vertical plane and collide in mid-air.

a By considering the horizontal motion of each stone, prove that $\cos\alpha = \frac{4}{5}$.

b Find the time which elapses between the instant when the stones are projected and the instant when they collide.

Solution:



It helps you to draw a sketch to illustrate the data of the question. This shows that, for A and B to collide, two conditions have to be met. Firstly, the horizontal distances moved by A and B (here labelled x_A and x_B) have to be equal. Secondly the distance fallen by $A(y_A)$ and the distance moved upwards by $B(y_B)$ must add up to the height of the cliff, 73.5 m.

a For A,

$$\text{R}(\rightarrow)\ \text{distance} = \text{speed} \times \text{time}$$
$$x_A = 28t$$

For B,

$$\text{R}(\rightarrow)\ u_x = 35\cos\alpha$$
$$\text{R}(\uparrow)\ u_y = 35\sin\alpha$$
$$\text{R}(\rightarrow)\ \text{distance} = \text{speed} \times \text{time}$$
$$x_B = 35\cos\alpha \times t$$
$$x_A = x_B$$
$$28\not{t} = 35\cos\alpha\,\not{t}$$
$$\cos\alpha = \frac{28}{35} = \frac{4}{5}, \text{ as required}$$

For collision, the horizontal distance travelled by the stones must be the same.

As $t \neq 0$ when the stones collide, you can divide both sides of the equation by t.

b For A,

$$\text{R}(\downarrow)\ s = ut + \frac{1}{2}at^2$$
$$y_A = 0 + 4.9t^2$$

For B,

$$\text{R}(\uparrow)\ s = ut + \frac{1}{2}at^2$$
$$y_B = 35\sin\alpha \times t - 4.9t^2$$
$$\cos\alpha = \frac{4}{5} \Rightarrow \sin\alpha = \frac{3}{5}$$
$$y_B = 35 \times \frac{3}{5} \times t - 4.9t^2 = 21t - 4.9t^2$$
$$y_A + y_B = 73.5$$
$$\not{4.9}t^2 + 21t - \not{4.9}t^2 = 73.5$$
$$t = \frac{73.5}{21} = 3.5$$

$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$ implies $\sin\alpha = \frac{3}{5}$ or you could use a 3, 4, 5 triangle to see this.

The two distances moved vertically must add up to the height of the cliff.

The time elapsed is 3.5 s.

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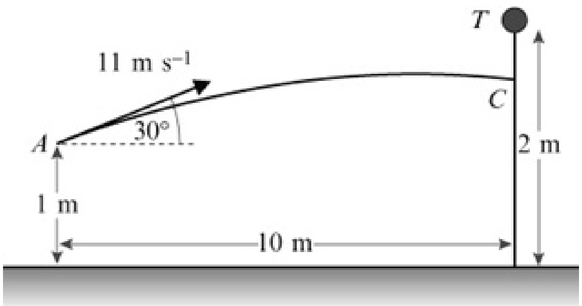
Review Exercise
Exercise A, Question 6

Question:



The object of a game is to throw a ball B from a point A to hit a target T which is placed at the top of a vertical pole, as shown in the figure above. The point A is 1 m above horizontal ground and the height of the pole is 2 m. The pole is a horizontal distance of 10 m from A.

The ball B is projected from A with speed 11 m s^{-1} at an angle of elevation of $30°$.

The ball hits the pole at C. The ball B and the target T are modelled as particles.

a Calculate, to 2 decimal places, the time taken for B to move from A to C.

b Show that C is approximately 0.63 m below T.

The ball is thrown again from A.

The speed of projection of B is increased to $V\text{ m s}^{-1}$, the angle of elevation remaining $30°$. This time B hits T.

c Calculate the value of V.

d Explain why, in practice, a range of values of V would result in B hitting the target.

Solution:

$\text{R}(\rightarrow)\quad u_x = 11\cos 30° = 5.5\sqrt{3}$

$\text{R}(\uparrow)\quad u_y = 11\sin 30° = 5.5$

a $\text{R}(\rightarrow)$ distance = speed × time

$$10 = 5.5\sqrt{3} \times t$$

$$t = \frac{10}{5.5\sqrt{3}} = 1.049\,727\ldots$$

The time taken to move from A to C is 1.05 seconds (2 d.p.).

> If the question specifies a particular accuracy, you must give your answer to that accuracy to gain full marks.

b

$\text{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$= 5.5 \times 1.05 - 4.9 \times 1.05^2 \approx 0.374$

The distance below T is $(1 - 0.374)\text{ m} \approx 0.63\text{ m}$, as required.

> The ball starts 1 m above the ground and hits the pole approximately 0.37 m higher than it started. That leaves another 0.63 m to reach 2 m above the ground.

c

$\text{R}(\rightarrow)\quad u_x = V\cos 30°$

$\text{R}(\uparrow)\quad u_y = V\sin 30°$

$\text{R}(\rightarrow)$ distance = speed × time

$$10 = V\cos 30° \times t$$

$$Vt = \frac{10}{\cos 30°} = \frac{20}{\sqrt{3}} \quad (1)$$

$\text{R}(\uparrow)\quad s = ut + \frac{1}{2}at^2$

$$1 = V\sin 30° \times t - 4.9t^2 \ldots (2)$$

$$1 = \frac{20}{\sqrt{3}} \times \frac{1}{2} - 4.9\left(\frac{20}{V\sqrt{3}}\right)^2$$

$$\frac{4.9 \times 400}{3V^2} = \frac{10}{\sqrt{3}} - 1 = 4.773\ldots$$

$$V^2 = \frac{4.9 \times 400}{3 \times 4.773\ldots} = 136.866\ldots$$

$$V = 11.699\ldots = 12 \text{ (2 s.f.)}.$$

> These two equations in V and t enable either to be found by elimination. In this case t has to be eliminated.
> From (1), $t = \frac{20}{V\sqrt{3}}$, and this is substituted into (2).

d B and T are not particles but take up space; they have extension. This would allow a range of values of V resulting in hitting the target.

> For example, the target takes up space and can be hit at its top, its bottom or anywhere in between.

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Review Exercise
Exercise A, Question 7

Question:

A particle P, projected from a point O on horizontal ground, moves freely under gravity and hits the ground again at A.

Taking O as origin, OA as the x-axis and the upward vertical at O as the y-axis, the equation of the path of P is

$$y = x - \frac{x^2}{500}$$

where x and y are measured in metres.

a By finding $\frac{dy}{dx}$, show that P was projected from O at an angle of 45° to the horizontal.

b Find the distance OA and the greatest vertical height attained by P above OA.

c Find the speed of projection of P.

d Find, to the nearest second, the time taken by P to move from O to A.

Solution:

a $y = x - \frac{x^2}{500}$

$\frac{dy}{dx} = 1 - \frac{x}{250}$

When $x = 0, \frac{dy}{dx} = 1$

The gradient of the direction of motion at O is 1 and the angle is given by $\tan\theta = 1$. Hence $\theta = 45°$, as required.

> $\frac{dy}{dx}$ is the gradient of the path of the projectile and, at any point of the path, this is the direction of motion of the particle.

b P is at A when $y = 0$

$0 = x - \frac{x^2}{500} = x\left(1 - \frac{x}{500}\right)$

At A, $x \neq 0$

$1 - \frac{x}{500} = 0 \Rightarrow x = 500$

$OA = 500$ m

The greatest height is reached when $\frac{dy}{dx} = 0$

$1 - \frac{x}{250} = 0 \Rightarrow x = 250$

$y = 250 - \frac{250^2}{500} = 125$

The greatest height reached is 125 m.

> The greatest height corresponds to the maximum point on the curve and, as this is a stationary value, using the knowledge you learnt in the C2 module, this is found by putting $\frac{dy}{dx} = 0$. The M2 module requires knowledge of the C1, C2 and C3 specifications.

c Let the speed of projection be U m s^{-1}.

Initially R(↑) $u_y = U \sin 45° = \frac{U}{\sqrt{2}}$

At the greatest height

R(↑) $u = \frac{U}{\sqrt{2}}, v = 0, s = 125, a = -9.8$

$v^2 = u^2 + 2as$

$0^2 = \frac{U^2}{2} - 2 \times 9.8 \times 125$

$U^2 = 4 \times 9.8 \times 125 = 4900 \Rightarrow U = 70$

The speed of projection is 70 m s^{-1}.

> At the greatest height the vertical component of the velocity is 0. The value of s is the height found in **b**.

d R(→) $u_x = 70 \sin 45° = \frac{70}{\sqrt{2}}$

R(→) distance = speed × time

$500 = \frac{70}{\sqrt{2}} t \Rightarrow t = \frac{500\sqrt{2}}{70} = 10.101\ldots$

The time taken for P to move from O to A is 10 s (nearest second).

> The horizontal component of the velocity is $u_x = \frac{70}{\sqrt{2}}$ throughout the motion.

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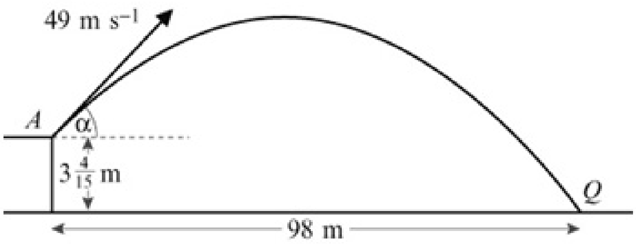
Review Exercise
Exercise A, Question 8

Question:



A golf ball is projected with speed 49 m s^{-1} at an angle of elevation α from a point A on the first floor of a driving range. Point A is at a height of $3\frac{4}{15}$ m above horizontal ground. The ball first strikes the ground at a point Q which is at a horizontal distance of 98 m from the point A, as shown in the figure above.

a Show that

$$6\tan^2\alpha - 30\tan\alpha + 5 = 0.$$

b Hence find the two possible angles of elevation.

c Find the smallest possible time of direct flight from A to Q.

Solution:

a $R(\rightarrow)$ $u_x = 49\cos\alpha$

$R(\uparrow)$ $u_y = 49\sin\alpha$

$R(\rightarrow)$ distance = speed × time

$98 = 49\cos\alpha \times t \Rightarrow t = \dfrac{2}{\cos\alpha}$

These two equations are simultaneous equations in α and t. You have to eliminate t first and then use trigonometric identities to show that the printed answer is correct.

At Q

$R(\uparrow)$ $s = ut + \frac{1}{2}at^2$

$-\dfrac{49}{15} = 49\sin\alpha \times t - 4.9t^2$

From A to Q, the ball descends a vertical distance of $3\frac{4}{15}$ m $= \frac{49}{15}$ m.

Dividing by 4.9

$-\frac{10}{15} = -\frac{2}{3} = 10\sin\alpha t - t^2$

Multiplying by 3 and rearranging

$3t^2 - 30\sin\alpha t - 2 = 0$

Substituting $t = \dfrac{2}{\cos\alpha}$

$\dfrac{12}{\cos^2\alpha} - 30\sin\alpha \times \dfrac{2}{\cos\alpha} - 2 = 0$

Using $\frac{1}{\cos\alpha} = \sec\alpha$ and $\sec^2\alpha = 1 + \tan^2\alpha$.

$12\sec^2\alpha - 60\tan\alpha - 2 = 0$

$12(\tan^2\alpha + 1) - 60\tan\alpha - 2 = 0$

$12\tan^2\alpha - 60\tan\alpha + 10 = 0$

Dividing by 2

$6\tan^2\alpha - 30\tan\alpha + 5 = 0$, as required.

b $\tan\alpha = \dfrac{30 \pm \sqrt{(900 - 120)}}{12} = 4.827\ldots, 0.1726\ldots$

$\alpha \approx 78.3°, 9.79°$

Using the formula $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$ to solve the quadratic in $\tan\alpha$.

To the nearest degree, the possible angles of elevation are 10° and 78°.

c The smallest possible time is given by

$t = \dfrac{2}{\cos 9.79°} \approx 2.029$

There are 2 possible angles. As $t = \dfrac{2}{\cos\alpha}$, the smaller value of t comes from the larger value of $\cos\alpha$, which corresponds to the smaller angle.

$\dfrac{2}{\cos 78.3°} \approx 9.86$.

The smallest possible time of direct flight from A to Q is 2.0 s (2 s.f.).

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Review Exercise
Exercise A, Question 9

Question:

A particle P moves on the x-axis. At time t seconds, its acceleration is $(5-2t)$ m s^{-2}, measured in the direction of x increasing. When $t=0$, its velocity is 6 m s^{-1} measured in the direction of x increasing. Find the time when P is instantaneously at rest in the subsequent motion.

Solution:

$$a = 5-2t$$
$$v = \int a\,dt = \int (5-2t)\,dt$$
$$= 5t - t^2 + C$$

When $t=0, v=6$

The initial condition, that when $t=0$ the velocity is 6 m s^{-1}, enables you to find the value of the constant of integration.

$6 = 0 - 0 + C \Rightarrow C = 6$
Hence
$v = 6 + 5t - t^2$
When P is at rest

$$0 = 6 + 5t - t^2$$

When P is at rest, its velocity is 0.

$$t^2 - 5t - 6 = (t-6)(t+1) = 0$$
$$t = 6, -1$$

$t > 0$
$$t = 6$$
P is at rest at time 6 seconds.

The question asks you for the value of t subsequent to, that is after, $t=0$. So you must pick the positive root of the quadratic.

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Review Exercise
Exercise A, Question 10

Question:

A particle P moves in a straight line in such a way that, at time t seconds, its velocity, v m s^{-1}, is given by

$$v = \begin{cases} 12t - 3t^2, & 0 \le t \le 5 \\ -\dfrac{375}{t^2}, & t > 5. \end{cases}$$

When $t = 0$, P is at the point O.

Calculate the displacement of P from O

a when $t = 5$,

b when $t = 6$.

Solution:

a For $0 \le t \le 5$

$$s = \int v \, dt = \int (12t - 3t^2) \, dt$$
$$= 6t^2 - t^3 + A$$

When $t = 0, s = 0$

$0 = 0 - 0 + A \Rightarrow A = 0$

This constant of integration is 0. Even when it seems obvious that a constant has this value, you should show sufficient working to justify the value 0.

Hence

$s = 6t^2 - t^3$

When $t = 5$

$s = 6 \times 5^2 - 5^3 = 25$

The displacement of P from O when $t = 5$ is 25 m.

b For $t > 5$

$$s = \int v \, dt = \int -375t^{-2} \, dt$$
$$= \frac{-375t^{-1}}{-1} + B = \frac{375}{t} + B$$

To integrate $-\frac{375}{t^2}$, write it as $-375t^{-2}$ and use the formula $\int t^n \, dt = \frac{t^{n+1}}{n+1} + A$.

From **a**, when $t = 5, s = 25$

The end point in **a** is the starting point in **b**.

$25 = \frac{375}{5} + B \Rightarrow B = 25 - 75 = -50$

Hence

$s = \frac{375}{t} - 50$

When $t = 6$

$s = \frac{375}{6} - 50 = 12.5$

The displacement of P from O when $t = 6$ is 12.5 m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 11

Question:

A particle is moving in a straight line Ox.

At time t seconds the acceleration of P is a m s^{-2} and the velocity v m s^{-1} of P is given by

$$v = 2 + 8\sin kt,$$

where k is a constant.

The initial acceleration of P is 4 m s^{-2}.

a Find the value of k.

Using the value of k found in **a**,

b find, in terms of π, the values of t in the interval $0 \le t \le 4\pi$ for which $a = 0$,

c show that $4a^2 = 64 - (v-2)^2$.

Solution:

a $v = 2 + 8\sin kt$

$a = \frac{dv}{dt} = 8k\cos kt$

For any constant k, $\frac{d}{dt}(\sin kt) = k\cos kt$.

When $t = 0, a = 4$

$4 = 8k \Rightarrow k = \frac{1}{2}$

The initial condition, that the acceleration is 4 m s^{-1}, gives an equation in k which you solve.

b $a = 8 \times \frac{1}{2}\cos\frac{1}{2}t = 4\cos\frac{1}{2}t$

When $a = 0$

$\cos\frac{1}{2}t = 0 \Rightarrow \frac{1}{2}t = \frac{\pi}{2}, \frac{3\pi}{2}$

Hence

$t = \pi, 3\pi$

In all differentiation and integration of trigonometric functions, it is assumed that angles are measured in radians. $\cos\theta = 0$ when θ is an odd multiple of $\frac{\pi}{2}$.

c

$$\begin{aligned} 64 - (v-2)^2 &= 64 - \left(8\sin\tfrac{1}{2}t\right)^2 \\ &= 64 - 64\sin^2\tfrac{1}{2}t = 64\left(1 - \sin^2\tfrac{1}{2}t\right) \\ &= 64\cos^2\tfrac{1}{2}t = 4\left(4\cos\tfrac{1}{2}t\right)^2 \\ &= 4a^2, \text{ as required} \end{aligned}$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 12

Question:

An aircraft is situated at rest at a point A on a runway XY which is of length 1400 m. Point A is 77 m from X. The aircraft moves along the runway towards Y with acceleration $\left(10-\frac{4}{5}t\right)$ m s^{-2}, where t seconds is the time from the instant the aircraft started to move.

a Find the speed of the aircraft when $t=6$ and determine the distance travelled in the first 6 seconds of the aircraft's motion.

B is the point such that $AB=\frac{3}{10}AY$.

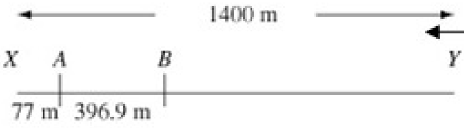
b Find the distance AB.

A safety regulation requires that the aircraft passes point B with a speed of 55 m s^{-1} or more.

c Given that $t=T$ when the aircraft passes B, form an equation for T.

d Show that $T=10.5$ satisfies the equation, and hence determine whether or not the aircraft satisfies this safety regulation as it passes B.

Solution:



A sketch is useful to help you sort out the various distances in this question.

a $v = \int a \, dt = \int \left(10 - \frac{4}{5}t\right) dt$

$= 10t - \frac{2}{5}t^2 + C$

When $t = 0, v = 0$

$0 = 0 + 0 + C \Rightarrow C = 0$

Hence

$v = 10t - \frac{2}{5}t^2$ (1)

When $t = 6$

$v = 10 \times 6 - \frac{2}{5} \times 6^2 = 45.6$

The speed of the aircraft when $t = 6$ is 45.6 m s^{-1}.

This is an exact answer and as g, or any similar approximation, has not been used, the answer should not be rounded.

$s = \int v \, dt = \int \left(10t - \frac{2}{5}t^2\right) dt$

$= 5t^2 - \frac{2}{15}t^3 + D$

Let $s = 0$, when $t = 0$

In principle, the displacement could be measured from any point but it is sensible to measure the displacement from the starting point, which is A.

$0 = 0 - 0 + D \Rightarrow D = 0$

Hence

$s = 5t^2 - \frac{2}{15}t^3$ (2)

When $t = 6$

$s = 5 \times 6^2 - \frac{2}{15} \times 6^3 = 151.2$

The distance travelled in the first 6 s of motion is 151.2 m.

b $AY = (1400 - 77)\text{ m} = 1323\text{ m}$

$AB = \frac{3}{10} AY = 396.9\text{ m}$

c Substituting $s = 396.9$ and $t = T$ into (2)

$396.9 = 5T^2 - \frac{2}{15}T^3$

$\frac{2}{15}T^3 - 5T^2 + 396.9 = 0$

This cubic equation would be very difficult to solve directly and the question only asks you to show that 10.5 satisfies the equation. To do that, you substitute $T = 10.5$ into the left hand side of the equation and show that the calculation gives the value 0.

d Substituting $T = 10.5$ into the left hand side of the answer in **c**

$\frac{2}{15} \times 10.5^3 - 5 \times 10.5^2 + 396.9$

$= 154.35 - 551.25 + 396.9$

$= 551.25 - 551.25 = 0$

$T = 10.5$ satisfies the equation in **c**, as required.

