Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise A, Question 1

Question:

Integrate the following with respect to x.

a $\sinh x + 3\cosh x$

b $5\operatorname{sech}^2 x$

c $\dfrac{1}{\sinh^2 x}$

d $\cosh x - \dfrac{1}{\cosh^2 x}$

e $\dfrac{\sinh x}{\cosh^2 x}$

f $\dfrac{3}{\sinh x \tanh x}$

g $\operatorname{sech} x(\operatorname{sech} x + \tanh x)$

h $(\operatorname{sech} x + \operatorname{cosech} x)(\operatorname{sech} x - \operatorname{cosech} x)$

Solution:

a $\int (\sinh x + 3\cosh x)\,dx = \cosh x + 3\sinh x + C$

b $\int 5\operatorname{sech}^2 x\,dx = 5\tanh x + C$

c $\int \dfrac{1}{\sinh^2 x}\,dx = \int \operatorname{cosech}^2 x\,dx = -\coth x + C$

d $\int \left(\cosh x - \dfrac{1}{\cosh^2 x}\right) dx = \int (\cosh x - \operatorname{sech}^2 x)\,dx = \sinh x - \tanh x + C$

e $\int \dfrac{\sinh x}{\cosh^2 x}\,dx = \int \dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x}\,dx = \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$

f $\int \dfrac{3}{\sinh x \tanh x}\,dx = 3\int \operatorname{cosech} x \coth x\,dx = -3\operatorname{cosech} x + C$

g $\int \operatorname{sech} x(\operatorname{sech} x + \tanh x)\,dx = \int (\operatorname{sech}^2 x + \operatorname{sech} x \tanh x)\,dx = \tanh x - \operatorname{sech} x + C$

h $\int (\operatorname{sech}^2 x - \operatorname{cosech}^2 x)\,dx = \tanh x + \coth x + C$

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Integration
Exercise A, Question 2

Question:

Find

a $\int \sinh 2x \, dx$

b $\int \cosh\left(\frac{x}{3}\right) dx$

c $\int \text{sech}^2 (2x-1) dx$

d $\int \text{cosech}^2 5x \, dx$

e $\int \text{cosech}\, 2x \coth 2x \, dx$

f $\int \text{sech}\left(\frac{x}{\sqrt{2}}\right) \tanh\left(\frac{x}{\sqrt{2}}\right) dx$

g $\int \left(5\sinh 5x - 4\cosh 4x + 3\text{sech}^2\left(\frac{x}{2}\right)\right) dx$

Solution:

a $\int \sinh 2x \, dx = \frac{1}{2}\cosh 2x + C$

b $\int \cosh\left(\frac{x}{3}\right) dx = \frac{1}{\left(\frac{1}{3}\right)} \sinh\left(\frac{x}{3}\right) + C = 3\sinh\left(\frac{x}{3}\right) + C$

c $\int \text{sech}^2 (2x-1) dx = \frac{1}{2}\tanh(2x-1) + C$

d $\int \text{cosech}^2 5x dx = -\frac{1}{5}\coth 5x + C$

e $\int \text{cosech}\, 2x \coth 2x \, dx = -\frac{1}{2}\text{cosech}\, 2x + C$

f $\int \text{sech}\left(\frac{x}{\sqrt{2}}\right) \tanh\left(\frac{x}{\sqrt{2}}\right) dx = -\frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \text{sech}\left(\frac{x}{\sqrt{2}}\right) + C = \sqrt{2}\text{sech}\left(\frac{x}{\sqrt{2}}\right) + C$

g $\int 5\sinh 5x - 4\cosh 4x + 3\text{sech}^2\left(\frac{x}{2}\right) dx = 5\times\frac{1}{5}\cosh 5x - 4\times\frac{1}{4}\sinh 4x + 3\times\frac{1}{\left(\frac{1}{2}\right)}\tanh\left(\frac{x}{2}\right) + C$

$= \cosh 5x - \sinh 4x + 6\tanh\left(\frac{x}{2}\right) + C$

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Integration
Exercise A, Question 3

Question:

Write down the results of the following. (This is a recognition exercise and also involve some integrals from C4.)