Substituting $T = 10.5$ into equation (1) in **a**

$v = 10 \times 10.5 - \frac{2}{5} \times 10.5^2 = 60.9 > 55$

The aircraft satisfies the safety condition.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 13

Question:

A particle P moves along the x-axis. It passes through the origin O at time $t = 0$ with speed 7 m s^{-1} in the direction of x increasing.

At time t seconds the acceleration of P in the direction of x increasing is $(20 - 6t) \text{ m s}^{-2}$.

a Show that the velocity $v \text{ m s}^{-1}$ of P at time t seconds is given by

$$v = 7 + 20t - 3t^2.$$

b Show that $v = 0$ when $t = 7$ and find the greatest speed of P in the interval $0 \leq t \leq 7$.

c Find the distance travelled by P in the interval $0 \leq t \leq 7$.

Solution:

a

$$v = \int a \, dt = \int (20 - 6t) \, dt$$
$$= 20t - 3t^2 + A$$

When $t = 0, v = 7$

$7 = 0 - 0 + A \Rightarrow A = 7$

Hence

$v = 7 + 20t - 3t^2$, as required.

b When

$t = 7$

$v = 7 + 20 \times 7 - 3 \times 7^2$

$= 7 + 140 - 147 = 0$, as required

For the greatest speed of P

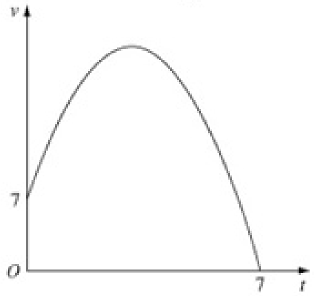
$$\frac{dv}{dt} = a = 20 - 6t = 0$$

$$t = \frac{20}{6} = \frac{10}{3}$$

When $t = \frac{10}{3}$

$$v = 7 + 20 \times \frac{10}{3} - 3 \times \left(\frac{10}{3}\right)^2 = 40\frac{1}{3}$$

A sketch of v against t is



This illustrates that the greatest value of the speed occurs at a maximum point.

The greatest speed of P in the interval $0 \le t \le 7$ is $40\frac{1}{3}$ m s^{-1}.

c

$$s = \int v \, dt = \int (7 + 20t - 3t^2) \, dt$$
$$= 7t + 10t^2 - t^3 + B$$

When $t = 0, s = 0$

$0 = 0 + 0 - B \Rightarrow B = 0$

Hence

$s = 7t + 10t^2 - t^3$

When $t = 7$

$s = 7 \times 7 + 10 \times 7^2 - 7^3 = 196$

Finding the distance travelled is not straightforward if the particle turns round. This happens when $v = 0$. However the sketch in **b** shows that P does not turn round until $t = 7$, so the distance travelled in this interval is found by substituting $t = 7$ into the equation for s.

The distance travelled by P in the interval $0 \le t \le 7$ is 196 m.

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Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 14

Question:

A particle P moves along a straight line. Initially, P is at rest at a point O on the line. At time t seconds (where $t \geq 0$) the acceleration of P is proportional to $(7 - t^2)$ and the displacement of P from O is s metres. When $t = 3$, the speed of P is 6 m s^{-1}.

a Show that

$$s = \frac{1}{24}t^2(42 - t^2).$$

b Find the total distance that P moves before returning to O.

Solution:

a

$$a = k(7-t^2) = 7k - kt^2$$

$$v = \int a \, dt = \int (7k - kt^2) \, dt$$

$$= 7kt - \frac{k}{3}t^3 + A$$

If $a \propto (7-t^2)$ then $a = k(7-t^2)$, where k is the constant of proportionality. You will need to use the information that the speed of P is 6 m s^{-1} when $t = 3$ to evaluate k.

When

$$t = 0, v = 0$$

$$0 = 0 - 0 + A \Rightarrow A = 0$$

Hence

$$v = 7kt - \frac{k}{3}t^3$$

You will need to integrate twice to obtain s from a.

When $t = 3, v = 6$

$$6 = 21k - 9k \Rightarrow 12k = 6 \Rightarrow k = \frac{1}{2}$$

$$v = \frac{7}{2}t - \frac{1}{6}t^3 \qquad (1)$$

$$s = \int v \, dt = \int \left(\frac{7}{2}t - \frac{1}{6}t^3\right) dt$$

$$= \frac{7}{4}t^2 - \frac{1}{24}t^4 + B$$

When $t = 0, s = 0$

$$0 = 0 - 0 + B \Rightarrow B = 0$$

$$s = \frac{7}{4}t^2 - \frac{1}{24}t^4 = \frac{42}{24}t^2 - \frac{1}{24}t^4$$

$$= \frac{1}{24}t^2(42 - t^2), \text{ as required}$$

b Substituting $v = 0$ into (1)

$$0 = \frac{7}{2}t - \frac{1}{6}t^3 = \frac{21}{6}t - \frac{1}{6}t^3$$

$$= \frac{1}{6}t\left(21 - t^2\right)$$

To find the total distance P moves, you will need to find the point where P reverses direction. That is where $v = 0$.

For $t > 0$

$t^2 = 21$

$t = \sqrt{21}$ but it is the value of t^2 you need to substitute into the expression for s. Using a decimal approximation for t would lose accuracy.

Substituting $t^2 = 21$ into the result of **a**

$$s = \frac{1}{24} \times 21 \times (42 - 21) = \frac{21^2}{24} = \frac{441}{24}$$

The total distance P moves before returning to O is $\left(2 \times \frac{441}{24}\right) \text{m} = \frac{441}{12} \text{ m}$.

P moves to a point $\frac{441}{24}$ m from O and then returns to O. So the total distance moved is twice this distance.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 15

Question:

A particle P of mass 0.3 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} of P is given by

$$\mathbf{v} = 3t^2\mathbf{i} + (6t-4)\mathbf{j}.$$

a Find the magnitude of $\mathbf{F}$ when $t = 2$.

When $t = 0$, P is at the point A. The position vector of A with respect to a fixed origin O is $(3\mathbf{j} - 4\mathbf{j})$ m. When $t = 4$, P is at the point B.

b Find the position vector of B.

Solution:

a

$$\text{Acceleration, } \mathbf{a} = \dot{\mathbf{v}} = 6t\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{F} = m\mathbf{a} = 0.3(6t\mathbf{i} + 6\mathbf{j}) = 1.8t\mathbf{i} + 1.8\mathbf{j}$$

You find $\mathbf{F}$ using Newton's Second Law $\mathbf{F} = m\mathbf{a}$, so you begin this question by differentiating the velocity to find the acceleration.

When $t = 2$

$$\mathbf{F} = 3.6\mathbf{i} + 1.8\mathbf{j}$$

$$|\mathbf{F}|^2 = 3.6^2 + 1.8^2 = 16.2$$

$$|\mathbf{F}| = \sqrt{16.3} = 4.0249\ldots$$

The magnitude of the vector $\mathbf{F}$, often written as F, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, is given by $F^2 = |\mathbf{F}|^2 = x^2 + y^2$.

The magnitude of $\mathbf{F}$ when $t = 2$ is 4.02 (2 d.p.).

b

$$\mathbf{r} = \int \mathbf{v}\,dt = \int \left(3t^2\mathbf{i} + (6t-4)\mathbf{j}\right) dt$$

$$= t^3\mathbf{i} + (3t^2 - 4t)\mathbf{j} + \mathbf{A}$$

When you integrate vectors the constant of integration is a vector.

When $t = 0, \mathbf{r} = 3\mathbf{i} - 4\mathbf{j}$

$3\mathbf{i} - 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$

Hence

$$\mathbf{r} = \left(t^3 + 3\right)\mathbf{i} + \left(3t^2 - 4t - 4\right)\mathbf{j}$$

When $t = 4$

$$\mathbf{r} = \left(4^3 + 3\right)\mathbf{i} + \left(3 \times 4^2 - 4 \times 4 - 4\right)\mathbf{j} = 67\mathbf{i} + 28\mathbf{j}$$

The position vector of B is $(67\mathbf{i} + 28\mathbf{j})$ m.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 16

Question:

Referred to a fixed origin O, the position vector of a particle P at time t seconds is $\mathbf{r}$ metres, where

$$\mathbf{r} = 6t^2\mathbf{i} + t^{\frac{5}{2}}\mathbf{j},\ t \geq 0.$$

At the instant when $t = 4$, find

a the speed of P,

b the acceleration of P, giving your answer as a vector.

Solution:

a

$\mathbf{v} = \dot{\mathbf{r}} = 12t\mathbf{i} + \frac{5}{2}t^{\frac{3}{2}}\mathbf{j}$

When $t = 4$

$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = 2^3 = 8$

$$\begin{aligned} \mathbf{v} &= 48\mathbf{i} + \frac{5}{2} \times 4^{\frac{3}{2}}\mathbf{j} = 48\mathbf{i} + 20\mathbf{j} \\ |\mathbf{v}|^2 &= 48^2 + 20^2 = 2704 \\ |\mathbf{v}| &= \sqrt{2704} = 52 \end{aligned}$$

The speed of P when $t = 4$ is 52 m s^{-1}.

You need to know that $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$.

b $\mathbf{a} = \dot{\mathbf{v}} = 12\mathbf{i} + \frac{5}{2} \times \frac{3}{2}t^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{4}t^{\frac{1}{2}}\mathbf{j}$

When $t = 4$

$\mathbf{a} = 12\mathbf{i} + \frac{15}{4} \times 4^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{2}\mathbf{j}$

The acceleration of P when $t = 4$ is $\left(12\mathbf{i} + \frac{15}{2}\mathbf{j}\right)\text{ m s}^{-2}$.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 17

Question:

A particle P moves in a horizontal plane. At time t seconds, the position vector of P is $\mathbf{r}$ metres relative to a fixed origin O where $\mathbf{r}$ is given by

$$\mathbf{r} = (18t - 4t^3)\mathbf{i} + ct^2\mathbf{j},$$

where c is a positive constant. When $t = 1.5$, the speed of P is 15 m s^{-1}. Find

a the value of c,

b the acceleration of P when $t = 1.5$.

Solution:

a

$\mathbf{v} = \dot{\mathbf{r}} = (18 - 12t^2)\mathbf{i} + 2ct\mathbf{j}$

When $t = 1.5$

$\mathbf{v} = (18 - 12 \times 1.5^2)\mathbf{i} + 3c\mathbf{j} = -9\mathbf{i} + 3c\mathbf{j}$

$|\mathbf{v}|^2 = (-9)^2 + (3c)^2 = 15^2$

$9c^2 = 15^2 - 9^2 = 144 \Rightarrow c^2 = \frac{144}{9} = 16$

As c is positive, $c = 4$

The speed of P is the magnitude of the velocity $\mathbf{v}$. $|\mathbf{v}|^2$ is both $(-9)^2 + (3c)^2$ and the speed squared. This gives you an equation in c.

b

$\mathbf{a} = \dot{\mathbf{v}} = -24t\mathbf{i} + 2c\mathbf{j}$

Using $c = 4$ and $t = 1.5$

$\mathbf{a} = -36\mathbf{i} + 8\mathbf{j}$

The acceleration of P when $t = 1.5$ is $(-36\mathbf{i} + 8\mathbf{j}) \text{ m s}^{-2}$.

Acceleration is a vector and the answer should be given in vector form.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 18

Question:

A particle P of mass 0.4 kg moves under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity of P, $\mathbf{v}$ m s^{-1}, is given by

$$\mathbf{v} = (6t + 4)\mathbf{i} + (t^2 + 3t)\mathbf{j}.$$

When $t = 0$, P is at the point with position vector $(-3\mathbf{i} + 4\mathbf{j})$ m with respect to a fixed origin O. When $t = 4$, P is at the point S.

a Calculate the magnitude of $\mathbf{F}$ when $t = 4$.

b Calculate the distance OS.

Solution:

a

Acceleration, $\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (2t + 3)\mathbf{j}$

You find **F** using Newton's Second Law $\mathbf{F} = m\mathbf{a}$, so you begin this part by differentiating the velocity to find the acceleration.

When $t = 4$

$$\mathbf{a} = 6\mathbf{i} + 11\mathbf{j}$$
$$\mathbf{F} = m\mathbf{a}$$
$$= 0.4(6\mathbf{i} + 11\mathbf{j}) = 2.4\mathbf{i} + 4.4\mathbf{j}$$
$$|\mathbf{F}|^2 = 2.4^2 + 4.4^2 = 25.12$$
$$|\mathbf{F}| = \sqrt{25.12} = 5.011\ldots$$

No accuracy is specified in this question and any sensible accuracy is acceptable.

The magnitude of **F** is 5.01 (2 d.p.).

b

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int \left((6t + 4)\mathbf{i} + (t^2 + 3t)\mathbf{j}\right)\mathrm{d}t$$
$$= \left(3t^2 + 4t\right)\mathbf{i} + \left(\frac{1}{3}t^3 + \frac{3}{2}t^2\right)\mathbf{j} + \mathbf{A}$$

When you integrate vectors the constant of integration is a vector.

When $t = 0, \mathbf{r} = -3\mathbf{i} + 4\mathbf{j}$

$-3\mathbf{i} + 4\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + \mathbf{A} \Rightarrow \mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$

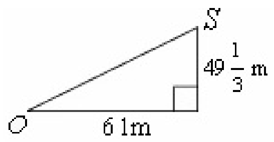
Hence

$$\mathbf{r} = \left(3t^2 + 4t - 3\right)\mathbf{i} + \left(\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4\right)\mathbf{j}$$

When $t = 4$

$$\mathbf{r} = \left(3\times 4^2 + 4\times 4 - 3\right)\mathbf{i} + \left(\frac{1}{3}\times 4^3 + \frac{3}{2}\times 4^2 + 4\right)\mathbf{j}$$
$$= 61\mathbf{i} + 49\frac{1}{3}\mathbf{j}$$
$$|\mathbf{r}|^2 = 61^2 + \left(49\frac{1}{3}\right)^2 = 6154\frac{7}{9} \Rightarrow |\mathbf{r}| = 78.452\ldots$$
$$OS = 78.45\text{ m (2 d.p.)}$$

The distance OS is found using Pythagoras' Theorem.



Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 19

Question:

Two particles P and Q move in a plane so that at time t seconds, where $t \geq 0$, P and Q have position vectors $\mathbf{r}_P$ metres and $\mathbf{r}_Q$ metres respectively, relative to a fixed origin O, where

$$\mathbf{r}_P = (3t^2 + 4)\mathbf{i} + (2t - \frac{1}{2})\mathbf{j},$$

$$\mathbf{r}_P = (t + 6)\mathbf{i} + \frac{3t^2}{2}\mathbf{j}.$$

Find

a the velocity vectors of P and Q at time t seconds,

b the speed of P when $t = 2$,

c the value of t at the instant when the particles are moving parallel to one another.

d Show that the particles collide and find the position vector of their point of collision.

Solution:

a

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = 6t\mathbf{i} + 2\mathbf{j}$$
$$\mathbf{v}_Q = \dot{\mathbf{r}}_Q = \mathbf{i} + 3t\mathbf{j}$$

> $\frac{d}{dt}((t+6)\mathbf{i}) = 1\mathbf{i} = \mathbf{i}$

The velocity of P at time t seconds is $(6t\mathbf{i} + 2\mathbf{j})$ m s^{-1} and the velocity of Q is $(\mathbf{i} + 3t\mathbf{j})$ m s^{-1}.

b When $t = 2$

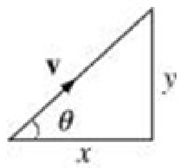
$$\mathbf{v}_P = 12\mathbf{i} + 2\mathbf{j}$$
$$|\mathbf{v}_P|^2 = 12^2 + 2^2 = 148 \Rightarrow \mathbf{v}_P = \sqrt{148} = 12.165\ldots$$

The speed of P when $t = 2$ is 12.2 m s^{-1} (3 s.f.).

c When P is moving parallel to Q

$$\frac{2}{6t} = \frac{3t}{1} \Rightarrow 18t^2 = 2 \Rightarrow t^2 = \frac{1}{9}$$
$$t \geq 0, t = \frac{1}{3}$$

> When the particles are moving parallel to each other, the angle each makes with $\mathbf{i}$ is the same.
>
> If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}, \tan\theta = \frac{y}{x}$ must be the same for both velocities.



d $\mathbf{i}$ components

$$3t^2 + 4 = t + 6$$
$$3t^2 - t - 2 = (t-1)(3t+2) = 0$$
$$t = 1, -\frac{2}{3}$$

$\mathbf{j}$ components

$$2t - \frac{1}{2} = \frac{3t^2}{2}$$

> For the particles to collide, both the $\mathbf{i}$ and $\mathbf{j}$ components of their position vectors must be the same for the same value of t. The appropriate method is to equate the $\mathbf{i}$ components and solve the resulting quadratic, and then do the same for the $\mathbf{j}$ components. If one of the roots of the quadratics is the same, then the particles collide.

Multiplying by 2 and rearranging

$$3t^2 - 4t + 1 = (t-1)(3t-1) = 0$$
$$t = 1, \frac{1}{3}$$

1 is a common root of the equations and, hence, P and Q collide at the point with position vector $\left(7\mathbf{i} + \frac{3}{2}\mathbf{j}\right)$ m.

> $t = 1$ can be substituted into either $\mathbf{r}_P$ or $\mathbf{r}_Q$ to find the position vector of the point where the particles collide.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 20

Question:

Referred to a fixed origin O, the particle R has position vector $\mathbf{r}$ metres at time t seconds, where

$$\mathbf{r} = (6\sin\omega t)\mathbf{i} + (4\cos\omega t)\mathbf{j}$$

and ω is a positive constant.

a Find $\dot{\mathbf{r}}$ and hence show that

$$v^2 = 2\omega^2(13 + 5\cos 2\omega t),$$

where v m s^{-1} is the speed of R at time t seconds.

b Deduce that

$$4\omega \le v \le 6\omega.$$

c Find $\ddot{\mathbf{r}}$.

d At the instant when $t = \frac{\pi}{3\omega}$, find the angle between $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, giving your answer in degrees to one decimal place.

Solution:

a

$$\mathbf{v} = \dot{\mathbf{r}} = (6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j}$$

$$v^2 = |\mathbf{v}|^2 = 36\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t$$

$$= 36\omega^2\left(\frac{1}{2}+\frac{1}{2}\cos 2\omega t\right) + 16\omega^2\left(\frac{1}{2}-\frac{1}{2}\cos 2\omega t\right)$$

$$= 18\omega^2 + 18\omega^2\cos 2\omega t + 8\omega^2 - 8\omega^2\cos 2\omega t$$

$$= 26\omega^2 + 10\omega^2\cos 2\omega t$$

$$= 2\omega^2(13 + 5\cos 2\omega t), \text{ as required}$$

Using the double angle formulae $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$.

b As $-1 \le \cos 2\omega t \le 1$

$$2\omega^2(13-5) \le 2\omega^2(13+5\cos 2\omega t) \le 2\omega^2(13+5)$$

$$16\omega^2 \le v^2 \le 36\omega^2$$

As $v, \omega > 0$

$4\omega \le v \le 6\omega$, as required

Both the $\cos x$ and $\sin x$ have the range $-1 \le x \le 1$. In this case the limits of $\cos 2\omega t$ limit the possible values of v^2 and, hence, of v.

c

$$\ddot{\mathbf{r}} = \frac{d}{dt}((6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j})$$

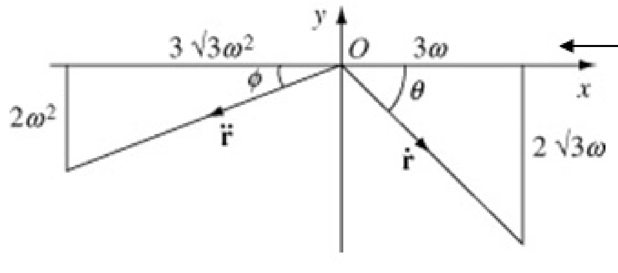
$$= -6\omega^2\sin\omega t\,\mathbf{i} - 4\omega^2\cos\omega t\,\mathbf{j}$$

d When $t = \frac{\pi}{3\omega}$

$$\dot{\mathbf{r}} = \left(6\omega\cos\frac{\pi}{3}\right)\mathbf{i} - \left(4\omega\sin\frac{\pi}{3}\right)\mathbf{j} = 3\omega\mathbf{i} - 2\sqrt{3}\omega\mathbf{j}$$

$$\ddot{\mathbf{r}} = -6\omega^2\sin\frac{\pi}{3}\mathbf{i} - 4\omega^2\cos\frac{\pi}{3}\mathbf{j} = -3\sqrt{3}\omega^2\mathbf{i} - 2\omega^2\mathbf{j}$$

Using $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.



A diagram is essential here. Once the diagram has been drawn, the problem reduces to basic trigonometry. You find the angles using the inverse tangent button on your calculator. The final angle could be given in degrees or radians (1.92ᶜ).

$$\tan\theta = \frac{2\sqrt{3}\omega}{3\omega} = \frac{2\sqrt{3}}{3} \Rightarrow \theta = 49.106\ldots^\circ$$

$$\tan\phi = \frac{2\omega^2}{3\sqrt{3}\omega^2} = \frac{2}{3\sqrt{3}} \Rightarrow \phi = 21.051\ldots^\circ$$

The angle between $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$ is $(180 - 49.106\ldots - 21.501\ldots)^\circ = 109.8^\circ$ (1 d.p.).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 21

Question:

Three particles of mass $3m$, $5m$ and λm are placed at the points with coordinates $(4,0)$, $(0,-3)$ and $(4, 2)$ respectively.

The centre of mass of the three particles is at $(2, k)$.

a Show that $\lambda = 2$.

b Calculate the value of k.

Solution:

The total mass is

$3m + 5m + \lambda m = (8+\lambda)m$

a $M(Oy)$

$(8+\lambda)m \times 2 = 3m \times 4 + 5m \times 0 + \lambda m \times 4$

$16 + 2\lambda = 12 + 4\lambda \Rightarrow 4 = 2\lambda$

$\lambda = 2$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis. The distance of a particle from the y-axis is its x-coordinate.

Divide throughout by m, simplify and remove the bracket.

b $M(Ox)$

$(8+\lambda)m \times k = 3m \times 0 + 5m \times -3 + \lambda m \times 2$

$10k = -15 + 4 \Rightarrow 10k = -11$

$k = -1.1$

Divide by m and simplify using $\lambda = 2$.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 22

Question:

Particles of mass $2M$, xM and yM are placed at points whose coordinates are (2, 5), (1, 3) and (3, 1) respectively. Given that the centre of mass of the three particles is at the point (2, 4), find the values of x and y.

Solution:

The total mass is

$2M + xM + yM = (2 + x + y)M$

M(Oy)

$(2 + x + y)M \times 2 = 2M \times 2 + xM \times 1 + yM \times 3$

$4 + 2x + 2y = 4 + x + 3y$

$x - y = 0 \quad (1)$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis. The distance of a particle from the y-axis is its x-coordinate.

Divide by M, simplify and collect together like terms to give the first of two simultaneous equations.

M(Ox)

$(2 + x + y)M \times 4 = 2M \times 5 + xM \times 3 + yM \times 1$

$8 + 4x + 4y = 10 + 3x + y$

$x + 3y = 2 \quad (2)$

This is the second of the simultaneous equations. You can solve the simultaneous equations using any correct method.

$(2) - (1)$

$4y = 2 \Rightarrow y = \frac{1}{2}$

From (1)

$x = y = \frac{1}{2}$

Solutionbank M2

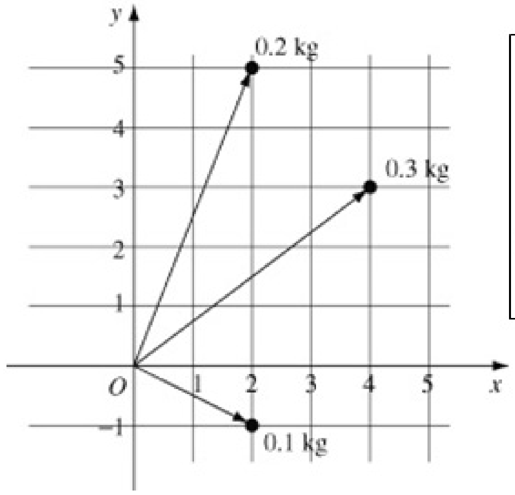
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 23

Question:

Three particles of mass 0.1 kg, 0.2 kg and 0.3 kg are placed at the points with position vectors $(2\mathbf{i}-\mathbf{j})$ m, $(2\mathbf{i}+5\mathbf{j})$ m and $(4\mathbf{i}+2\mathbf{j})$ m respectively. Find the position vector of the centre of mass of the particles.

Solution:



This diagram illustrates the positions of the particles with respect to Cartesian axes. You take moments about the y-axis and the x-axis to find the x- and y-coordinates, respectively, of the centre of mass of the system of particles.

Let the position vector of the centre of mass be $(\bar{x}\mathbf{i}+\bar{y}\mathbf{j})$ m.

$\bar{x}$ and $\bar{y}$ are the standard symbols for the coordinates of a centre of mass.

The total mass is

$(0.1+0.2+0.3)\text{kg} = 0.6 \text{ kg}$

M(Oy)

$$0.6\bar{x} = 0.1\times 2+0.2\times 2+0.3\times 4 = 1.8$$
$$\bar{x} = \frac{1.8}{0.6} = 3$$

The x-coordinates are the distances of the points from the y-axis.

M(Ox)

$$0.6\bar{y} = 0.1\times -1+0.2\times 5+0.3\times 2 = 1.5$$
$$\bar{y} = \frac{1.5}{0.6} = 2.5$$

The y-coordinates are the distances of the points from the x-axis.

The position vector of the centre of mass is $(3\mathbf{i}+2.5\mathbf{j})$ m.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 24

Question:

Three particles of mass $2M$, M and kM, where k is a constant, are placed at points with position vectors $6\mathbf{i}$ m, $4\mathbf{j}$ m and $(2\mathbf{i}-2\mathbf{j})$ m respectively. The centre of mass of the three particles has position vector $(3\mathbf{i}+c\mathbf{j})$ m, where c is a constant.

a Show that $k=3$.

b Hence find the value of c.

Solution:

The total mass is
$2M+M+kM=(3+k)M$

a M(Oy)
$(3+k)M\times 3=2M\times 6+M\times 0+kM\times 2$
$9+3k=12+2k$
$k=3$, as required

The moment of the mass of a particle about an axis is the mass multiplied by the perpendicular distance from the particle to the axis. The particle of mass M has position vector $4\mathbf{j}$ m and so lies on Oy. So its moment about Oy is zero.

b M(Ox)

$(3+k)M\times c=2M\times 0+M\times 4+kM\times -2$
Using $k = 3$
$6c = 4-6=-2$
$c = -\frac{1}{3}$

Divide by M and use the result to **a**.

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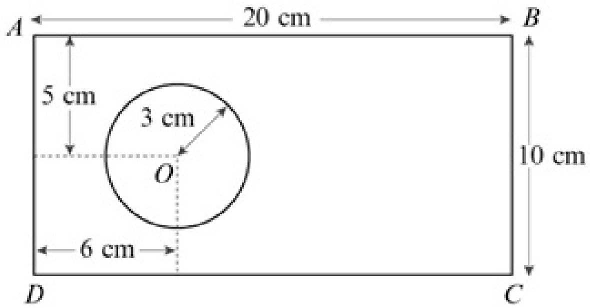
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 25

Question:



The figure shows a metal plate that is made by removing a circle of centre O and radius 3 cm from a uniform rectangular lamina $ABCD$, where $AB = 20$ cm and $BC = 10$ cm. The point O is 5 cm from both AB and CD and is 6 cm from AD.

a Calculate, to 3 significant figures, the distance of the centre of mass of the plate from AD.

The plate is freely suspended from A and hangs in equilibrium.

b Calculate, to the nearest degree, the angle between AB and the vertical.

Solution:

a

The area of the lamina is $20 \times 10 = 200$ cm^2.

The area of the circle is $\pi \times 3^2 = 9\pi$ cm^2.

The area of the plate is $(200 - 9\pi)$ cm^2.

Let the distance of the centre of mass of the plate from AB be $\bar{x}$ cm.

	Lamina	Circle	Plate
Mass ratios	200	9π	$200 - 9\pi$
Distances	10	6	$\bar{x}$

As the lamina is uniform, masses are proportional to areas.

By symmetry, the centre of mass of the rectangular lamina is 10 cm from AB and the centre of mass of the circle is 6 cm from AB.

M(AD)

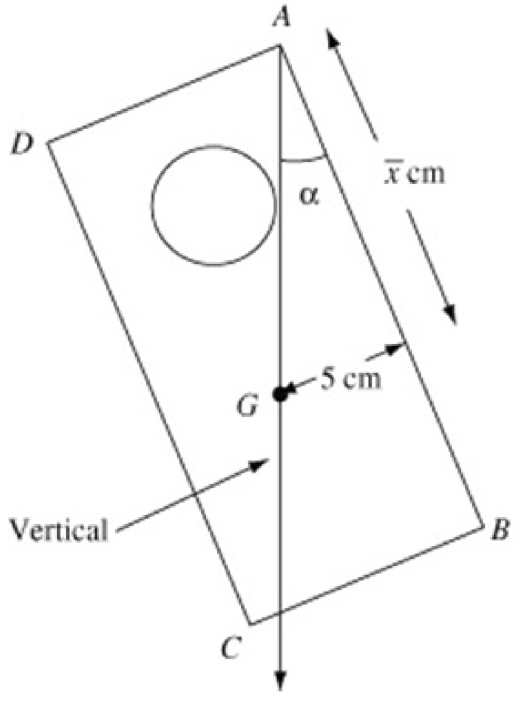
$$200 \times 100 = 9\pi \times 6 + (200 - 9\pi) \times \bar{x}$$

$$\bar{x} = \frac{2000 - 54\pi}{200 - 9\pi} = 10.658\ldots$$

The moment of the total mass (in this case the rectangle) about AD equals the sum of the moments of the circle and the plate about AD.

The distance of the centre of mass of the plate from AB is 10.7 cm (3 s.f.).

b Let the angle between AB and the vertical be α.



When the plate is suspended freely from A, its centre of mass G is vertically below the point of suspension A. The distance of G from AD was found in **a** and the distance of G from AB is 5 cm by symmetry. You calculate α using trigonometry.

$$\tan\alpha = \frac{5}{\bar{x}} = \frac{5}{10.658\ldots} = 0.4691\ldots$$

$$\alpha = 25.13\ldots^\circ$$

The angle between AB and the vertical is 25° (nearest degree).

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 26

Question:

A triangular frame ABC is made by bending a piece of wire of length 24 cm, so that AB, BC and AC are of lengths 6 cm, 8 cm and 10 cm respectively. Given that the wire is uniform, find the distance of the centre of mass of the frame from

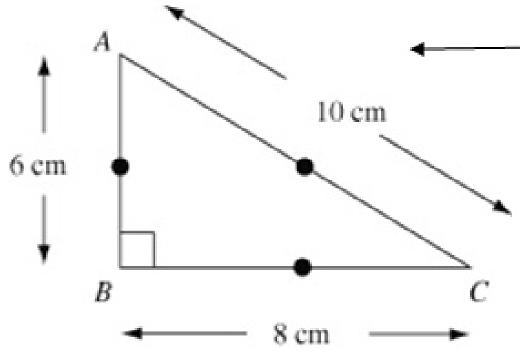
a AB,

b BC.

The frame is suspended from the corner A and hangs in equilibrium.

c Find, to the nearest degree, the acute angle made by AB with the downward vertical.

Solution:



When there is no diagram, it is a good idea to sketch one. You are expected to recognise that a triangle with sides in the ratio of 3 : 4 : 5 has a right angle.

As the wire is uniform, each side has a mass proportional to its length and the centre of mass of each side is at the middle of the side.

It is a common error to confuse a triangular frame with a solid triangle. A triangular frame consists of three straight line sections, each of which is considered separately.

Let the centre of mass G of the framework be $(\bar{x}, \bar{y})$.

	Framework	AB	BC	AC
Mass ratios	24	6	8	10
Distances (x)	$\bar{x}$	0	4	4
Distances (y)	$\bar{y}$	3	0	3

Displaying the masses and coordinates in a table helps you to form correct equations.

The centre of mass of each side is at the middle of the side.

a M(AB)

$$24\bar{x} = 6\times 0 + 8\times 4 + 10\times 4 = 72$$

$$\bar{x} = \frac{72}{24} = 3$$

b M(BC)

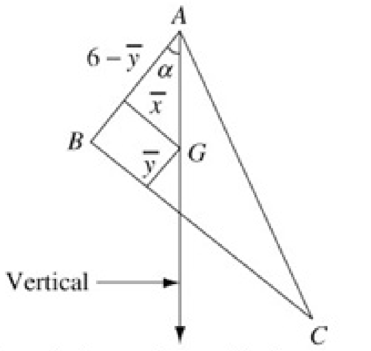
$$24\bar{y} = 6\times 3 + 8\times 0 + 10\times 3 = 48$$

$$\bar{y} = \frac{48}{24} = 2$$

c Let the acute angle made by AB with the downward vertical be α.

$$\tan\alpha = \frac{\bar{x}}{6-\bar{y}} = \frac{3}{6-2} = \frac{3}{4}$$

$$\alpha = 37^\circ \text{ (nearest degree).}$$



When the frame hangs freely from A, its centre of mass G is vertically below A.

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Review Exercise
Exercise A, Question 27

Question:

Three uniform rods AB, BC and CA of mass $2m$, m and $3m$ respectively have lengths l, l and $l\sqrt{2}$ respectively. The rods are rigidly joined to form a right-angled triangular framework.

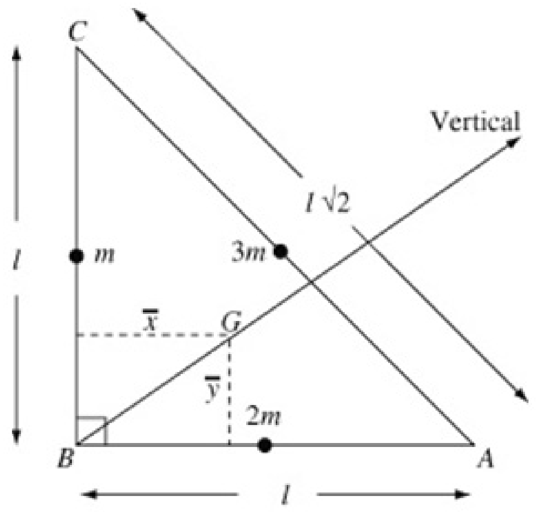
a Calculate, in terms of l, the distance of the centre of mass of the framework from

i BC,

ii AB.

b Calculate the angle, to the nearest degree, that BC makes with the vertical when the framework is freely suspended from the point B.

Solution:



As $l^2 + l^2 = (l\sqrt{2})^2$, the angle at B is a right angle.

As each rod is uniform, the centre of mass of each rod is at its mid-point. You can think of this as replacing each rod by a particle of the appropriate mass at the mid-point of the rod.

a The total mass is

$2m + m + 3m = 6m$

	Total	AB	BC	CA
Mass	$6m$	$2m$	m	$3m$
Distances (x)	$\bar{x}$	$\frac{l}{2}$	0	$\frac{l}{2}$
Distances (y)	$\bar{y}$	0	$\frac{l}{2}$	$\frac{l}{2}$

A table displays the masses and coordinates in a concise form and helps you form correct equations.

Let the centre of mass G of the framework be $(\bar{x}, \bar{y})$.

i M(AB)

$$6m \times \bar{x} = 2m \times \frac{l}{2} + m \times 0 + 3m \times \frac{l}{2} = \frac{5ml}{2}$$

$$\bar{x} = \frac{5ml}{12m} = \frac{5l}{12}$$

The moment of the total mass about any axis equals the sum of the moments of the individual masses about that axis.

ii M(BC)

$$6m \times \bar{y} = 2m \times 0 + m \times \frac{l}{2} + 3m \times \frac{l}{2} = 2ml$$

$$\bar{y} = \frac{2ml}{6m} = \frac{l}{3}$$

b Let the angle that BC makes with the vertical be α.

$$\tan \alpha = \frac{\bar{y}}{\bar{x}} = \frac{\frac{l}{3}}{\frac{5l}{12}} = \frac{4}{5}$$

$$\alpha = 39^\circ \text{ (nearest degree).}$$

When the framework hangs freely from A, its centre of mass G is vertically below A.
The vertical line has been drawn in the diagram.

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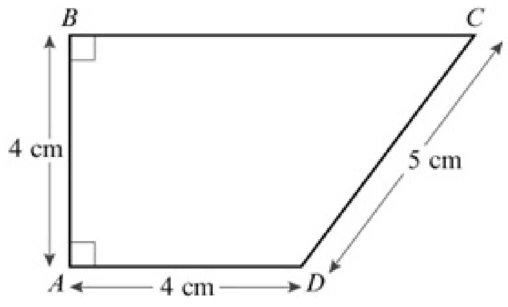
Review Exercise
Exercise A, Question 28

Question:



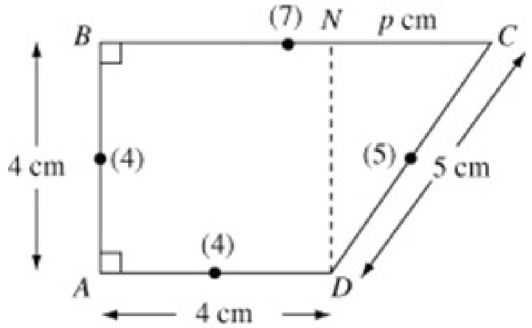
A thin uniform wire of total length 20 cm, is bent to form a frame. The frame is in the shape of a trapezium $ABCD$, where $AB = AD = 4$ cm, $CD = 5$ cm and AB is perpendicular to BC and AD, as shown in the figure.

a Find the distance of the centre of mass of the frame from AB.

The frame has mass M. A particle of mass kM is attached to the frame at C. When the frame is freely suspended from the midpoint of BC, the frame hangs in equilibrium with BC horizontal.

b Find the value of k.

Solution:



a Let N be the foot of the perpendicular from D to BC and $NC = p$ cm.

By Pythagoras' Theorem

$p^2 = 5^2 - 4^2 = 9 \Rightarrow p = 3$

Hence $BC = BN + NC = (4+3)$ cm $= 7$ cm

The total length of the frame is $(4+7+5+4)$ cm $= 20$ cm

Let the distance of the centre of mass of the frame from AB be $\bar{x}$ cm

	Total	AB	BC	CD	DA
Mass ratios	20	4	7	5	4
Distances	$\bar{x}$	0	3.5	5.5	2

As the wire is uniform, each section of the wire has a mass proportional to its length. As the ratios of the lengths are 4 : 7 : 5 : 4, the mass ratios are also 4 : 7 : 5 : 4.

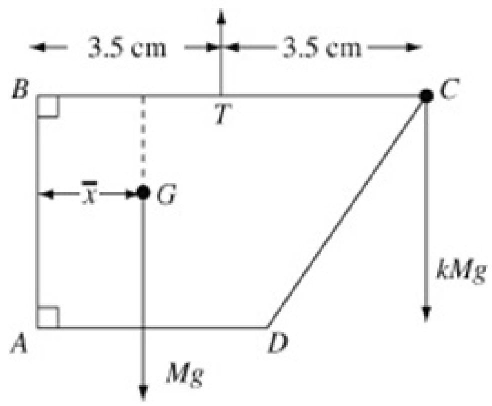
The distance of the centre of mass of CD from AB is $BN + \frac{1}{2}NC = (4 + 1.5)$ cm $= 5.5$ cm.