a $\int \frac{1}{1+x^2}\,dx$

b $\int \frac{1}{\sqrt{1+x^2}}\,dx$

c $\int \frac{1}{1+x}\,dx$

d $\int \frac{2x}{1+x^2}\,dx$

e $\int \frac{1}{\sqrt{1-x^2}}\,dx, |x| < 1$

f $\int \frac{1}{\sqrt{x^2-1}}\,dx$

g $\int \frac{3x}{\sqrt{x^2-1}}\,dx$

h $\int \frac{3}{(1+x)^2}\,dx$

Solution:

a $\arctan x + C$

b $\text{arsinh}x + C$

c $\ln|1+x| + C$

d $\ln(1+x^2) + C$

e $\arcsin x + C$

f $\text{arcosh}x + C$

g $3\sqrt{x^2-1} + C$

h $-\frac{3}{(1+x)} + C$

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Integration
Exercise A, Question 4

Question:

Find

a $\int \frac{2x+1}{\sqrt{1-x^2}}\,dx$

b $\int \frac{1+x}{\sqrt{x^2-1}}\,dx$

c $\int \frac{x-3}{1+x^2}\,dx$

Solution:

a $\int \frac{2x+1}{\sqrt{(1-x^2)}}\,dx = \int \frac{2x}{\sqrt{(1-x^2)}}\,dx + \int \frac{1}{\sqrt{(1-x^2)}}\,dx$

$= 2\int x(1-x^2)^{-\frac{1}{2}}\,dx + \int \frac{1}{\sqrt{(1-x^2)}}\,dx$

$= -2\sqrt{(1-x^2)} + \text{ar}\sin x + C$

b $\int \frac{1+x}{\sqrt{(x^2-1)}}\,dx = \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int \frac{x}{\sqrt{(x^2-1)}}\,dx$

$= \int \frac{1}{\sqrt{(x^2-1)}}\,dx + \int x(x^2-1)^{-\frac{1}{2}}\,dx$

$= \text{arcosh}x + \sqrt{(x^2-1)} + C$

c $\int \frac{x-3}{\sqrt{(1+x^2)}}\,dx = \int \frac{x}{\sqrt{(1+x^2)}}\,dx - \int \frac{3}{\sqrt{(1+x^2)}}\,dx$

$= \int x(1+x^2)^{-\frac{1}{2}}\,dx - \int \frac{3}{(1+x^2)}\,dx$

$= \sqrt{(1+x^2)} - 3\text{arsinh}x + C$

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Integration
Exercise A, Question 5

Question:

a Show that $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$

b Hence find $\displaystyle\int \frac{x^2}{1+x^2}\,dx$

Solution:

a $\dfrac{x^2}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{1+x^2}{1+x^2} - \dfrac{1}{1+x^2} = 1 - \dfrac{1}{1+x^2}$

b $$\begin{aligned}\int \frac{x^2}{1+x^2}\,dx &= \int\left\{1 - \frac{1}{1+x^2}\right\}dx \\ &= x - \arctan x + C\end{aligned}$$

Using a.

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Integration
Exercise B, Question 1

Question:

Find

a $\int \sinh^3 x \cosh x \, dx$

b $\int \tanh 4x \, dx$

c $\int \tanh^5 x \operatorname{sech}^2 x \, dx$

d $\int \operatorname{cosech}^7 x \coth x \, dx$

e $\int \sqrt{\cosh 2x} \sinh 2x \, dx$

f $\int \operatorname{sech}^{10} 3x \tanh 3x \, dx$

Solution:

a $\int \sinh^3 x \cosh x \, dx = \int (\sinh x)^3 \cosh x \, dx = \frac{1}{4}\sinh^4 x + C$

b $\int \tanh 4x \, dx = \int \frac{\sinh 4x}{\cosh 4x} \, dx = \frac{1}{4}\ln \cosh 4x + C$

c $\int \tanh^5 x \operatorname{sech}^2 x \, dx = \int (\tanh x)^5 \operatorname{sech}^2 x \, dx = \frac{1}{6}\tanh^6 x + C$

d $\int \operatorname{cosech}^7 x \coth x dx = \int \operatorname{cosech}^6 x (\operatorname{cosech} x \coth x) dx$

$$= -\int (\operatorname{cosech})^6 (-\operatorname{cosech} x \coth x) dx$$
$$= -\frac{1}{7}\operatorname{cosech}^7 x + C$$

e $\int \sqrt{\cosh 2x} \sinh 2x \, dx = \frac{1}{2}\int (\cosh 2x)^{\frac{1}{2}} (2\sinh 2x) dx$

$$= \frac{1}{2}\left\{\frac{(\cosh 2x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right\} + C$$
$$= \frac{1}{3}(\cosh 2x)^{\frac{3}{2}} + C$$

f $\int \operatorname{sech}^{10} 3x \tanh 3x \, dx = -\frac{1}{3}\int \operatorname{sech}^9 3x (-3\operatorname{sech} 3x \tanh 3x) dx$

$$= -\frac{1}{3}\left\{\frac{\operatorname{sech}^{10} 3x}{10}\right\} + C$$
$$= -\frac{1}{30}\operatorname{sech}^{10} 3x + C$$

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Integration
Exercise B, Question 2

Question:

Find

a $\displaystyle\int \frac{\sinh x}{2+3\cosh x}\,dx$

b $\displaystyle\int \frac{1+\tanh x}{\cosh^2 x}\,dx$

c $\displaystyle\int \frac{5\cosh x + 2\sinh x}{\cosh x}\,dx$.

Solution:

a $$\int \frac{\sinh x}{2+3\cosh x}\,dx = \frac{1}{3}\int \frac{3\sinh x}{2+3\cosh x}\,dx$$
$$= \frac{1}{3}\ln(2+3\cosh x) + C$$

b $$\int \frac{1+\tanh x}{\cosh^2 x}\,dx = \int (1+\tanh x)\,\text{sech}^2 x\,dx$$
$$= \int (\text{sech}^2 x + \tanh x\,\text{sech}^2 x)\,dx$$
$$= \tanh x + \frac{1}{2}\tanh^2 x + C \quad \text{or} \quad \tanh x - \frac{1}{2}\text{sech}^2 x + C$$

c $$\int \frac{5\cosh x + 2\sinh x}{\cosh x}\,dx = \int (5 + 2\tanh x)\,dx$$
$$= 5x + 2\ln\cosh x + C$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise B, Question 3

Question:

a Show that $\int \coth x \, dx = \ln \sinh x + C$.

b Show that $\int_1^2 \coth 2x \, dx = \ln \sqrt{\left(e^2 + \frac{1}{e^2}\right)}$.

Solution:

a $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} \, dx = \ln \sinh x + C$

b $\int \coth 2x \, dx = \frac{1}{2} \ln \sinh 2x + C$

So $\int_1^2 \coth 2x = \left[\frac{1}{2} \ln \sinh 2x\right]_1^2 = \frac{1}{2}(\ln \sinh 4 - \ln \sinh 2)$

$$= \frac{1}{2} \ln\left(\frac{\sinh 4}{\sinh 2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{e^4 - e^{-4}}{e^2 - e^{-2}}\right)$$

$$= \frac{1}{2} \ln(e^2 + e^{-2})$$

Using $a^2 - b^2 = (a+b)(a-b)$ with $a = e^2, b = e^{-2}$

$$= \ln \sqrt{e^2 + \frac{1}{e^2}}$$

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Integration
Exercise B, Question 4

Question:

Use integration by parts to find

a $\int x \sinh 3x \, dx$

b $\int x \operatorname{sech}^2 x \, dx$.

Solution:

a $\int x \sinh 3x \, dx = \frac{1}{3} x \cosh 3x - \int \frac{1}{3} \cosh 3x dx$

$= \frac{1}{3} x \cosh 3x - \frac{1}{9} \sinh 3x + C$

Using $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ with $u = x$ and $\frac{dv}{dx} = \sinh 3x$

b $\int x \operatorname{sech}^2 x \, dx = x \tanh x - \int \tanh x \, dx$

$= x \tanh x - \ln \cosh x + C$

Using integration by parts with $u = x$ and $\frac{dv}{dx} = \operatorname{sech}^2 x$

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Integration
Exercise B, Question 5

Question:

Find

a $\int e^x \cosh x \, dx$

b $\int e^{-2x} \sinh 3x \, dx$

c $\int \cosh x \cosh 3x \, dx$.