M(AB)

$$20\bar{x} = 4\times0+7\times3.5+5\times5.5+4\times2 = 60$$

$$\bar{x} = \frac{60}{20} = 3$$

b



The whole weight of the framework acts at the centre of mass G. There is also a force acting upwards at the mid-point, say T, of BC, but, when you take moments about T, this force has zero moment about T.

Let T be the mid-point of BC M(T)

$$\cancel{Mg}(3.5-\bar{x}) = k\cancel{Mg}\times3.5$$

$$3.5-3 = 3.5k$$

$$k = \frac{0.5}{3.5} = \frac{1}{7}$$

The perpendicular distance from T to the line of action of the weight of the framework is $3.5-\bar{x} = 0.5$ cm.

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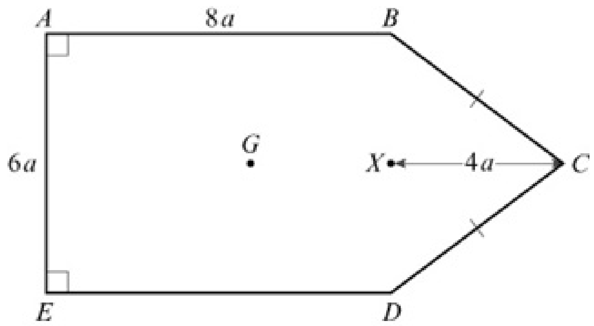
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 29

Question:



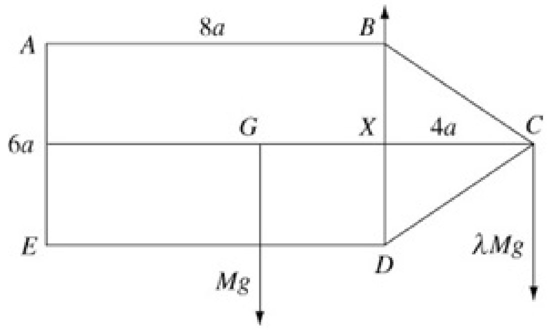
The figure shows a uniform lamina $ABCDE$ such that $ABDE$ is a rectangle, $BC = CD$, $AB = 8a$ and $AE = 6a$. The point X is the mid-point of BD and $XC = 4a$. The centre of mass of the lamina is at G.

a Show that $GX = \frac{44}{15}a$.

The mass of the lamina is M. A particle of mass λM is attached to the lamina at C. The lamina is suspended from B and hangs freely under gravity with AB horizontal.

b Find the value of λ.

Solution:



a The area of rectangle $ABDE$ is $6a \times 8a = 48a^2$

The area of ΔBCD is $\frac{1}{2} \times 6a \times 4a = 12a^2$

The area of lamina $ABCDE$ is $48a^2 + 12a^2 = 60a^2$

You break the lamina up into two parts. The rectangle $ABDE$, with dimensions $8a \times 6a$, and the triangle BCD, with base $6a$ and height $4a$.

	Lamina	Rectangle	Triangle
Mass ratios	$60a^2$	$48a^2$	$12a^2$
Displace-ments from X	GX	$4a$	$-\frac{4}{3}a$

The centre of mass of the triangle is $\frac{1}{3}h = \frac{1}{3} \times 4a$ from the base BD of the triangle.

M(X)

$$60a^2 \times GX = 48a^2 \times 4a + 12a^2 \times \left(-\frac{4}{3}a\right)$$

$$= 192a^3 - 16a^3 = 176a^3$$

$$GX = \frac{176a^3}{60a^2} = \frac{44}{15}a, \text{ as required}$$

Taking moments about X, the rectangle is on one side of X, here taken as positive, and the triangle is on the other side of X, here taken as negative.

b With the particle at C

M(X)

$$Mg \times GX = \lambda Mg \times 4a$$

$$\cancel{Mg} \times \frac{44}{15}\cancel{a} = \lambda \cancel{Mg} \times 4\cancel{a}$$

$$\lambda k = \frac{44}{15 \times 4} = \frac{11}{15}$$

There is a vertical force at B but, as the line of action of this force passes through X, its moment about X is zero.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 30

Question:

A uniform square plate $ABCD$ has mass $10M$ and the length of a side of the plate is $2l$. Particles of mass M, $2M$, $3M$ and $4M$ are attached at A, B, C and D respectively. Calculate, in terms of l, the distance of the centre of mass of the loaded plate from

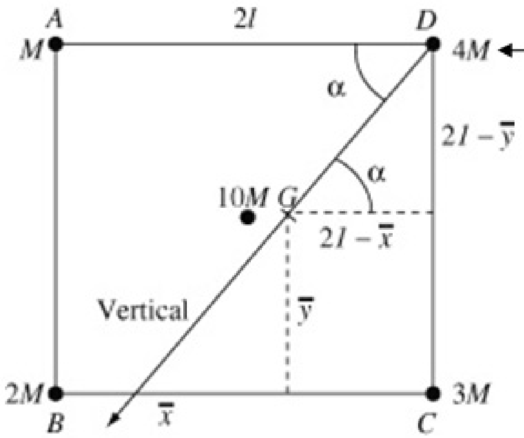
a AB,

b BC.

The loaded plate is freely suspended from the vertex D and hangs in equilibrium.

c Calculate, to the nearest degree, the angle made by DA with the downward vertical.

Solution:



The angle required in **c** is marked α in the diagram. It has been brought into a right angled triangle, in which you can use trigonometry, using alternate (Z) angles.

The total mass is

$M + 2M + 3M + 4M + 10M = 20M$

Let the distance of the centre of mass, G say, of the loaded plate from AB and BC be $\bar{x}$ cm and $\bar{y}$ cm respectively.

Total	Plate	A	B	C	D	
Mass	$20M$	$10M$	M	$2M$	$3M$	$4M$
Distances (x)	$\bar{x}$	l	0	0	$2l$	$2l$
Distances (y)	$\bar{y}$	l	$2l$	0	0	$2l$

A table displays the masses and coordinates in a concise form and helps you form correct equations.

a M(AB)

$20M \times \bar{x} \quad = \quad 10M \times l + 3M \times 2l + 4M \times 2l = 24Ml$

$\bar{x} = \dfrac{24Ml}{20M} \quad = \quad \dfrac{6l}{5}$

b M(BC)

$20M \times \bar{y} \quad = \quad 10M \times l + M \times 2l + 4M \times 2l = 20Ml$

$\bar{y} = \dfrac{20Ml}{20M} \quad = \quad l$

c Let the angle made by DA with the downward vertical be α.

$\tan\alpha \quad = \quad \dfrac{2l - \bar{y}}{2l - \bar{x}} = \dfrac{2l - l}{2l - \frac{6l}{5}} = \dfrac{l}{\frac{4l}{5}} = \dfrac{5}{4}$

$\alpha \quad = \quad 51^\circ$ (nearest degree).

When the framework hangs freely from D, its centre of mass G is vertically below D. The vertical line has been drawn in the diagram.

Solutionbank M2

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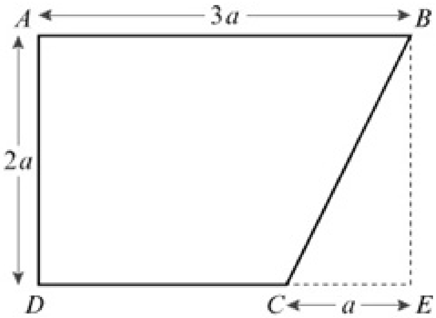
Review Exercise
Exercise A, Question 31

Question:



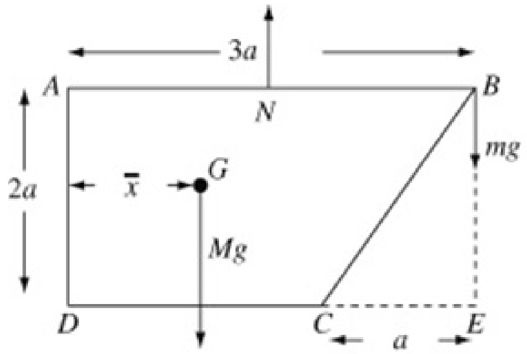
A uniform lamina $ABCD$ is made by taking a uniform sheet of metal in the form of a rectangle $ABED$, with $AB = 3a$ and $AD = 2a$, and removing the triangle BCE, where C lies on DE and $CE = a$, as shown in the figure.

a Find the distance of the centre of mass of the lamina from AD.

The lamina has mass M. A particle of mass m is attached to the lamina at B. When the loaded lamina is freely suspended from the midpoint of AB, it hangs in equilibrium with AB horizontal.

b Find m in terms of M.

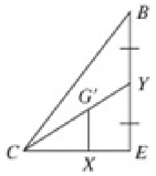
Solution:



a The area of rectangle $ABCD$ is $3a \times 2a = 6a^2$.
The area of triangle BCE is $\frac{1}{2}a \times 2a = a^2$.
The area of lamina $ABCD$ is $6a^2 - a^2 = 5a^2$.

Let the distance of the centre of mass of the lamina, say G, from AD be $\bar{x}$ cm.

	Rectangle	Lamina	Triangle
Mass ratio	$6a^2$	$5a^2$	a^2
Distances	$\frac{3}{2}a$	$\bar{x}$	$\frac{8}{3}a$



If G' is the centre of mass of the triangle and Y is the mid-point of BE then $CG' : G'Y = 2:1$.
Using similar triangles,
$\frac{CX}{XE} = \frac{CG'}{G'Y} = \frac{2}{1}$ and, as $CE = a$, $XE = \frac{1}{3}a$. So G' is $\frac{1}{3}a$ from BE and $\left(3a - \frac{1}{3}a\right) = \frac{8}{3}a$ from AD.

M(AD)

$$6a^2 \times \frac{3}{2}a = 5a^2 \times \bar{x} + a^2 \times \frac{8}{3}a$$
$$5a^2\bar{x} = 9a^3 - \frac{8}{3}a^3 = \frac{19}{3}a^3$$
$$\bar{x} = \frac{19}{15}a$$

The moment of the complete rectangle about AD is the sum of the moments of the lamina and the triangle which has been removed from the rectangle.

b Let N be the mid-point of AB
M (N)

$$Mg\left(\frac{3}{2}a - \bar{x}\right) = mg \times \frac{3}{2}a$$
$$\frac{3}{2}Ma - \frac{19}{15}Ma = \frac{3}{2}ma$$
$$\frac{3}{2}m = \left(\frac{3}{2} - \frac{19}{15}\right)M = \frac{7}{30}M$$
$$m = \frac{2}{3} \times \frac{7}{30}M = \frac{7}{45}M$$

There is a vertical force at N holding the loaded lamina in equilibrium but this has no moment about N.

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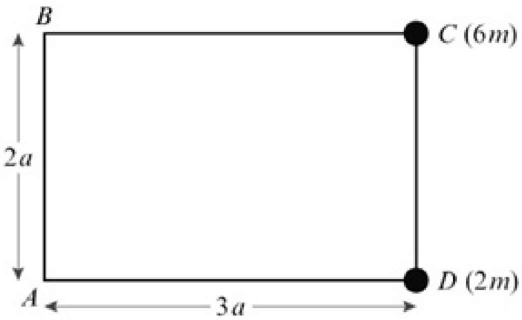
Review Exercise
Exercise A, Question 32

Question:



The figure shows four uniform rods joined to form a rectangular framework $ABCD$, where $AB = CD = 2a$ and $BC = AD = 3a$. Each rod has mass m. Particles of mass $6m$ and $2m$ are attached to the framework at points C and D respectively.

a Find the distance of the centre of mass of the loaded framework from

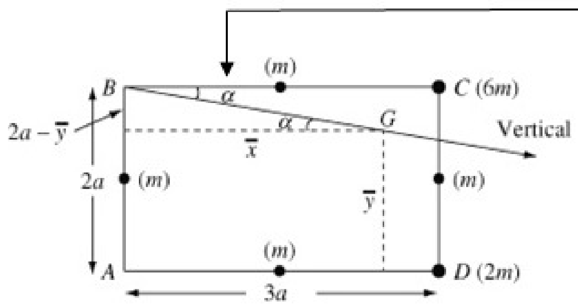
i AB,

ii AD.

The loaded framework is freely suspended from B and hangs in equilibrium.

b Find the angle which BC makes with the vertical.

Solution:



The angle required in **b** is marked α in the diagram. It has been brought into a right angled triangle, in which you can use trigonometry, using alternate (Z) angles.

a The total mass is

$m + m + m + m + 2m + 6m = 12m$

Let the distance of the centre of mass of the loaded framework, say G, from AB and AD be $\bar{x}$ and $\bar{y}$ respectively.

It is a common error to confuse a rectangular frame with a solid rectangle. The frame consists of four straight sections, each of which is considered separately.

		Rods				Particles	
	Total	AB	BC	CD	DA	C	D
Masses	$12m$	m	m	m	m	$6m$	$2m$
Distances (x)	$\bar{x}$	0	$\frac{3}{2}a$	$3a$	$\frac{3}{2}a$	$3a$	$3a$
Distances (y)	$\bar{y}$	a	$2a$	a	0	$2a$	0

There is a lot of information in this question and summarising it in a table helps you both to check that you have not overlooked anything and to obtain correct moments equations.

i M (AB)

$$12m \times \bar{x} = m \times \frac{3}{2}a + m \times 3a + m \times \frac{3}{2}a + 6m \times 3a + 2m \times 3a = 30ma$$

$$\bar{x} = \frac{30ma}{12m} = \frac{5}{2}a$$

ii M (AD)

$$12m\times\bar{y} = m\times a + m\times 2a + m\times a + 6m\times 2a = 16ma$$

$$\bar{y} = \frac{16ma}{12m} = \frac{4}{3}a$$

b Let α be the angle BC makes with the vertical.

When the loaded framework hangs freely from B, its centre of mass G is vertically below B. The downward vertical has been drawn in the diagram.

$$\tan\alpha = \frac{2a-\bar{y}}{\bar{x}}$$

$$= \frac{2a-\frac{4}{3}a}{\frac{5}{2}a} = \frac{2}{3}\times\frac{2}{5} = \frac{4}{15}$$

$\alpha = 15°$ (nearest degree)

It is a good idea to write down an expression for the angle in terms of $\bar{x}$ and $\bar{y}$ and not to immediately use the expressions obtained in **a** in terms of a. Everyone makes mistakes from time to time and writing the expression using the general terms $\bar{x}$ and $\bar{y}$ makes your method clear.

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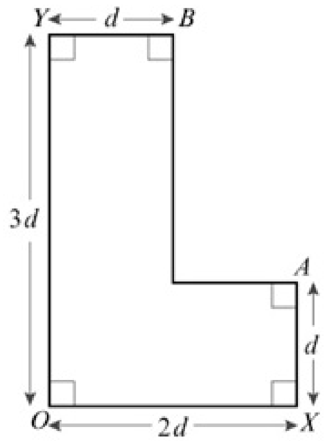
Review Exercise
Exercise A, Question 33

Question:



The figure shows a uniform L-shaped lamina with $OX = 2d, OY = 3d$ and $OX = YB = d$. The angles at O, A, B, X and Y are all right angles.

Find, in terms of d, the distance of the centre of mass of the lamina

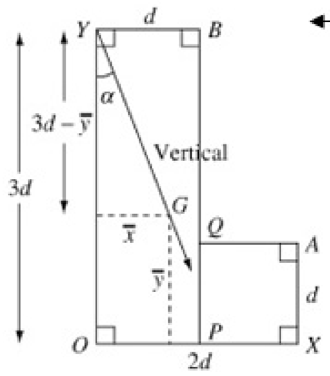
a from OX,

b from OY.

The lamina is suspended from the point Y and hangs freely in equilibrium.

c Find, to the nearest degree, the angle that OY makes with the vertical.

Solution:



You divide the L-shaped lamina into parts, each with a known centre of mass. This can be done in different ways. Here the lamina has been divided up into a rectangle *OXBP* of dimensions $3d \times d$ and a square *AXPQ* of side *d*.

The area of rectangle *OXBP* is $3d \times d = 3d^2$.
The area of square *AXQP* is $d \times d = d^2$.
The area of the L-shaped lamina is $3d^2 + d^2 = 4d^2$.
Let the distances of the centre of mass of the lamina, say *G*, from *OX* and *OY* be $\bar{x}$ and $\bar{y}$ respectively.

	Lamina	*OYBP*	*AXPQ*
Mass ratios	$4d^2$	$3d^2$	d^2
Distances (x)	$\bar{x}$	$\frac{1}{2}d$	$\frac{3}{2}d$
Distances (y)	$\bar{y}$	$\frac{3}{2}d$	$\frac{1}{2}d$

As the lamina is uniform, the masses of the lamina, rectangle and square are proportional to their areas. You could 'cancel' the d^2 here and just use 4 : 3 : 1. This would shorten the working a little.

a M (*OX*)

The distance of the centre of mass from *OX* is $\bar{y}$, not $\bar{x}$.

$$4d^2 \times \bar{y} = 3d^2 \times \frac{3}{2}d + d^2 \times \frac{1}{2}d = 5d^3$$

$$\bar{y} = \frac{5d^3}{4d^2} = \frac{5}{4}d$$

b M(*OY*)

$$4d^2 \times \bar{x} = 3d^2 \times \frac{1}{2}d + d^2 \times \frac{3}{2}d = 3d^3$$

$$\bar{x} = \frac{3d^3}{4d^2} = \frac{3}{4}d$$

c Let α be the angle *OY* makes with the vertical.

When the L-shaped lamina hangs freely from *Y*, its centre of mass *G* is vertically below *Y*. The vertical has been drawn in the diagram. The arrow is pointing downwards.

$$\tan\alpha = \frac{\bar{x}}{3d - \bar{y}} = \frac{\frac{3}{4}d}{3d - \frac{5}{4}d} = \frac{3}{4} \times \frac{4}{7} = \frac{3}{7}$$

$$\alpha = 23^\circ \text{ (nearest degree)}$$

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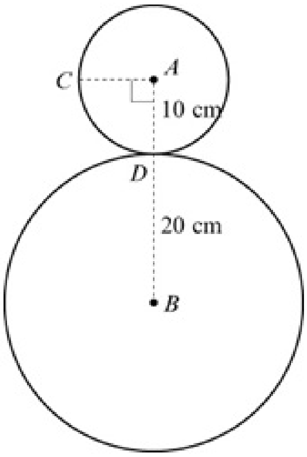
Review Exercise
Exercise A, Question 34

Question:



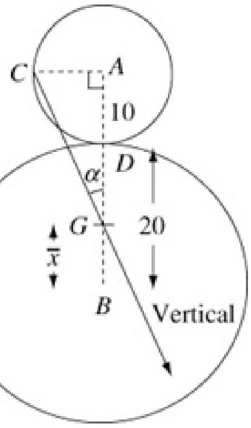
The figure shows a decoration which is made by cutting 2 circular discs from a sheet of uniform card. The discs are joined so that they touch at a point D on the circumference of both discs. The discs are coplanar and have centres A and B with radii 10 cm and 20 cm respectively.

a Find the distance of the centre of mass of the decoration from B.

The point C lies on the circumference of the smaller disc and $\angle CAB$ is a right angle. The decoration is freely suspended from C and hangs in equilibrium.

b Find, in degrees to one decimal place, the angle between AB and the vertical.

Solution:



a The area of the smaller circle is $\pi 10^2 = 100\pi \text{ cm}^2$.

The area of the larger circle is $\pi 20^2 = 400\pi \text{ cm}^2$.

The area of the decoration is $100\pi + 400\pi = 500\pi \text{ cm}^2$.

Let the distance of the centre of mass, say G, from B be $\bar{x}$ cm.