Solution:

a $\int e^x \cosh x \, dx = \int e^x \left(\frac{e^x + e^{-x}}{2} \right) dx$

You cannot use integration by parts.

$= \frac{1}{2} \int (e^{2x} + 1) dx$

$= \frac{1}{4} e^{2x} + \frac{1}{2} x + C$

b $\int e^{-2x} \sinh 3x \, dx = \int e^{-2x} \left(\frac{e^{3x} - e^{-3x}}{2} \right) dx$

You could use integration by parts twice.

$= \frac{1}{2} \int (e^x - e^{-5x}) dx$

$= \frac{1}{2} e^x + \frac{1}{10} e^{-5x} + C$

c $\int \cosh x \cosh 3x \, dx = \int \left(\frac{e^x + e^{-x}}{2} \right) \left(\frac{e^{3x} + e^{-3x}}{2} \right) dx$

or write as $\frac{1}{2}(\cosh 4x + \cosh 2x)$

$= \frac{1}{4} \int (e^{4x} + e^{-4x} + e^{2x} + e^{-2x}) dx$

$= \frac{1}{16} e^{4x} - \frac{1}{16} e^{-4x} + \frac{1}{8} e^{2x} - \frac{1}{8} e^{-2x} + C$ or $\frac{1}{8} \sinh 4x + \frac{1}{4} \sinh 2x + C$

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Integration
Exercise B, Question 6

Question:

By writing $\cosh 3x$ in exponential form, find $\int \cosh^2 3x \, dx$ and show that it is equivalent to the result found in Example 5**b**.

Solution:

$$\int \cosh^2 3x \, dx = \frac{1}{4}\int (e^{3x} + e^{-3x})^2 \, dx$$
$$= \frac{1}{4}\int (e^{6x} + 2 + e^{-6x}) \, dx$$
$$= \frac{1}{24}e^{6x} - \frac{1}{24}e^{-6x} + \frac{1}{2}x + C$$
$$= \frac{1}{12}\sinh 6x + \frac{1}{2}x + C \quad \text{which was result in Example 5b}$$

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Integration
Exercise B, Question 7

Question:

Evaluate $\int_0^1 \frac{1}{\sinh x + \cosh x}\,dx$, giving your answer in terms of e.

Solution:

$$\sinh x + \cosh x = \frac{1}{2}\left(e^x - e^{-x}\right) + \frac{1}{2}\left(e^x + e^{-x}\right) = e^x$$

So $\int_0^1 \left(\frac{1}{\sinh x + \cosh x}\right)dx = \int_0^1 e^{-x}\,dx = \left[-e^{-x}\right]_0^1 = 1 - \frac{1}{e}$

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Integration
Exercise B, Question 8

Question:

Use appropriate identities to find

a $\int \sinh^2 x \, dx$

b $\int (\text{sech } x - \tanh x)^2 \, dx$

c $\int \frac{\cosh^2 3x}{\sinh^2 3x} \, dx$

d $\int \sinh^2 x \cosh^2 x \, dx$

e $\int \cosh^5 x \, dx$

f $\int \tanh^3 2x \, dx$.

Solution:

a $\int \sinh^2 x \, dx = \frac{1}{2}\int (\cosh 2x - 1)dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + C$

b $\int (\text{sech}\, x - \tanh x)^2 \, dx = \int (\text{sech}^2 x - 2\text{sech} x \tanh x + \tanh^2 x)dx$

$$= \int (\text{sech}^2 x - 2\text{sech} x \tanh x + 1 - \text{sech}^2 x)dx$$

$$= \int (1 - 2\text{sech} x \tanh x)dx$$

$$= x + 2\text{sech} x + C$$

c $\int \frac{\cosh^2 3x}{\sinh^2 3x} dx = \int \coth^2 3x \, dx$

$$= \int (1 + \text{cosech}^2 3x)dx$$

$$= x - \frac{1}{3}\coth 3x + C$$

d $\int \sinh^2 x \cosh^2 x \, dx = \int \left(\frac{1}{2}\sinh 2x\right)^2 dx$

$$= \frac{1}{4}\int \sinh^2 2x \, dx$$

$$= \frac{1}{4}\int \left(\frac{\cosh 4x - 1)}{2}\right)dx$$ Using $\cosh 2u = 1 + 2\sinh^2 u$

$$= -\frac{1}{8}x + \frac{1}{32}\sinh 4x + C$$

e $\int \cosh^5 x dx = \int \cosh^4 x \cosh x \, dx$

$$= \int (1 + \sinh^2 x)^2 \cosh x \, dx$$

$$= \int (1 + 2\sinh^2 x + \sinh^4 x) \cosh x dx$$

$$= \int (\cosh x + 2\sinh^2 x \cosh x + \sinh^4 x \cosh x)dx$$

$$= \sinh x + \frac{2}{3}\sinh^3 x + \frac{1}{5}\sinh^5 x + C$$

f $\int \tanh^3 2x \, dx = \int \tanh^2 2x \tanh 2x \, dx$

$$= \int (1 - \text{sech}^2 2x) \tanh 2x \, dx$$

$$= \int (\tanh 2x - \tanh 2x \text{sech}^2 2x)dx$$

$$= \frac{1}{2}\ln \cosh 2x - \frac{1}{4}\tanh^2 2x + C$$

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Integration
Exercise B, Question 9

Question:

Show that $\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx = \frac{1}{8}(3 + \ln 16)$.

Solution:

$$\int_0^{\ln 2} \cosh^2\left(\frac{x}{2}\right) dx = \int_0^{\ln 2} \left(\frac{1 + \cosh x}{2}\right) dx$$

$$= \frac{1}{2}\left[x + \sinh x\right]_0^{\ln 2}$$

$$= \frac{1}{2}\left[\ln 2 + \left(\frac{e^{\ln 2} - e^{-\ln 2}}{2}\right)\right]$$

$$e^{\ln 2} = 2, e^{-\ln 2} = e^{\ln \frac{1}{2}} = \frac{1}{2}$$

$$= \frac{1}{2}\left[\ln 2 + \frac{3}{4}\right]$$

$$= \frac{1}{8}\left[3 + 4\ln 2\right]$$

$$= \frac{1}{8}(3 + \ln 16)$$

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Integration
Exercise B, Question 10

Question:

The region bounded by the curve $y = \sinh x$, the line $x = 1$ and the positive x-axis is rotated through $360°$ about the x-axis. Show that the volume of the solid of revolution formed is $\frac{\pi}{8e^2}\left(e^4 - 4e^2 - 1\right)$.

Solution:

$$\text{Volume} = \pi\int_0^1 \sinh^2 x \, dx = \frac{\pi}{2}\int_0^1 (\cosh 2x - 1)dx$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\sinh 2 - 1\right]$$

$$= \frac{\pi}{2}\left[\frac{1}{4}\left(e^2 - e^{-2}\right) - 1\right]$$

$$= \frac{\pi}{8}\left[e^2 - 4 - e^{-2}\right]$$

$$= \frac{\pi}{8e^2}\left(e^4 - 4e^2 - 1\right).$$

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Integration
Exercise B, Question 11

Question:

Using the result for $\int \text{sech } x \, dx$ given in Example 7, find

a $\int \frac{2}{\cosh x} dx$

b $\int \text{sech } 2x \, dx$

c $\int \sqrt{1 - \tanh^2\left(\frac{x}{2}\right)} dx$.