	Decoration	Small circle	Large circle
Mass ratios	500π	100π	400π
Distances from B (cm)	$\bar{x}$	30	0

As the card is uniform, the masses of the decoration and circles are proportional to their areas. You could simplify the ratio $500\pi : 100\pi : 400\pi$ to $5 : 1 : 4$ and this would simplify your working.

The centre of the larger circle is at B.

M(B)

$$500\pi \times \bar{x} = 100\pi \times 30 + 400\pi \times 0$$

$$\bar{x} = \frac{3000\not{\pi}}{500\not{\pi}} = 6$$

The distance of the centre of mass of the decoration from B is 6 cm.

b Let the angle between AB and the vertical be α.

$$AC = 10 \text{ cm}$$

$$AG = (10 + (20 - 6)) \text{ cm} = 24 \text{ cm}$$

$$\tan \alpha = \frac{10}{24} = \frac{5}{12}$$

$$\alpha = 22.6° \text{ (1 d.p.)}$$

When the decoration is freely suspended from C, its centre of mass G is vertically below C.
The vertical has been drawn in the diagram with the arrow pointing directly down. You use the result of **a** and trigonometry in the right angled triangle GAC to complete the question.

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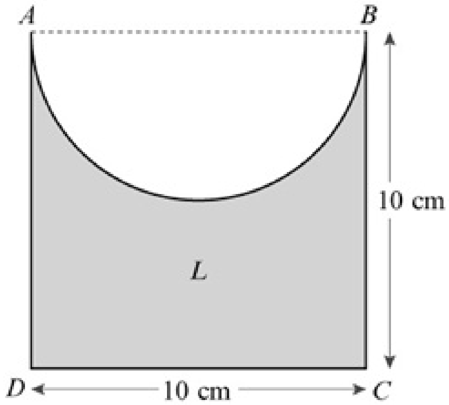
Review Exercise
Exercise A, Question 35

Question:



A uniform lamina L is formed by taking a uniform square sheet of material $ABCD$ of side 10 cm and removing a semicircle with diameter AB from the square, as shown in the figure.

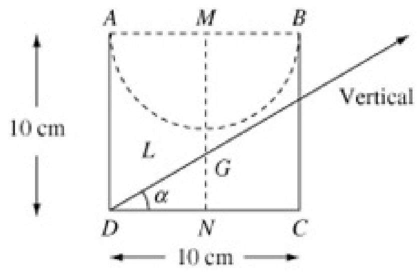
a Find, in cm to 2 decimal places, the distance of the centre of mass of the lamina from the midpoint of AB.

[The centre of mass of a uniform semicircular lamina, radius a, is at a distance $\frac{4a}{3\pi}$ from the centre of the bounding diameter.]

The lamina is freely suspended from D and hangs at rest.

b Find, in degrees to one decimal place, the angle between CD and the vertical.

Solution:



The area radius of the semi-circle is $\frac{1}{2}\pi r^2$ with $r = 5$.

a The area of the semi-circle is $\frac{1}{2}\pi \times 5^2 = \frac{25}{2}\pi \text{ cm}^2$.

The area of the square is $10 \times 10 = 100 \text{ cm}^2$.

The area of L is $\left(100 - \frac{25}{2}\right)\pi \text{ cm}^2$.

Let M be the mid-point of AB and N the mid-point of DC.

Let G be the centre of mass of L and $MG = \bar{x}$ cm

	Square	Semi-circle	L
Mass ratios	100	$\frac{25}{2}\pi$	$100 - \frac{25}{2}\pi$
Distances from M (cm)	5	$\frac{20}{3\pi}$	$\bar{x}$

The question gives the expression $\frac{4a}{3\pi}$ for the centre of mass of semi-circle and $a = 5$.

M (M)

$$100 \times 5 = \frac{25}{2}\not\pi \times \frac{20}{3\not\pi} + \left(100 - \frac{25}{2}\pi\right)\bar{x}$$

$$\left(100 - \frac{25}{2}\pi\right)\bar{x} = 500 - \frac{250}{3}$$

$$\bar{x} = \frac{500 - \frac{250}{3}}{100 - \frac{25}{2}\pi} = 6.860\,958\ldots$$

The question, by asking for 2 decimal places, implies there are no neat answers to this question and you must use your calculator to find the answers.

The distance of the centre of mass of L from the mid-point of AB is 6.86 cm (2 d.p.).

b Let the angle between CD and the vertical be α.

$$GN = (10 - \bar{x}) \text{ cm}, DN = 5 \text{ cm}$$

$$\tan\alpha = \frac{GN}{DN} = \frac{10 - 6.860}{5} = 0.6278\ldots$$

$$\alpha = 32.1^\circ \text{ (1 d.p.)}$$

When L is suspended freely from D, the centre of mass G hangs vertically below D. The downward vertical is drawn in the diagram.

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Review Exercise
Exercise A, Question 36

Question:

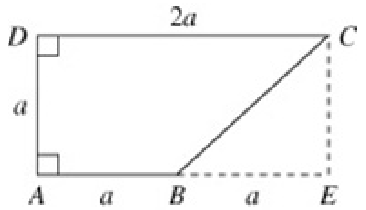
A uniform lamina $ABCD$ is in the form of a trapezium in which $AB = AD = a$, $CD = 2a$ and $\angle BAD = \angle ADC = 90°$

a Find the distance of the centre of mass of the lamina from AD and from AB.

The lamina stands with the edge AB on a plane inclined at an angle α to the horizontal with A higher than B. The lamina is in a vertical plane through a line of greatest slope of the plane.

b Given that the lamina is on the point of overturning about B, find the value of $\tan \alpha$.

Solution:



When there is no diagram in the question, sketch the diagram yourself. This makes the layout of the question clearer.

a Let the perpendicular from C to AB produced meet AB produced at E.

The area of rectangle $AECD = 2a \times a = 2a^2$.

The area of triangle $BEC = \frac{1}{2}a \times a = \frac{1}{2}a^2$.

The area of the lamina $= 2a^2 - \frac{1}{2}a^2 = \frac{3}{2}a^2$.

This solution treats the lamina as if it was made from a uniform rectangle, of dimensions a by $2a$, by removing the right angled triangle BEC. There are other equally valid alternatives.

Let the distance of the centre of mass of the lamina, say G, from AD and AB be $\bar{x}$ and $\bar{y}$ respectively.

	Rectangle	Triangle	Lamina
Mass ratios	$2a^2$	$\frac{1}{2}a^2$	$\frac{3}{2}a^2$
Distances (x)	a	$\frac{5}{3}a$	$\bar{x}$
Distances (y)	$\frac{1}{2}a$	$\frac{1}{3}a$	$\bar{y}$

G is $\frac{1}{3}a$ from CE and so it is $2a - \frac{1}{3}a = \frac{5}{3}a$ from AD.

M (AD)

$$2a^2 \times a = \frac{1}{2}a^2 \times \frac{5}{3}a + \frac{3}{2}a^2 \times \bar{x}$$

$$2a^3 = \frac{5}{6}a^3 + \frac{3}{2}a^2\bar{x} \Rightarrow \frac{3}{2}a^2\bar{x} = 2a^3 - \frac{5}{6}a^3 = \frac{7}{6}a^3$$

$$\bar{x} = \frac{7}{6} \times \frac{2}{3}a = \frac{7}{9}a$$

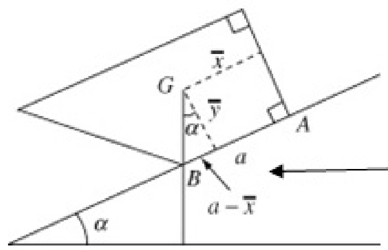
M (AD)

$$2a^2 \times \frac{1}{2}a = \frac{1}{2}a^2 \times \frac{1}{3}a + \frac{3}{2}a^2 \times \bar{y}$$

$$a^3 = \frac{1}{6}a^3 + \frac{3}{2}a^2\bar{y} \Rightarrow \frac{3}{2}a^2\bar{y} = a^3 - \frac{1}{6}a^3 = \frac{5}{6}a^3$$

$$\bar{y} = \frac{5}{6} \times \frac{2}{3}a = \frac{5}{9}a$$

b



When the lamina is on the point of overturning about B, the centre of mass G of the lamina is vertically above B.

$$\tan\alpha = \frac{a - \bar{x}}{\bar{y}} = \frac{\frac{2}{9}a}{\frac{5}{9}a} = \frac{2}{5}$$

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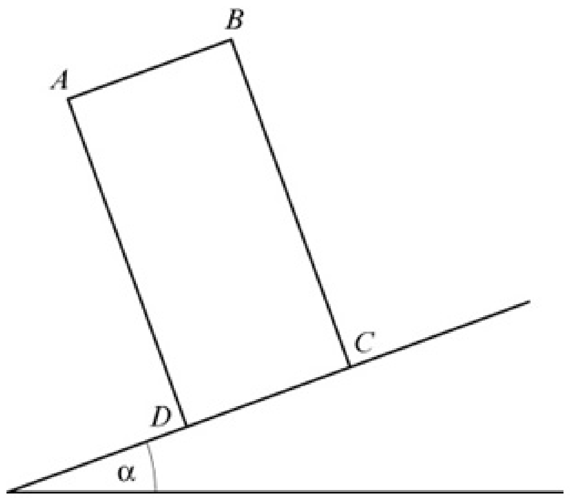
Review Exercise
Exercise A, Question 37

Question:



A thin uniform rectangular metal plate $ABCD$ of mass M rests on a rough plane inclined at an angle α to the horizontal. The plate lies in a vertical plane containing a line of greatest slope of the inclined plane, with the edge CD in contact with the plane and C further up the plane than D, as shown in the figure. The lengths of AB and BC are 10 cm and 30 cm respectively. The plane is sufficiently rough to prevent the plate from slipping.

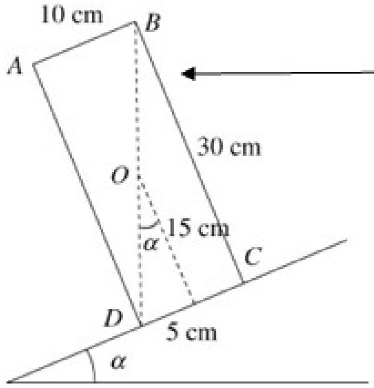
a Find, to the nearest degree, the greatest value which α can have if the plate does not topple.

A small stud of mass m is fixed to the plate at the point C.

b Given that $\tan\alpha = \frac{1}{2}$, find, in terms of M, the smallest value of m which will enable the plate to stay in equilibrium without toppling.

Solution:

a



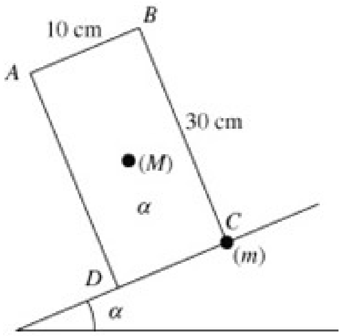
The greatest value which α can take without toppling about D corresponds to the situation where the centre of plate, here labelled O, is vertically above D. As drawn here, if O was to the left of D, the weight of the plate would have an anticlockwise moment about D and the plate would rotate about D.

$$\tan\alpha = \frac{5}{15} = \frac{1}{3}$$
$$\alpha = 18^\circ \text{ (nearest degree)}$$

The point O is 5 cm from AD and 15 cm from CD.

The greatest value α can have if the plate does not topple is 18° (nearest degree).

b



The greatest value of α occurs when the centre of mass, say G, of the plate combined with the stud is vertically above D. The first step in this part must, therefore, be finding the position of G.

Let the centre of mass, say G, of the plate and stud combined be $\bar{x}$ cm and $\bar{y}$ cm from AD and DC respectively.

	Total	Plate	Stud
Mass	$M+m$	M	m
Distances (x)	$\bar{x}$	5	10
Distances (y)	$\bar{y}$	15	0

M (AD)

$$(M+m)\bar{x} = M\times 5 + m\times 10$$

$$\bar{x} = \frac{5M+10m}{M+m}$$

M (DC)

$$(M+m)\bar{y} = M\times 15$$

$$\bar{y} = \frac{15M}{M+m}$$

Given that $\tan\alpha = \frac{1}{2}$

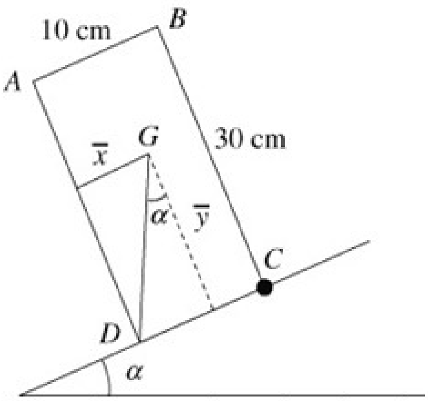
$$\tan\alpha = \frac{\bar{x}}{\bar{y}} = \frac{1}{2} \Rightarrow 2\bar{x} = \bar{y}$$

$$\frac{2(5M+10m)}{\cancel{M+m}} = \frac{15M}{\cancel{M+m}}$$

$15M$

$$10M + 20m = 15M \Rightarrow 20m = 5M \Rightarrow m = \frac{1}{4}M$$

The smallest value of m which will enable the plate to stay in equilibrium without toppling is $\frac{1}{4}M$.



When the system is on the point of toppling about D, the centre of mass of the plate and stud combined, G, is vertically above O.
The diagram above illustrates that $\tan\alpha = \frac{\bar{x}}{\bar{y}}$ and, as you are given that $\tan\alpha = \frac{1}{2}$, you can form an equation relating m and M.

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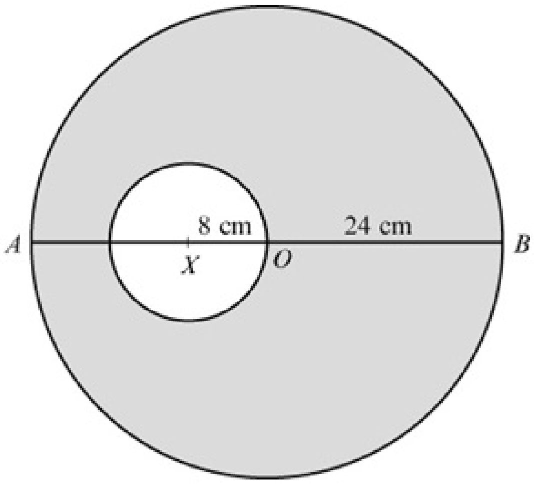
Review Exercise
Exercise A, Question 38

Question:



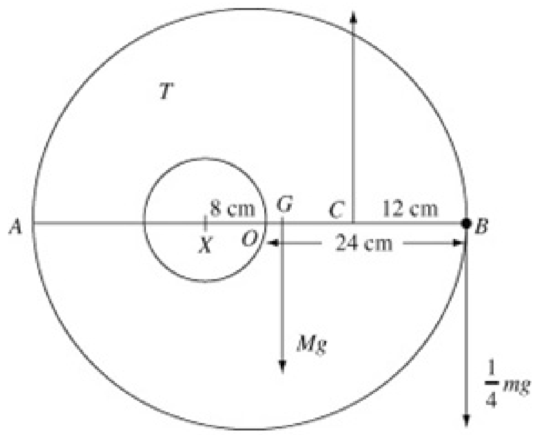
The figure shows a template T made by removing a circular disc, of centre X and radius 8 cm, from a uniform circular lamina, of centre O and radius 24 cm. The point X lies on the diameter AOB of the lamina and $AX = 16$ cm . The centre of mass of T lies at the point G.

a Find AG.

The template T is free to rotate about a smooth fixed horizontal axis, perpendicular to the plane of T, which passes through the midpoint of OB. A small stud of mass $\frac{1}{4}m$ is fixed at B, and T and the stud are in equilibrium with AB horizontal.

b Modelling the stud as a particle, find the mass of T in terms of m.

Solution:



a Let $AG = \bar{x}$ cm .

The area of the larger circle is $\pi \times 24^2 = 576\pi$ cm^2.

The area of the smallest circle is $\pi \times 8^2 = 64\pi$ cm^2.

The area of T is $576\pi - 64\pi = 64\pi$ cm^2.

$AX = (24 - 8)$ cm $= 16$ cm

	Larger circle	Smaller circle	T
Mass ratios	576π	64π	512π
Distance (cm)	24	16	$\bar{x}$

The large circle is uniform so the masses of the large circle, small circle and T are proportional to their areas.

The mass ratios $576\pi : 64\pi : 512\pi$ can be simplified to 9 : 1 : 8 and using these reduced ratios would ease the working.

M (A)

$$576\not{\pi} \times 24 = 64\not{\pi} \times 16 + 512\not{\pi} \times \bar{x}$$

$$\bar{x} = \frac{576 \times 24 - 64 \times 16}{512} = 25$$

$AG = 25$ cm

b Let C be the mid-point of OB and the mass of T be M.

$BC = 12$ cm, $CG = 11$ cm

M (C)

$$M\not{g} \times 11 = \frac{1}{4} m\not{g} \times 12$$

$$11M = 3m \Rightarrow M = \frac{3}{11}m$$

There is a force acting vertically upwards on T at C but, when moments are taken about C, this has zero moment.

The mass of T is $\frac{3}{11}m$.

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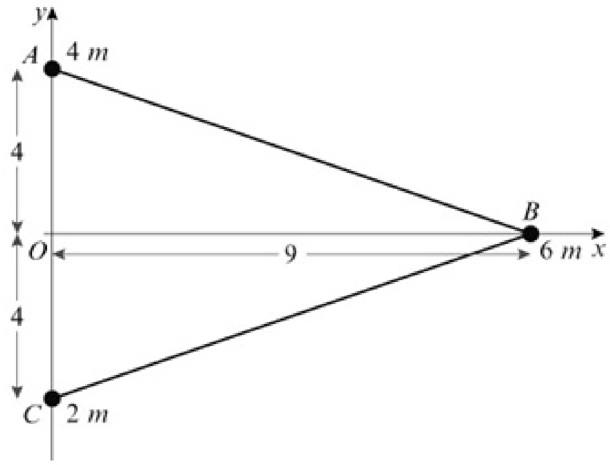
Review Exercise
Exercise A, Question 39

Question:



The figure shows a triangular lamina ABC. The coordinates of A, B and C are (0, 4), (9, 0) and (0, −4) respectively. Particles of mass $4m$, $6m$ and $2m$ are attached at A, B and C respectively.

a Calculate the coordinates of the centre of mass of the three particles, *without the lamina*.

The lamina ABC is uniform and of mass km. The centre of mass of the combined system consisting of the three particles and the lamina has coordinates $(4, \lambda)$.

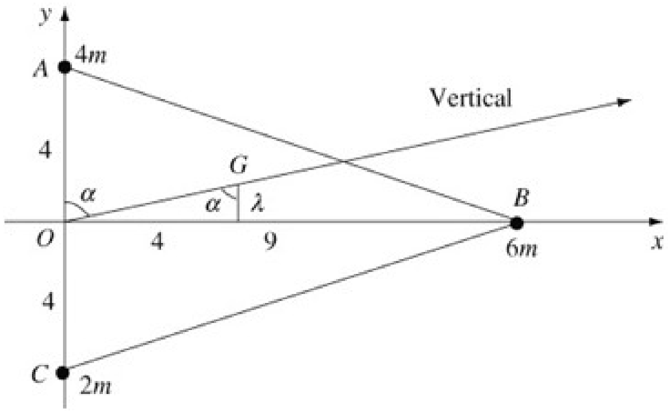
b Show that $k = 6$.

c Calculate the value of λ.

The combined system is freely suspended from O and hangs at rest.

d Calculate, in degrees to one decimal place, the angle between AC and the vertical.