Solution:

Using $\int \text{sech } x \, dx = 2 \arctan(e^x) + C$

a $\int \frac{2}{\cosh x} dx = \int 2 \text{ sech } x \, dx = 4 \arctan(e^x) + C$

b Using the substitution $u = 2x$,

$\left(\text{or using } \int f'(ax+b) dx = \frac{1}{a} f(ax+b) + C 6x \right)$

$\int \text{sech } 2x dx = \frac{1}{2} \int \text{sech } u \, du = \arctan\left(e^u\right) + C = \arctan\left(e^{2x}\right) + C$

c $\int \sqrt{1 - \tanh^2\left(\frac{x}{2}\right)} dx = \int \text{sech}\left(\frac{x}{2}\right) dx = \frac{1}{\left(\frac{1}{2}\right)} 2 \arctan\left(e^{\frac{x}{2}}\right) + C$

$= 4 \arctan\left(e^{\frac{x}{2}}\right) + C$

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Integration
Exercise B, Question 12

Question:

Using the substitution $u = x^2$, or otherwise, find

a $\int x\cosh^2(x^2)\,dx$

b $\int \frac{x}{\cosh^2(x^2)}\,dx$.

Solution:

Using the substitution $u = x^2$, $du = 2x\,dx$,

a So $\int x\cosh^2(x^2)\,dx = \frac{1}{2}\int \cosh^2 u\,du$

$$= \frac{1}{4}\int (\cosh 2u + 1)\,du$$

$$= \frac{1}{8}\sinh 2u + \frac{u}{4} + C$$

$$= \frac{1}{8}\sinh(2x^2) + \frac{x^2}{4} + C$$

b So $\int \frac{x}{\cosh^2(x^2)}\,dx = \int x\,\mathrm{sech}^2(x^2)\,dx$

$$= \frac{1}{2}\int \mathrm{sech}^2 u\,du$$

$$= \frac{1}{2}\tanh u + C$$

$$= \frac{1}{2}\tanh(x^2) + C$$

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Integration
Exercise C, Question 1

Question:

Use the substitution $x = a\tan\theta$ to show that $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$.

Solution:

Using $x = a\tan\theta$, $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$

so
$$\begin{aligned}\int \frac{1}{a^2+x^2}\,\mathrm{d}x &= \int \frac{1}{a^2+a^2\tan^2\theta}\,a\sec^2\theta\,\mathrm{d}\theta \\ &= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\,\mathrm{d}\theta \\ &= \frac{1}{a}\int d\theta \\ &= \frac{1}{a}\theta + C \\ &= \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C\end{aligned}$$

$$\boxed{x = a\tan\theta \Rightarrow \theta = \arctan\left(\frac{x}{a}\right)}$$

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Integration
Exercise C, Question 2

Question:

Use the substitution $x = \cos\theta$ to show that $\int \frac{1}{\sqrt{1-x^2}}\,dx = -\arccos x + C$.

Solution:

Using $x = \cos\theta$, $dx = -\sin\theta\, d\theta$

so $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\,d\theta$

$$= -\int d\theta$$

$$= -\theta + C$$

$$= -\arccos x + C$$

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Integration
Exercise C, Question 3

Question:

Use suitable substitutions to find

a $\int \frac{3}{\sqrt{4-x^2}}\,dx$

b $\int \frac{1}{\sqrt{x^2-9}}\,dx$

c $\int \frac{4}{5+x^2}\,dx$

d $\int \frac{1}{\sqrt{4x^2+25}}\,dx$.

Solution:

a Let $x = 2\sin\theta$, so $dx = 2\cos\theta \, d\theta$

$$\int \frac{3}{\sqrt{4-x^2}}\,dx = \int \frac{3}{\sqrt{4-4\sin^2\theta}} 2\cos\theta \, d\theta$$

$$= \int \frac{6\cos\theta}{2\cos\theta}\,d\theta$$

$$= 3\int d\theta$$

$$= 3\theta + C$$

$$= 3\arcsin\left(\frac{x}{2}\right) + C$$

b Let $x = 3\cosh u$, so $dx = 3\sinh u \, du$

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{9\cosh^2 u - 9}} 3\sinh u \, du$$

$$= \int \frac{1}{3\sqrt{\cosh^2 u - 1}} 3\sinh u \, du$$

$$= \int \frac{3\sinh u}{3\sinh u}\,du$$

$$= \int 1 \, du$$

$$= u + C$$

$$= \operatorname{arcosh}\left(\frac{x}{3}\right) + C$$

c Let $x = \sqrt{5}\tan\theta$, so $dx = \sqrt{5}\sec^2\theta \, d\theta$

$$\int \frac{4}{5+x^2}\,dx = \int \frac{4}{5+5\tan^2\theta}\sqrt{5}\sec^2\theta \, d\theta$$

$$= \int \frac{4\sqrt{5}\sec^2\theta}{5\sec^2\theta}\,d\theta \longleftarrow \boxed{5+5\tan^2\theta = 5\left(1+\tan^2\theta\right) = 5\sec^2\theta}$$

$$= \frac{4\sqrt{5}}{5}\int d\theta$$

$$= \frac{4\sqrt{5}}{5}\theta + C$$

$$= \frac{4\sqrt{5}}{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$$

d You need $4x^2 = 25\sinh^2 u$, or $2x = 5\sinh u$, then $dx = \frac{5}{2}\cosh u\, du$

$$\begin{aligned}
\int \frac{1}{\sqrt{4x^2+25}}\,dx &= \int \frac{1}{\sqrt{25\sinh^2 u + 25}}\left(\frac{5}{2}\cosh u\right)du \\
&= \frac{5}{2}\int \frac{\cosh u}{5\sqrt{\sinh^2 u + 1}}\,du \\
&= \frac{1}{2}\int \frac{\cosh u}{\cosh u}\,du \\
&= \frac{1}{2}\int 1\,du \\
&= \frac{1}{2}u + C \\
&= \frac{1}{2}\operatorname{arsinh}\left(\frac{2x}{5}\right) + C
\end{aligned}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 4

Question:

Write down the results for the following:

a $\int \frac{1}{\sqrt{25-x^2}}\,dx$

b $\int \frac{3}{\sqrt{x^2+9}}\,dx$

c $\int \frac{1}{\sqrt{x^2-2}}\,dx$

d $\int \frac{2}{16+x^2}\,dx$.

Solution:

a $\int \frac{1}{\sqrt{25-x^2}}\,dx = \arcsin\left(\frac{x}{5}\right)+C$

Using $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$

b $\int \frac{3}{\sqrt{x^2+9}}\,dx = 3\text{arsinh}\left(\frac{x}{3}\right)+C$

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \text{arsinh}\left(\frac{x}{a}\right)+C$.

c $\int \frac{1}{\sqrt{x^2-2}}\,dx = \text{arcosh}\left(\frac{x}{\sqrt{2}}\right)+C$

Using $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \text{arcosh}\left(\frac{x}{a}\right)+C$

d $\int \frac{2}{16+x^2}\,dx = 2\int \frac{1}{16+x^2}\,dx$

$= 2\left\{\frac{1}{4}\arctan\left(\frac{x}{4}\right)\right\}+C$

Using $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$

$= \frac{1}{2}\arctan\left(\frac{x}{4}\right)+C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 5

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Find

a $\int \frac{1}{\sqrt{4x^2-12}}\,dx$

b $\int \frac{1}{4+3x^2}\,dx$

c $\int \frac{1}{\sqrt{9x^2+16}}\,dx$

d $\int \frac{1}{\sqrt{3-4x^2}}\,dx$.

Solution:

a $\displaystyle\int \frac{1}{\sqrt{4x^2-12}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4\left(x^2-3\right)}}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(x^2-3\right)}}\,\mathrm{d}x$$

$$= \frac{1}{2}\operatorname{arcosh}\left(\frac{x}{\sqrt{3}}\right)+C$$

b $\displaystyle\int \frac{1}{4+3x^2}\,\mathrm{d}x = \int \frac{1}{3\left\{\frac{4}{3}+x^2\right\}}\,\mathrm{d}x$

$$= \frac{1}{3}\left\{\frac{1}{\left(\frac{2}{\sqrt{3}}\right)}\arctan\left(\frac{x}{\left(\frac{2}{\sqrt{3}}\right)}\right)\right\}+C$$

$$= \frac{\sqrt{3}}{6}\arctan\left(\frac{\sqrt{3}x}{2}\right)+C$$

c $\displaystyle\int \frac{1}{\sqrt{9x^2+16}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,\mathrm{d}x$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left\{x^2+\left(\frac{16}{9}\right)\right\}}}\,\mathrm{d}x$$