Solution:



a Let the centre of mass of the three particles, without the lamina, be $(\bar{x}, \bar{y})$.
The total mass of the three particles is
$4m + 6m + 2m = 12m$

	Total	A	B	C
Mass	$12m$	$4m$	$6m$	$2m$
Distances (x)	$\bar{x}$	0	9	0
Distances (y)	$\bar{y}$	4	0	–4

You must take into account the signs of the coordinates.

M (Oy)

$12m \times \bar{x} = 6m \times 9$

$\bar{x} = \frac{54m}{12m} = \frac{9}{2}$

M (Ox)

$12m \times \bar{y} = 4m \times 4 + 2m \times (-4) = 8m$

$\bar{y} = \frac{8m}{12m} = \frac{2}{3}$

The centre of mass of the three particles, without the lamina, has coordinates $\left(\frac{9}{2}, \frac{2}{3}\right)$.

b The centre of mass of the lamina is (3, 0). Let the centre of mass of the combined system of the lamina and the three particles be at the point G.

The question gives you that the coordinates of G are $(4, \lambda)$.

The total mass of the system is

$12m + km = (12 + k)m$

	Combined system	Particles	Lamina
Mass	$(12+k)m$	$12m$	km
Distances (x)	4	$\frac{9}{2}$	3
Distances (y)	λ	$\frac{2}{3}$	0

Although you could start again, considering the three particles separately, it is sensible to use the result from **a** for the centre of mass of the three particles.

M (Oy)

$$(12+k)\cancel{m}\times 4 = 12\cancel{m}\times\frac{9}{2} + k\cancel{m}\times 3$$

$$48 + 4k = 54 + 3k$$

$k = 6$, as required

c M (Ox)

$$(12+k)m\times\lambda = 12m\times\frac{2}{3} + km\times 0$$

Using $k = 6$

$$18\lambda = 8$$

$$\lambda = \frac{8}{18} = \frac{4}{9}$$

d Let the angle between AC and the vertical be α.

$$\tan\alpha = \frac{4}{\lambda} = 4 \div \frac{4}{9} = 9$$

$$\alpha = 83.7^\circ \text{ (1 d.p.)}$$

When the particle is freely suspended from O, the centre of mass of the combined system G hangs vertically below O.

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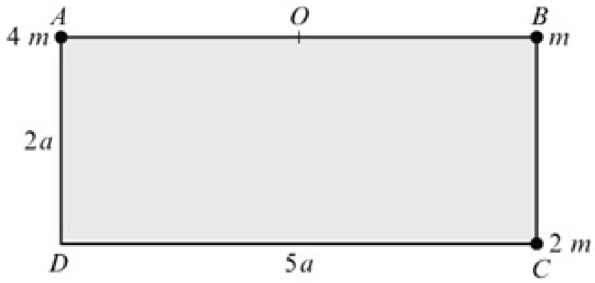
Review Exercise
Exercise A, Question 40

Question:



A loaded plate L is modelled as a uniform rectangular lamina $ABCD$ and three particles. The sides CD and AD of the lamina have length $5a$ and $2a$ respectively and the mass of the lamina is $3m$. The three particles have mass $4m$, m and $2m$ and are attached at the points A, B and C respectively, as shown in the figure.

a Show that the distance of the centre of mass of L from AD is $2.25a$.

b Find the distance of the centre of mass of L from AB.

The point O is the midpoint of AB. The loaded plate L is freely suspended from O and hangs at rest under gravity.

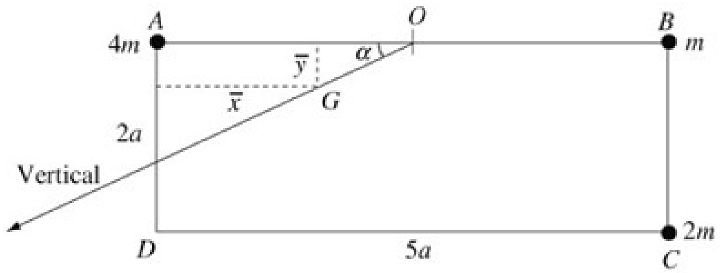
c Find, to the nearest degree, the size of the angle that AB makes with the horizontal.

A horizontal force of magnitude P is applied at C in the direction CD. The loaded plate L remains suspended from O and rests in equilibrium with AB horizontal and C vertically below B.

d Show that $P = \frac{5}{4}mg$.

e Find the magnitude of the force on L at O.

Solution:



Let the distances of the centre of mass of L, say G, from AD and AB be $\bar{x}$ and $\bar{y}$ respectively.

The distance $\bar{y}$ is measured from AB, not DC.

The mass of L is $3m + 4m + m + 2m = 10m$.

		Rectangle	Particles		
	L	$ABCD$	A	B	C
Mass	$10m$	$3m$	$4m$	m	$2m$
Distances (x)	$\bar{x}$	$2.5a$	0	$5a$	$5a$
Distances (y)	$\bar{y}$	a	0	0	$2a$

a M (AD)

$$10m \times \bar{x} = 3m \times 2.5a + m \times 5a + 2m \times 5a = 22.5ma$$

$$\bar{x} = \frac{22.5ma}{10m} = 2.25a, \text{ as required}$$

b M (AB)

$$10m \times \bar{y} = 3m \times a + 2m \times 2a = 7ma$$

$$\bar{y} = \frac{7ma}{10m} = 0.7a$$

c Let α be the angle between OA and the vertical

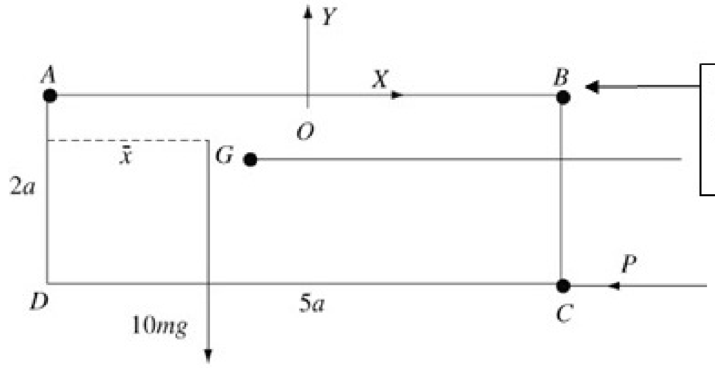
$$\tan\alpha = \frac{\bar{y}}{2.5a - \bar{x}} = \frac{0.7a}{0.25a} = 2.8$$

$$\alpha = 70° \text{ (nearest degree)}$$

When L is freely suspended from O, the centre of mass G of the complete system hangs vertically below O.

The angle that AB makes with the horizontal is $(90 - 70)° = 20°$ (nearest degree).

d



The total weight $10mg$ of the loaded plate L acts at the centre of mass G of the plate.

M (O)

$$P \times 2a = 10mg \times (2.5a - \bar{x})$$
$$= 10mg \times 0.25a$$
$$P = \frac{2.5mg\,a}{2a} = \frac{5}{4}mg, \text{ as required}$$

Using $\bar{x} = 2.25a$ from **a**.

e Let the horizontal and vertical components of the force acting on L at O be X and Y respectively.

$$\text{R}(\rightarrow) \quad X = P = \frac{5}{4}mg$$
$$\text{R}(\uparrow) \quad Y = 10mg$$

Let the magnitude of the force acting on L at O be R.

$$R^2 = X^2 + Y^2$$
$$= \left(\frac{5}{4}mg\right)^2 + (10mg)^2 = \frac{1625}{4}m^2g^2$$
$$R = \frac{\sqrt{1625}}{4}mg = \frac{5\sqrt{65}}{4}mg$$

This exact answer or approximate answers, such as $10.1mg$, would be accepted.

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Review Exercise
Exercise A, Question 41

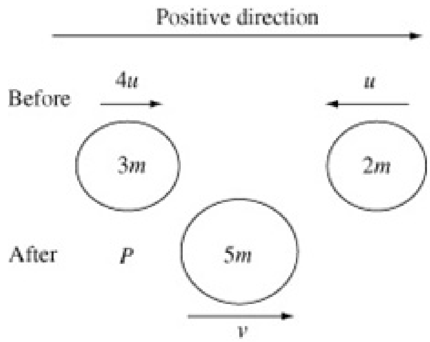
Question:

Two particles, of mass $3m$ and $2m$, are moving in opposite directions in a straight horizontal line with speeds $4u$ and u respectively. The particles collide and coalesce to form a single particle P. Calculate

a the speed of P in terms of u,

b the loss in kinetic energy, in terms of m and u, due to the collision.

Solution:



a Let the speed of P be v.

Conservation of linear momentum

$$4m\times 3m + 2m\times(-u) = 5mv$$
$$10mu = 5mv$$
$$v = \frac{10u}{5} = 2u$$

Part **a** of this question is in the M1 specification. The prerequisites in the M2 specification includes 'A knowledge of the specification for M1 … is assumed and may be tested'.

b The kinetic energy before impact is

$$\frac{1}{2}3m\times(4u)^2 + \frac{1}{2}2m\times u^2 = 24mu^2 + mu^2 = 25mu^2$$

The direction of motion of the particle of mass $2m$ is not relevant when working out its kinetic energy. In **a**, the direction was needed for calculating the linear momentum. Linear momentum is a vector quantity; kinetic energy is not.

The kinetic energy after impact is

$$\frac{1}{2}5mv^2 = \frac{1}{2}5m\times(2u)^2 = 10mu^2$$

The loss in kinetic energy is

$$25mu^2 - 10mu^2 = 15mu^2$$

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Review Exercise
Exercise A, Question 42

Question:

A particle of mass 4 kg is moving in a straight horizontal line. There is a constant resistive force of magnitude R newtons. The speed of the particle is reduced from 25 m s^{-1} to rest over a distance of 200 m.

Use the work–energy principle to calculate the value of R.

Solution:

The loss in kinetic energy is

$$\frac{1}{2}mu^2 = \frac{1}{2}4 \times 25^2 = 1250 \text{ J}$$

The initial kinetic energy of the particle is 1250 J. Its final kinetic energy is 0, as the particle has been brought to rest.

The work done by the resistive force is given by

$$\text{work done} = \text{force} \times \text{distance moved}$$
$$W = R \times 200$$

Using the work–energy principle

$$200R = 1250$$
$$R = \frac{1250}{200} = 6.25$$

The work done by the resistive force is equal to the loss of energy of the particle.

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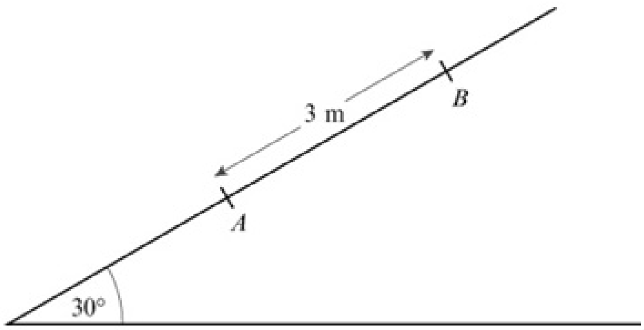
Review Exercise
Exercise A, Question 43

Question:



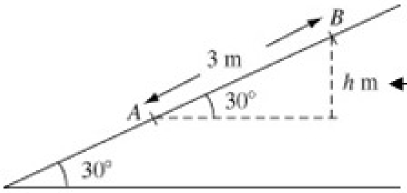
A particle P of mass 2 kg is projected from a point A up a line of greatest slope AB of a fixed plane. The plane is inclined at an angle of $30°$ to the horizontal and $AB = 3$ m with B above A, as shown in the figure. The speed of P at A is 10 m s^{-1}.

a Assuming the plane is smooth, find the speed of P at B.

The plane is now assumed to be rough. At A the speed of P is 10 m s^{-1} and at B the speed of P is 7 m s^{-1}.

b By using the work–energy principle, or otherwise, find the coefficient of friction between P and the plane.

Solution:



The change in the potential energy of P depends on the vertical distance it has moved. You find this using trigonometry.

a Let the vertical distance moved by P be h m.

$\frac{h}{3} = \sin 30° \Rightarrow h = 3\sin 30° = 1.5$

The potential energy gained by P is given by

$\text{PE} = mgh = 2 \times 9.8 \times 1.5 = 29.4$

Let the speed of P at B be v m s^{-1}.

The kinetic energy lost by P is given by

$$\text{K.E.} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}2\times 10^2 - \frac{1}{2}2v^2 = 100 - v^2$$

Using the principle of conservation of energy

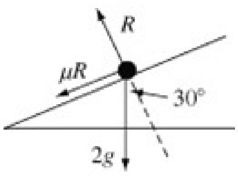
$$100 - v^2 = 29.4$$
$$v^2 = 100 - 29.4 = 70.6$$
$$v = \sqrt{70.6} = 8.402\ldots$$

If no forces other than gravity are acting on the particle, as mechanical energy is conserved, the loss of kinetic energy must equal the gain in potential energy.

The speed of P at B is 8.4 m s^{-1} (2 s.f.).

As a numerical value of g has been used, you should round your final answer to 2 significant figures. Three significant figures are also acceptable.

b



Let the normal reaction between the particle and the plane have magnitude R N.

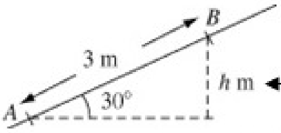
R(↖) $R = 2g\cos 30°$

The frictional force is given by

$F = \mu R = \mu 2g\cos 30°$

The kinetic energy lost by P is given by

$$\text{K.E.} = \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}2\times 10^2 - \frac{1}{2}2\times 7^2 = 51$$



The change in the potential energy of P depends on the vertical distance it has moved. You find this using

The potential energy gained by P is the same as in **a**.

The vertical height moved is the same as in **a**.

The total loss of mechanical energy, in J, is
$51 - 29.4 = 21.6$

The work done by friction is given by

$$\text{work done} = \text{force} \times \text{distance moved}$$
$$W = \mu R \times 3 = \mu 2g\cos 30° \times 3 = \mu \times 50.922\ldots$$

Using the work–energy principle

The work done by the friction is equal to the total loss of energy of the particle.

$\mu \times 50.922\ldots = 21.6 \Rightarrow \mu = 0.424\ldots$

The coefficient of friction is 0.42 (2 s.f.).

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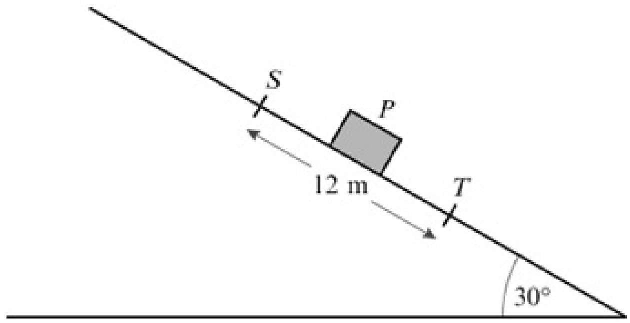
Review Exercise
Exercise A, Question 44

Question:



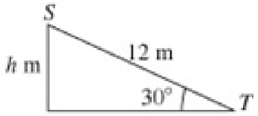
A small package is modelled as a particle P of mass 0.6 kg. The package slides down a rough plane from a point S to a point T, where $ST = 12$ m. The plane is inclined at $30°$ to the horizontal and ST is a line of greatest slope of the plane, as shown in the figure. The speed of P at S is 10 m s^{-1} and the speed of P at T is 9 m s^{-1}. Calculate

a the total loss of energy of P in moving from S to T,

b the coefficient of friction between P and the plane.

Solution:

a



In moving from S to T, P descends a vertical distance of h m, where

$\frac{h}{12} = \sin 30° \Rightarrow h = 12 \sin 30° = 6$

The potential energy, in J, lost by P is given by
$mgh = 0.6 \times 9.8 \times 6 = 35.28$

The change in the potential energy of P depends on the vertical distance it has moved. You find this using trigonometry.

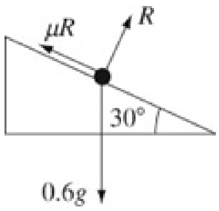
The kinetic energy, in J, lost by P is given by

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(u^2 - v^2)$$
$$= \frac{1}{2} \times 0.6 \times (10^2 - 9^2) = 5.7$$

The total loss of energy of P is
$(35.28 + 5.7)$ J $= 40.98$ J $= 41$ J (2 s.f.).

As P moves from S to T both kinetic and potential energy are lost.

b



Let the normal reaction between the particle and the plane have magnitude R N.
R(↗) $R = 0.6g \cos 30°$

The frictional force is given by
$F = \mu R = \mu 0.6g \cos 30° = \mu \times 5.09229\ldots$

The work done by friction is given by

$$\text{work done} = \text{force} \times \text{distance moved}$$
$$W = F \times 12 = \mu \times 61.106\ldots$$

Using the work–energy principle
$\mu \times 61.106\ldots = 40.98 \Rightarrow \mu = 0.6706$
The coefficient of friction is 0.67 (2 s.f.).

Friction opposes motion and acts up the plane. The work done by friction against the motion of the particle equals the total loss of energy of the particle. You should use the unrounded answer from **a** for the total energy loss.

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Review Exercise
Exercise A, Question 45

Question:

A particle P has mass 4 kg. It is projected from a point A up a line of greatest slope of a rough plane inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between P and the plane is $\frac{2}{7}$. The particle comes to rest instantaneously at the point B on the plane, where $AB = 2.5$ m. It then moves back down the plane to A.

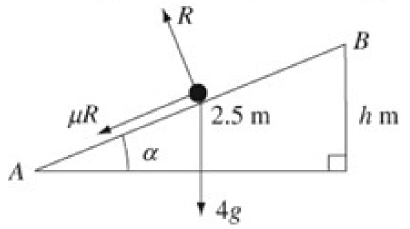
a Find the work done by friction as P moves from A to B.

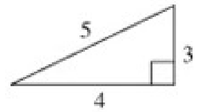
b Using the work–energy principle, find the speed with which P is projected from A.

c Find the speed of P when it returns to A.

Solution:

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$





> This diagram illustrates that if $\tan\alpha = \frac{3}{4}$, $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$.

Let the normal reaction between the particle and the plane have magnitude R N.

R(↖) $R = 4g\cos\alpha = 4 \times 9.8 \times \frac{4}{5} = 31.36$

The work done by friction is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= \mu R \times AB \\ &= \tfrac{2}{7} \times 31.36 \times 2.5 = 22.4 \end{aligned}$$

> The magnitude of the frictional force is given by $F = \mu R$ for the particle's motion both up and down the plane. The direction of the frictional force changes but its magnitude does not.

The work done by friction in moving from A to B is 22.4 J.

b Let the vertical distance moved by P in moving from A to B be h m.

$\frac{h}{2.5} = \sin\alpha \Rightarrow h = 2.5 \times \frac{3}{5} = 1.5$

The potential energy, in J, gained by P in moving from A to B is given by

$mgh = 4 \times 9.8 \times 1.5 = 58.8$

Let the speed of P at A be u m s^{-1}.

The kinetic energy, in J, lost by P in moving from A to B is given by

$\frac{1}{2}mu^2 = 2u^2$

> At B the particle is instantaneously at rest and has no kinetic energy. So all of the initial kinetic energy has been lost.

The mechanical energy, in J, lost by P in moving from A to B is given by

$2u^2 - 58.8$

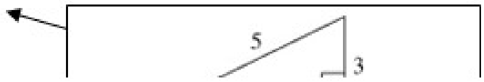
> In moving from A to B, kinetic energy has been lost and potential energy gained. The difference between the values is the net loss.

Using the work–energy principle

$$\begin{aligned} 22.4 &= 2u^2 - 58.8 \\ u^2 &= \frac{58.8 + 22.4}{2} = 40.6 \\ u &= \sqrt{40.6} = 6.371\ldots \end{aligned}$$

The speed of P at A is 6.4 m s^{-1} (2 s.f.).

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$



c Let the speed of P when it returns to A be v m s^{-1}.

The work done by friction as P moves from B to A is the same that it does as P moves from A to B. Hence the total work done by friction is

$2 \times 22.4 = 44.8$ J

The work done depends on the normal reaction and the distance moved. Both the reaction and the distance are the same as the particle moves from A to B and from B to A. So you can find the total work done by friction by doubling your answer to **a**.

By the work–energy principle

$$44.8 = \frac{1}{2}mu^2 - \frac{1}{2}mv^2$$
$$= 2 \times 40.6 - 2v^2$$
$$v^2 = \frac{81.2 - 44.8}{2} = 18.2$$
$$v = \sqrt{18.2} = 4.266\ldots$$

As the particle is now at the same level as it started, there is no change in its potential energy. So the change in mechanical energy is just the loss in kinetic energy.

The speed of P when it returns to A is 4.3 m s^{-1} (2 s.f.).

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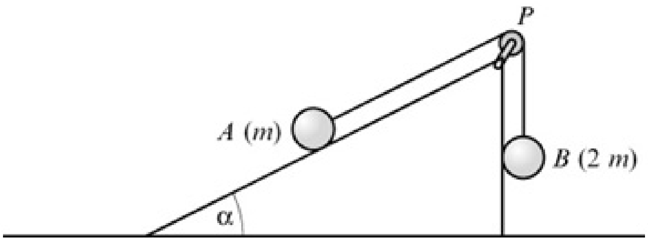
Review Exercise
Exercise A, Question 46

Question:



Two particles A and B of mass m and $2m$ respectively are attached to the ends of a light inextensible string. The particle A lies on a rough plane inclined at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$. The string passes over a small light pulley P fixed at the top of the plane. The particle B hangs freely below P, as shown in the figure. The particles are released from rest with the string taut and the section of the string from A to P parallel to a line of greatest slope of the plane. The coefficient of friction between A and the plane is $\frac{5}{8}$. When each particle has moved a distance h, B has not reached the ground and A has not reached P.

a Find an expression for the potential energy lost by the system when each particle has moved a distance h.

When each particle has moved a distance h, they are moving with speed v.

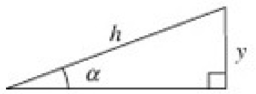
b Using the work–energy principle, find an expression for v^2, giving your answer in the form kgh where k is a number.

Solution:

a $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

You can sketch a 3, 4, 5 triangle to check these relations.

As B descends a distance h, A moves a distance h up the plane. Let the vertical displacement of A be y.



$\frac{y}{h} = \sin\alpha = \frac{3}{5} \Rightarrow y = \frac{3}{5}h$

The potential energy lost by B is $2mgh$.

B has mass $2m$ and descends a distance h.

The potential energy gained by A is

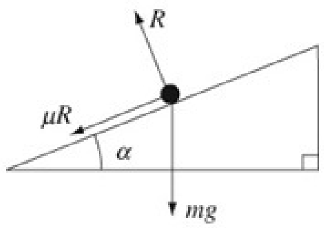
$mgy = mg \times \frac{3}{5}h = \frac{3}{5}mgh$

A has mass m and ascends a vertical distance $\frac{3}{5}h$.

The net loss in potential energy of the system is

$2mgh - \frac{3}{5}mgh = \frac{7}{5}mgh$

b For A



Let the normal reaction between the particle and the plane have magnitude R N.

R(↖) $R = mg\cos\alpha = mg \times \frac{4}{5} = \frac{4}{5}mg$

The work done by friction is given by

$$\begin{aligned} \text{work done} &= \text{force} \times \text{distance moved} \\ W &= \mu R \times h \\ &= \frac{5}{8} \times \frac{4}{5}mg \times h = \frac{1}{2}mgh \end{aligned}$$

The gain in kinetic energy is

$\frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 = \frac{3}{2}mv^2$

Both particles start from rest, so the system has no initial kinetic energy.

The net loss of mechanical energy is

$\frac{7}{5}mgh - \frac{3}{2}mv^2$

The total loss in mechanic energy is the difference between the loss in potential energy you worked out in **a** less the kinetic energy gained.

Using the work-energy principle

$$\frac{1}{2}mgh = \frac{7}{5}mgh - \frac{3}{2}mv^2$$

$$\frac{3}{2}mv^2 = \left(\frac{7}{5} - \frac{1}{2}\right)mgh = \frac{9}{10}mgh$$

$$v^2 = \frac{2}{3} \times \frac{9}{10}gh = \frac{3}{5}gh$$

The work done by friction against the motion of the particle A equals the total loss of energy of the system. Other than gravity, there is no force acting on B. The only force causing the loss of mechanical energy is the friction acting on A.

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Review Exercise
Exercise A, Question 47

Question:

The engine of a car is working at a constant rate of 6 kW in driving a car along a straight horizontal road at 54 km h^{-1}. Find, in N, the magnitude of the resistance to motion of the car.

Solution:

Let F N be the magnitude of the driving force produced by the engine and R N be the magnitude of the resistance to motion.

$R(\rightarrow) \quad F - R = 0 \Rightarrow R = F$

This is Newton's Second Law $\mathbf{F} = m\mathbf{a}$. As the car is travelling at a constant speed the acceleration is zero.

$6 \text{ kW} = 6000 \text{ W}$

$$54 \text{ km h}^{-1} = \frac{54 \times 1000}{3600} \text{ m s}^{-1} = 15 \text{ m s}^{-1}$$

Before you use the formula $P = Fv$, you have to convert the data in kilowatts and kilometres an hour into base SI units.

$$P = Fv$$

Power = force × speed

$$6000 = F \times 15 \Rightarrow F = \frac{6000}{15} = 400$$

Hence $R = F = 400$

The resistance to motion has magnitude 400 N.

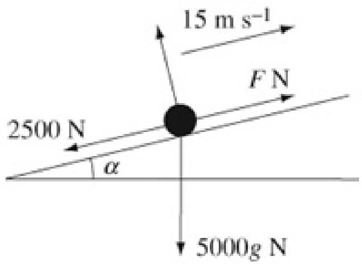
Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 48

Question:

A lorry of mass 5000 kg moves at a constant speed of 15 m s^{-1} up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{15}$. The resistance experienced by the lorry is constant and has magnitude 2500 N.
Find, in kW, the rate of working of the lorry's engine.

Solution:



Let F N be the magnitude of the driving force produced by the engine.

$\text{R}(\nearrow) \quad \mathbf{F} = m\mathbf{a}$

$F - 2500 - 5000g\sin\alpha = 0$

$F = 2500 + 5000 \times 9.8 \times \frac{1}{15} = 2826.\dot{6}$

$P = Fv$

$= 2826.\dot{6} \times 15 = 42\,400$

The rate of working of the lorry's engine is 42 kW (2 s.f.).

As the lorry is travelling at a constant speed, its acceleration is zero.

The component of the weight down the plane is $5000g\cos(90 - \alpha) = 5000g\sin\alpha$.

$P = Fv$ gives the answer in watts. To give the answer in kilowatts, as the question requires, you divide by 1000. Either 42 kW or 42.4 kW would be an acceptable answer.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

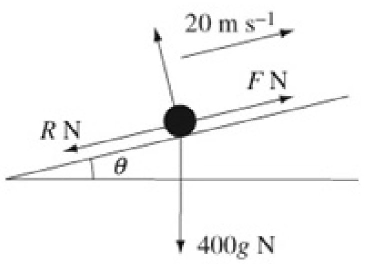
Review Exercise
Exercise A, Question 49

Question:

A car of mass 400 kg is moving up a straight road inclined at an angle θ to the horizontal where $\sin\theta = \frac{1}{14}$. The resistance to motion of the car from non-gravitational forces is modelled as a constant force of magnitude R newtons. When the car is moving at a constant speed of 20 m s^{-1}, the power developed by the car's engine is 10 kW.

Find the value of R.

Solution:



Let FN be the magnitude of the driving force produced by the engine.

$$10\text{ kW} = 10\,000\text{ W}$$

Before you use the formula $P = Fv$, you have to convert kilowatts to watts.

$$P = Fv$$
$$10\,000 = F \times 20 \Rightarrow F = 500$$

$$\text{R}(\nearrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - R - 400g\sin\theta = 0$$

As the car is travelling at a constant speed, its acceleration is zero.

$$R = F - 400g\sin\theta$$
$$= 500 - 400 \times 9.8 \times \frac{1}{14} = 220$$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 50

Question:

A lorry of mass 1500 kg moves along a straight horizontal road. The resistance to motion of the lorry has magnitude 750 N and the lorry's engine is working at a rate of 36 kW.

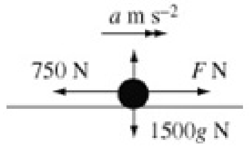
a Find the acceleration of the lorry when its speed is 20 m s^{-1}.

The lorry comes to a hill inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{10}$. The magnitude of the resistance to motion from non-gravitational forces remains 750 N. The lorry moves up the hill at a constant speed of 20 m s^{-1}.

b Find the rate at which the lorry is now working.

Solution:

a



Let the acceleration of the lorry be a m s^{-2} and the driving force of the engine have magnitude FN.

$$36\text{ kW} = 36\,000\text{ W}$$

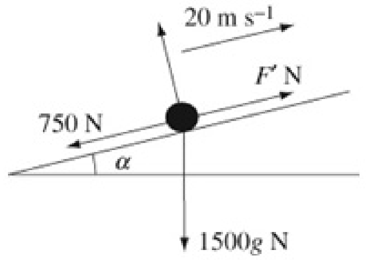
kW must be converted to W.

$$P = Fv$$
$$36\,000 = F \times 20 \Rightarrow F = 1800$$
$$R(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - 750 = 1500a$$
$$1800 - 750 = 1500a$$
$$a = \frac{1800-750}{1500} = 0.7$$

The acceleration of the lorry when the speed is 20 m s^{-1} is 0.7 m s^{-2}.

This result is only true at one instant in time. The speed would now increase and the driving force and acceleration decrease.

b



Let the driving force of the engine have magnitude F' N.

The driving forces in **a** and **b** are different and it is a good idea to avoid confusion by using different symbols for the forces.

In this part of the question the lorry is moving at a constant speed and the acceleration is zero.

$$R(\nearrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F' - 750 - 1500g\sin\alpha = 0$$
$$F' = 750 + 1500 \times 9.8 \times \frac{1}{10} = 2220$$
$$P = Fv$$
$$= 2220 \times 20 = 44\,400$$

The rate at which the lorry is now working is 44.4 kW.

This question asks for no particular form of the answer, so you could give your answer in either W or kW. Two or three significant figures are acceptable.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 51

Question:

A car of mass 1200 kg moves along a straight horizontal road. The resistance to motion of the car from non-gravitational forces is of constant magnitude 600 N. The car moves with constant speed and the engine of the car is working at a rate of 21 kW.

a Find the speed of the car.

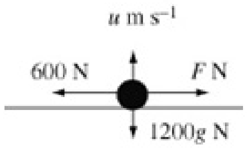
The car moves up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{14}$.

The car's engine continues to work at 21 kW and the resistance to motion from non-gravitational forces remains of magnitude 600 N.

b Find the constant speed at which the car moves up the hill.

Solution:

a



Let the speed of the car be u m s^{-1} and the driving force of the engine have magnitude F N.

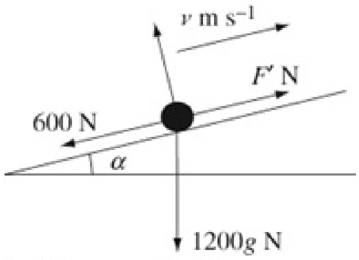
$21\text{ kW} = 21\,000\text{ W}$

$$R(\rightarrow)\quad F - 600 = 0 \Rightarrow F = 600$$
$$P = Fv$$
$$21\,000 = 600u \Rightarrow u = \frac{21\,000}{600} = 35$$

The speed of the car is 35 m s^{-1}.

In both parts of this question, as the car is moving with constant speed, the acceleration is zero. So the vector sum of the forces acting on the car is zero.

b



Let the speed of the car be u m s^{-1} and the driving force of the engine have magnitude F' N.

The driving forces in **a** and **b** are different and it is a good idea to avoid confusion by using different symbols for the forces.

$$R(\nearrow)\quad F' - 1200g\sin\alpha - 600 = 0$$
$$F' = 1200 \times 9.8 \times \frac{1}{14} + 600 = 1440$$
$$P = Fv$$
$$21\,000 = 1440v \Rightarrow v = 14.58\dot{3}$$

The constant speed of the car as it moves up the hill is 15 m s^{-1} (2 s.f.).

Although there is an exact answer, $14\frac{7}{12}$, a numerical value for g has been used in the question and the answer should be rounded to 2 significant figures. Three significant figures (14.6) is also acceptable.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 52

Question:

A car of mass 800 kg tows a caravan of mass 480 kg along a straight level road. The tow-bar connecting the car and the caravan is horizontal and of negligible mass. With the car's engine working at a rate of 24 kW, the car and caravan are travelling at a constant speed of 25 m s^{-1}.

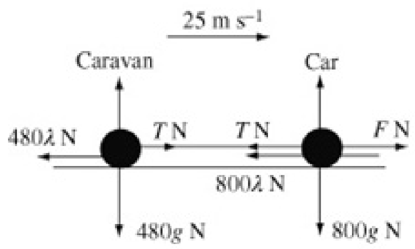
a Calculate the magnitude of the total resistance to the motion of the car and the caravan.

The resistance to the motion of the car has magnitude 800λ newtons and the resistance to the motion of the caravan has magnitude 480λ newtons, where λ is a constant. Find

b the value of λ,

c the tension in the tow-bar.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the car and R N be the total magnitude of the resistance to the motion of both the car and the caravan.

For the combined car and caravan
$R(\rightarrow)\ F - R = 0 \Rightarrow F = R$

When you consider the car and caravan combined, the tensions at the ends the tow-bar cancel one another out and can be ignored. When you consider the caravan and car separately, the tensions have to be included in your equations.

$$24\text{ kW} = 24\,000\text{ W}$$
$$P = Fv$$
$$24\,000 = F \times 25 \Rightarrow F = \frac{24\,000}{25} = 960$$
$$R = F = 960$$

The magnitude of the total resistance to the motion of the car and caravan is 960 N.

b

$$480\lambda + 800\lambda = 960$$
$$1280\lambda = 960 \Rightarrow \lambda = \frac{960}{1280} = 0.75$$

The sum of the resistances to the caravan and the car must equal the total resistance, 960 N.

c Let the magnitude of the tension in the tow-bar be T N.
For the caravan alone

You could consider the car alone. In that case your working would be
$F - T - 800\lambda = 0$
$T = F - 800\lambda = 960 - 800 \times 0.75 = 360$.

$$R(\rightarrow)\quad T - 480\lambda = 0$$
$$T = 480\lambda = 480 \times 0.75 = 360$$

The tension in the tow-bar is 360 N.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 53

Question:

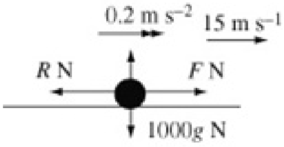
A car of mass 1000 kg is moving along a straight horizontal road. The resistance to motion is modelled as a constant force of magnitude R newtons. The engine of the car is working at a constant rate of 12 kW. When the car is moving with speed 15 m s^{-1}, the acceleration of the car is 0.2 m s^{-2}.

a Show that $R = 600$.

The car now moves with constant speed $U \text{ m s}^{-1}$ downhill on a straight road inclined at θ to the horizontal, where $\sin\theta = \frac{1}{40}$. The engine of the car is now working at a rate of 7 kW. The resistance to motion from non-gravitational forces remains of magnitude R newtons.

b Calculate the value of U.

Solution:

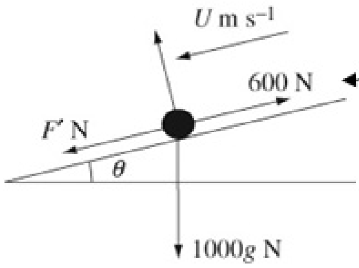


a Let F N be the magnitude of the driving force produced by the engine of the car.

$$12\text{ kW} = 12\,000\text{ W}$$
$$P = Fv$$
$$12\,000 = F \times 15$$
$$F = \frac{12\,000}{15} = 800$$
$$\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - R = 1000 \times 0.2$$
$$R = F - 1000 \times 0.2$$
$$= 800 - 200 = 600, \text{ as required}$$

Using Newton's second law, the vector sum of the forces on the car equals the mass times acceleration.

b



Resistance acts against motion. As the car is travelling down the hill, the resistance of 600 N acts up the hill.

Let the driving force of the engine have magnitude F' N.

$$\text{R}(\swarrow) F' + 1000g\sin\theta - 600 = 0$$
$$F' = 600 - 1000 \times 9.8 \times \frac{1}{40} = 355$$

As the car is travelling at a constant speed, there is no acceleration.

$$7\text{ kW} = 7000\text{ W}$$
$$P = Fv$$
$$7000 = 355U$$
$$U = \frac{7000}{355} = 19.718\ldots = 20\ (2\text{ s.f.})$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

A car of mass 1000 kg is moving along a straight road with constant acceleration f m s^{-2}. The resistance to motion is modelled as a constant force of magnitude 1200 N. When the car is travelling at 12 m s^{-1}, the power generated by the engine of the car is 24 kW.

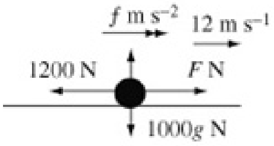
a Calculate the value of f.

When the car is travelling at 14 m s^{-1}, the engine is switched off and the car comes to rest without braking in a distance d metres.

b Assuming the same model for resistance, use the work–energy principle to calculate the value of d.

c Give a reason why the model used for resistance may not be realistic.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the car.

$$24\text{ kW} = 24\,000\text{ W}$$
$$P = Fv$$
$$24\,000 = F \times 12 \Rightarrow F = 2000$$

$$\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$$
$$F - 1200 = 1000f$$
$$2000 - 1200 = 1000f \Rightarrow f = \frac{800}{1000} = 0.8$$

b The kinetic energy, in J, lost as the car is brought to rest is

$$\frac{1}{2}mu^2 = \frac{1}{2}1000 \times 14^2 = 98\,000$$

The final kinetic energy is zero.

$$\text{Work done by resistance} = \text{Energy lost}$$
$$\text{Resistance} \times \text{distance} = \text{Energy lost}$$
$$1200d = 98\,000$$
$$d = \frac{98\,000}{1200} = 81\frac{2}{3}$$

You use the work-energy principle. The work done by the resistance (1200 N) in bringing the car to rest is equal to the kinetic energy lost.

c Resistance usually varies with speed.

As the speed slows down, the resistance to motion usually decreases. In this case, this might mean that the car would travel further.

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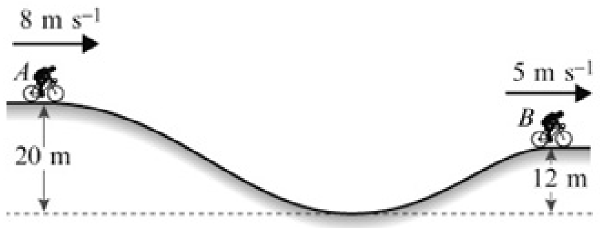
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:



The figure shows the path taken by a cyclist in travelling on a section of a road. When the cyclist comes to the point A on the top of the hill she is travelling at 8 m s^{-1}. She descends a vertical distance of 20 m to the bottom of the hill. The road then rises to the point B through a vertical distance of 12 m. When she reaches B her speed is 5 m s^{-1}. The total mass of the cyclist and the cycle is 80 kg and the total distance along the road from A to B is 500 m. By modelling the resistance to the motion of the cyclist as of constant magnitude 20 N,

a find the work done by the cyclist in moving from A to B.

At B the road is horizontal.

b Given that at B the cyclist is accelerating at 0.5 m s^{-2}, find the power generated by the cyclist at B.

Solution:

a From A to B, the cyclist descends

$(20-12)\text{ m} = 8\text{ m}$

> Whatever the path taken, the potential energy lost in travelling from A to B depends solely on the difference in levels between A and B.

The potential energy, in J, lost in travelling from A to B is given by

$mgh = 80 \times 9.8 \times 8 = 6272$

The kinetic energy, in J, lost in travelling from A to B is given by

$$\begin{aligned}\tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2 &= \tfrac{1}{2}m\left(u^2 - v^2\right)\\ &= 40\left(8^2 - 5^2\right) = 1560\end{aligned}$$

The total mechanical energy lost is
$(6272 + 1560)\text{ J} = 7832\text{ J}$

The work done by resistance due to non-gravitational forces is given by

$$\begin{aligned}W &= \text{force} \times \text{distance moved}\\ &= 20 \times 500 = 10\,000\end{aligned}$$

$(10\,000 - 7832)\text{ J} = 2168\text{ J}$

> The non-gravitational resistances to motion have worked 10 000 J against the motion. However, the mechanical energy lost is only 7832 J. The difference between these values is the work that has been done by the cyclist.

The work done by the cyclist in moving from A to B is 2200 J (2 s.f.).

b At B, let the force generated by the cyclist be F N.

$$\begin{aligned}R(\rightarrow)\quad \mathbf{F} &= m\mathbf{a}\\ F - 20 &= 80 \times 0.8\\ P &= Fv\\ &= 60 \times 5 = 300\end{aligned}$$

The power generated by the cyclist is 300 W.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 56

Question:

A van of mass 1500 kg is driving up a straight road inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{12}$. The resistance to motion due to non-gravitational forces is modelled as a constant force of magnitude 1000 N.

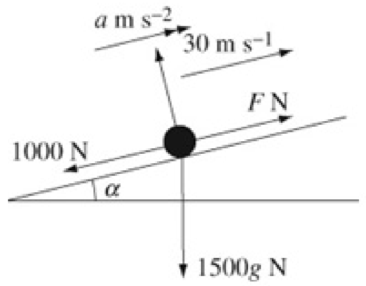
a Given that initially the speed of the van is 30 m s^{-1} and that the van's engine is working at a rate of 60 kW, calculate the magnitude of the initial deceleration of the van.

When travelling up the same hill, the rate of working of the van's engine is increased to 80 kW.

b Using the same model for the resistance due to non-gravitational forces, calculate in m s^{-1} the constant speed which can be sustained by the van at this rate of working.

c Give one reason why the use of this model for resistance may mean your answer to part **b** is too high.

Solution:



a Let F N be the magnitude of the driving force produced by the engine of the van and a m s^{-2} be the acceleration of the van.

From the wording of the question, you would expect a to be negative and this gives you a check on your working.

$$60\text{ kW} = 60\,000\text{ W}$$
$$P = Fv$$
$$60\,000 = F \times 30 \Rightarrow F = 2000$$

R(↗) $\mathbf{F} = m\mathbf{a}$

$$F - 1000 - 1500g\sin\alpha = 1500 \times a$$
$$1500a = 2000 - 1000 - 15 \times 9.8 \times \frac{1}{12} = -225$$
$$a = -\frac{225}{1500} = -0.15$$

The deceleration of the van is 0.15 m s^{-2}.

A common error is to leave the answer as -0.15. That is the acceleration and you have been asked for the deceleration.

b Let F' N be the magnitude of the driving force produced by the engine of the van at the increased power and u m s^{-1} be the constant speed.

R(↗) $F - 1000 - 1500g\sin\alpha = 0$

In **b**, the van is travelling at a constant speed and the acceleration is zero.

$$F' = 1000 + 1500 \times 9.8 \times \frac{1}{12} = 2225 *$$
$$80\text{ kW} = 80\,000\text{ W}$$
$$P = Fv$$
$$80\,000 = 2225u \Rightarrow u = \frac{80\,000}{2225} = 35.955\ldots$$

The constant speed is 36 m s^{-1} (2 s.f.).

c The resistance usually increases with speed. The equation $*$ in **b** would then give an increased value of F'.

Unless the speed is very high, resistances to motion are usually proportional to speed.

As $v = \frac{80\,000}{F'}$, an increased value of F' gives a lower value of v, and the answer in **b** would be too high.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 57

Question:

A model car has weight 200 N. It undergoes tests on a straight hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{10}$. The engine of the car works at a constant rate of P watts.

When the car goes up the hill it is observed to travel at a constant speed of 8 m s^{-1}. Given that the total resistance to the motion of the car from forces other than gravity is R newtons,

a express P in terms of R.

When the car runs down the same hill with the engine running at the same rate, it is observed to travel at a constant speed of 24 m s^{-1}.

In an initial model of the situation the resistance to motion due to non-gravitational forces is assumed to be constant whatever the speed of the car.

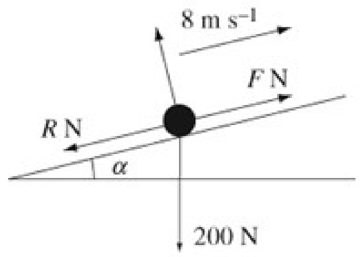
b Using this model, find an estimate for the value of P.

In a refined model the resistance to motion due to non-gravitational forces is assumed to be proportional to the speed of the car.

c Using this model, find a revised estimate for P.

Solution:

a



Let F N be the magnitude of the driving force produced by the engine of the model car.

$$P = Fv$$

$$P = F \times 8 \Rightarrow F = \frac{P}{8} \qquad (1)$$

$$\text{R}(\nearrow) F - R - 200\sin\alpha = 0$$

$$F = R + 200 \times \frac{1}{10} = R + 20 \qquad (2)$$

Eliminating F from (1) and (2)

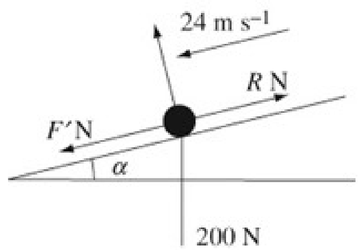
$$\frac{P}{8} = R + 20$$

$$P = 8(R + 20) \qquad (3)$$

To obtain an expression for P in terms of R, you will have to eliminate F from the two equations you get from $P = Fv$ and resolving up the hill.

You have been given the weight of the model car in newtons and not the mass in kilograms. g is not needed. It is a common error, when not reading the question carefully, to add g when weights are given and not masses.

b



Let F' N be the magnitude of the driving force produced by the engine of the model car as it moves downhill.

$$P = Fv$$

$$P = F' \times 24 \Rightarrow F' = \frac{P}{24} \qquad (4)$$

$$\text{R}(\swarrow) F' - R + 200\sin\alpha = 0$$

$$F' = R - 200 \times \frac{1}{10} = R - 20 \qquad (5)$$

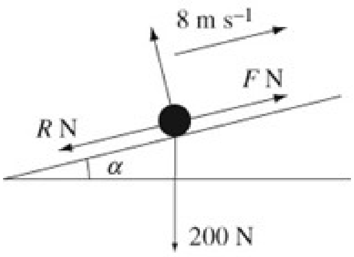
Eliminating F' from (4) and (5)

$$\frac{P}{24} = R - 20$$

$$P = 24(R - 20) \qquad (6)$$

You start **b** by repeating **a**, with the directions of the driving force and resistance reversed. You can then find R and P by solving simultaneous equations.

a



Let F N be the magnitude of the driving
Eliminating P from (3) and (6)

$$\begin{aligned} 8(R+20) &= 24(R-20) \\ 8R+160 &= 24R-480 \\ 16R &= 640 \Rightarrow R=40 \end{aligned}$$

Substituting $R=40$ into (6)

$$P = 24(40-20) = 480$$

c When the resistance is proportional to the speed, equation (4) is still correct.

$$R(\swarrow) F' - 3R + 200\sin\alpha = 0$$

$$F' = 3R - 200\times\frac{1}{10} = 3R - 20 \ldots 1.4 \text{ m s}^{-2} \qquad (5)'$$

The speed in **c**, 24 m s^{-1}, is three times the speed in **a** 8 m s^{-1}. If the resistance is proportional to the speed, the resistance in **c** must be three times the resistance in **a**. In **a** the resistance is R, so here it is $3R$. Solving **c**, is essentially a repeat of **b**, replacing R by $3R$ where appropriate.

Eliminating F' from (4) and (5)$'$

$$\begin{aligned} \frac{P}{24} &= 3R-20 \\ P &= 24(3R-20) \qquad (6)' \end{aligned}$$

Eliminating P from (3) and (6)$'$

$$\begin{aligned} 8(R+20) &= 24(3R-20) \\ 8R+160 &= 72R-480 \\ 64R &= 640 \Rightarrow R=10 \end{aligned}$$

Substituting $R=10$ into (6)$'$

$$P = 24(30-20) = 240$$

The answer here is half the answer in **b**. In many questions, it is assumed that resistance is a constant and you are sometimes asked to comment on this. This question shows the error which can follow from such an assumption.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 58

Question:

A car of mass 1000 kg is towing a trailer of mass 1500 kg along a straight horizontal road. The tow-bar joining the car to the trailer is modelled as a light rod parallel to the road. The total resistance to motion of the car is modelled as having constant magnitude 750 N. The total resistance to motion of the trailer is modelled as a force of magnitude R newtons, where R is a constant. When the engine is working at a rate of 50 kW, the car and the trailer travel at a constant speed of 25 m s^{-1}.

a Show that $R = 1250$.

When travelling at 25 m s^{-1} the driver of the car disengages the engine and applies the brakes. The brakes provide a constant braking force of magnitude 1500 N to the car. The resisting forces of magnitude 750 N and 1250 N are assumed to remain unchanged. Calculate

b the deceleration of the car while braking,

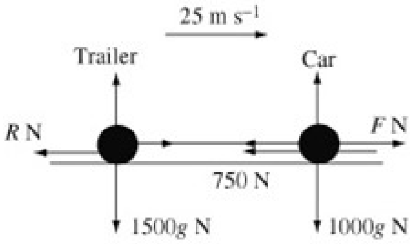
c the thrust in the tow-bar while braking,

d the work done, in kJ, by the braking force in bringing the car and the trailer to rest.

e Suggest how the modelling assumption that the resistances to motion are constant could be refined to be more realistic.

Solution:

a



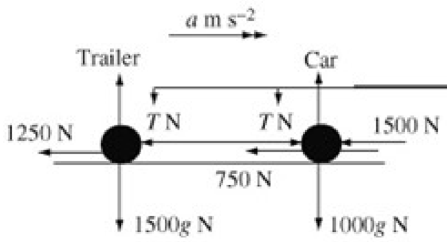
Let F N be the magnitude of the driving force produced by the engine of the car.

$$50\text{ kW} = 50\,000\text{ W}$$
$$P = Fv$$
$$50\,000 = F \times 25 \Rightarrow F = 2000$$

For the car and trailer combined
$R(\rightarrow)\ F - 750 - R = 0$
$R = F - 750 = 2000 - 750 = 1250$, as required

When you consider the car and trailer combined, the tensions at the ends the tow-bar cancel one another out and can be ignored.

b



As the car brakes, the forces in the tow-bar are thrusts and act in the directions shown in this diagram. The forces in the tow-bar in **a** are tensions and act in the opposite directions to thrusts.

Let the acceleration of the car while braking be a m s^{-2}.

For the car and trailer combined

$$R(\rightarrow)\ \mathbf{F} = m\mathbf{a}$$
$$-1500 - 750 - 1250 = 2500a$$
$$2500a = -3500 \Rightarrow a = -\frac{3500}{2500} = -1.4$$

The deceleration of the car while braking is 1.4 m s^{-2}.

c Let the magnitude of the thrust in the tow-bar while braking be T N.

For the trailer alone

$$\begin{aligned} R(\rightarrow) \quad \mathbf{F} &= m\mathbf{a} \\ -1250 - T &= 1500a = 1500 \times (-1.4) \\ T &= 1500 \times 1.4 - 1250 = 850 \end{aligned}$$

The magnitude of the thrust in the tow-bar while braking is 850 N.

d To find the distance travelled in coming to rest

$v^2 = u^2 + 2as$

$$\begin{aligned} 0^2 &= 25^2 + 2 \times (-1.4)s \\ s &= \frac{25^2}{2.8} \end{aligned}$$

The work done by a force is its magnitude multiplied by the distance moved by its point of application. This formula you learnt for the M1 module gives you a straightforward way of finding this distance.

The work done, in J, by the braking force of 1500 N is given by

$W = 1500 \times s = 1500 \times \frac{25^2}{2.8} = 334\,821$

Work done = force × distance moved

The work done by the braking force in bringing the car and the trailer to rest is 335 kJ (3 s.f.).

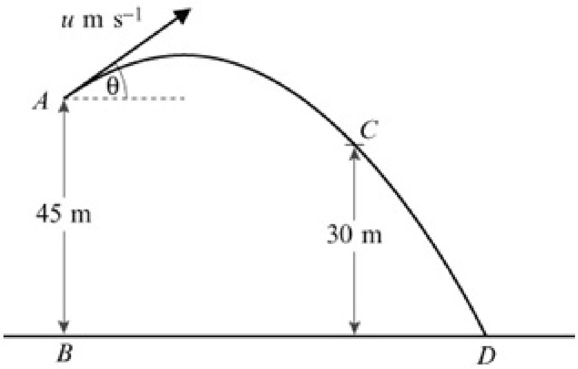
e The resistance could be modelled as varying with speed.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 59

Question:



A particle P is projected from a point A with speed u m s^{-1} at an angle of elevation θ, where $\cos\theta = \frac{4}{5}$. The point B, on horizontal ground, is vertically below A and $AB = 45$ m . After projection, P moves freely under gravity passing through a point C, 30 m above the horizontal ground, before striking the ground at the point D, as shown in the figure above.

Given that P passes through C with speed 24.5 m s^{-1},

a using conservation of energy, or otherwise, show that $u = 17.5$,

b find the size of the angle which the velocity of P makes with the horizontal as P passes through C,

c find the distance BD.

Solution:

a The vertical distance fallen by P in moving from A to C is $(45-30)\text{ m} = 15\text{ m}$.

Using the principle of conservation of energy,
Kinetic energy gained = Potential energy lost

$$\frac{1}{2}\cancel{m}v^2 - \frac{1}{2}\cancel{m}u^2 = \cancel{m}gh$$

The mass of the particle cancels throughout this equation. The calculations in this question are independent of the mass of P.

$$\frac{1}{2}\times 24.5^2 - \frac{1}{2}u^2 = 9.8\times 15$$

$$u^2 = 24.5^2 - 2\times 9.8\times 15 = 306.25$$

$$u = \sqrt{306.25} = 17.5\text{, as required}$$

This equation has a similar form to $v^2 = u^2 + 2as$. However, it would be an error to use this formula, which is a formula for motion in a straight line, as P is not moving in a straight line.

b $R(\rightarrow)$ $u_x = u\cos\theta = 17.5\times\frac{4}{5} = 14$

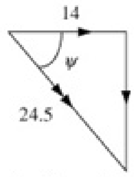
The horizontal component of the velocity is constant throughout the motion.

Let the required angle be ψ

$$\cos\psi = \frac{14}{24.5} = \frac{4}{7}$$

$$\psi = 55.15\ldots^\circ = 55^\circ \text{ (nearest degree)}$$

At C, the velocity of P and its components are illustrated in this diagram.



ψ can now be found using trigonometry. There is no need to find the vertical component of the velocity at C.

c $R(\uparrow)$ $u_y = u\sin\theta = 17.5\times\frac{3}{5} = 10.5$

To find the time taken for P to move from A to D

$$R(\uparrow)\quad s = ut + \frac{1}{2}at^2$$

$$-45 = 10.5t - 4.9t^2$$

$$4.9t^2 - 10.5t - 45 = 0$$

$$49t^2 - 105t - 450 = 0$$

$$(7t-30)(7t+15) = 0$$

$$t = \frac{30}{7}\text{, as } t > 0$$

These factors are difficult to spot and you can use the formula for a quadratic. You should, however, obtain an exact answer.

$$R(\rightarrow)\quad \text{distance} = \text{speed}\times\text{time}$$

$$= 14\times\frac{30}{7} = 60$$

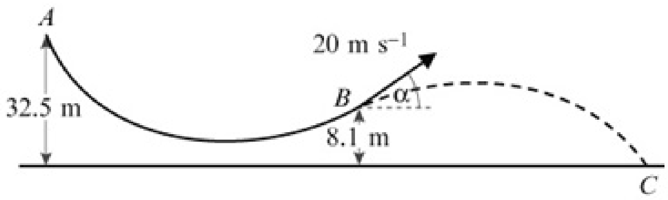
$$BD = 60\text{ m}$$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 60

Question:



In a ski-jump competition, a skier of mass 80 kg moves from rest at a point A on a ski-slope. The skier's path is an arc AB. The starting point A of the slope is 32.5 m above horizontal ground. The end B of the slope is 8.1 m above the ground. When the skier reaches B she is travelling at 20 m s^{-1} and moving upwards at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$, as shown in the figure. The distance along the slope from A to B is 60 m. The resistance to motion while she is on the slope is modelled as a force of constant magnitude R newtons.

a By using the work–energy principle, find the value of R.

On reaching B, the skier then moves through the air and reaches the ground at the point C. The motion of the skier in moving from B to C is modelled as that of a particle moving freely under gravity.

b Find the time the skier takes to move from B to C.

c Find the horizontal distance from B to C.

d Find the speed of the skier immediately before she reaches C.

Solution:

a The kinetic energy, in J, gained in moving from A to B is

$\frac{1}{2}mv^2 = \frac{1}{2}80 \times 20^2 = 16\,000$

The potential energy, in J, lost in moving from A to B is

$mgh = 40 \times 9.8 \times (32.5 - 8.1) = 19\,129.6$

The net loss of mechanical energy is
$(19\,129.6 - 16\,000)\text{ J} = 3129.6\text{ J}$

The net loss in mechanical energy is the work done by the resistance to motion.

The work done by the resisting force of R newtons, in J, is given by

$$\begin{aligned} \text{Work} &= \text{force} \times \text{distance} \\ &= R \times 60 \end{aligned}$$

By the work–energy principle

$$\begin{aligned} 60R &= 3129.6 \\ R &= \frac{3129.6}{60} = 52.15 = 52 \text{ (2 s.f.)} \end{aligned}$$

b $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

$R(\rightarrow)\quad u_x = 20\cos\alpha = 20\times\frac{4}{5} = 16$

$R(\uparrow)\quad u_y = 20\sin\alpha = 20\times\frac{3}{5} = 12$

You can sketch a 3, 4, 5 triangle to check these relations.

To find the time taken to move from B to C

$$R(\uparrow)\quad s = ut + \frac{1}{2}at^2$$
$$-8.1 = 12t - 4.9t^2$$
$$4.9t^2 - 12t - 8.1 = 0$$
$$49t^2 - 120t - 81 = 0$$
$$(t-3)(49t+27) = 0$$
$$t = 3, \text{ as } t > 0$$

Rearranging the quadratic and multiplying by 10.

The time taken to move from B to C is 3 s.

c $\rightarrow$ distance = speed × time
$= 16\times 3 = 48$

The horizontal distance from B to C is 48 m.

d Let the speed of the skier immediately before reaching C be w m s^{-1}.

Using conservation of energy

$$\frac{1}{2}\cancel{m}w^2 - \frac{1}{2}\cancel{m}v^2 = \cancel{m}gh$$
$$w^2 = v^2 + 2gh$$
$$= 20^2 + 2\times 9.8\times 8.1 = 558.76$$
$$w = \sqrt{558.76} = 23.638\ldots$$

Cancelling the m and rearranging the formula. This result is similar to $v^2 = u^2 + 2as$. However, it would be an error to use this formula, which is a formula for motion in a straight line, as the skier is not moving in a straight line. You must establish the result using the principle of conservation of energy.

The speed of the skier immediately before reaching C is 24 m s^{-1} (2 s.f.).