$$= \frac{1}{3}\operatorname{arsinh}\left(\frac{x}{\left(\frac{4}{3}\right)}\right)+C$$

$$= \frac{1}{3}\operatorname{arsinh}\left(\frac{3x}{4}\right)+C$$

d $\displaystyle\int \frac{1}{\sqrt{3-4x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4\left\{\frac{3}{4}-x^2\right\}}}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left\{\frac{3}{4}-x^2\right\}}}\,\mathrm{d}x$$

$$= \frac{1}{2}\arcsin\left(\frac{x}{\frac{\sqrt{3}}{2}}\right)+C \qquad \boxed{a^2=\frac{3}{4}\ \text{so}\ a=\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right)+C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 6

Question:

Unless a substitution is given or asked for, use the standard results 7 to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Evaluate

a $\int_1^3 \frac{2}{1+x^2}\,dx$

b $\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx$

c $\int_{-1}^2 \frac{1}{\sqrt{21-3x^2}}\,dx$.

Solution:

a $\int_1^3 \frac{2}{1+x^2}\,dx = 2\left[\arctan x\right]_1^3$

$= 2(\arctan 3 - \arctan 1)$

$= 0.927 \quad (3\text{ s.f.})$

Remember that you need to be in radian mode.

b $\int_1^2 \frac{3}{\sqrt{1+4x^2}}\,dx = 3\int_1^2 \frac{1}{2\sqrt{\frac{1}{4}+x^2}}\,dx$

$= \frac{3}{2}\left[\text{arsinh}\,\frac{x}{\left(\frac{1}{2}\right)}\right]_1^2$

$= \frac{3}{2}\left[\text{arsinh}\,(2x)\right]_1^2$

$= \frac{3}{2}\left[\text{arsinh}4 - \text{arsinh}\,2\right]$

$= 0.977 \quad (3\text{ s.f.})$

c $\int_{-1}^2 \frac{1}{\sqrt{21-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_{-1}^2 \frac{1}{\sqrt{7-x^2}}\,dx$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{x}{\sqrt{7}}\right)\right]_{-1}^2$

$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{2}{\sqrt{7}}\right) - \arcsin\left(-\frac{1}{\sqrt{7}}\right)\right]$

$= \frac{1}{\sqrt{3}}\left[0.85707\ldots - (-0.38759\ldots)\right]$

You need to be in radian mode

$= 0.719 \quad (3\text{ s.f.})$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 7

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Evaluate, giving your answers in terms of π or as a single natural logarithm, whichever is appropriate.

a $\displaystyle\int_0^4 \frac{1}{\sqrt{x^2+16}}\,dx$

b $\displaystyle\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,dx$

c $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

Solution:

Reminder: The logarithmic form of an inverse hyperbolic function is in the Edexcel formulae booklet.

a $\int_0^4 \frac{1}{\sqrt{x^2+16}}\,dx = \left[\operatorname{arsinh}\left(\frac{x}{4}\right)\right]_0^4$

$= \operatorname{arsinh}1 - \operatorname{arsinh}0$

$= \ln\left\{1+\sqrt{2}\right\}$

Using $\operatorname{arsinh}x = \ln\left\{x+\sqrt{x^2+1}\right\}$

b $\int_{13}^{15} \frac{1}{\sqrt{x^2-144}}\,dx = \left[\operatorname{arcosh}\left(\frac{x}{12}\right)\right]_{13}^{15}$

$= \operatorname{arcosh}\left(\frac{5}{4}\right) - \operatorname{arcosh}\left(\frac{13}{12}\right)$

Using $\operatorname{arcosh}x = \ln\left\{x+\sqrt{x^2-1}\right\}$

$= \ln\left\{\frac{5}{4}+\sqrt{\frac{25}{16}-1}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{169}{144}-1}\right\}$

$= \ln\left\{\frac{5}{4}+\sqrt{\frac{9}{16}}\right\} - \ln\left\{\frac{13}{12}+\sqrt{\frac{25}{144}}\right\}$

$= \ln 2 - \ln\left(\frac{3}{2}\right)$

$= \ln\left(\frac{4}{3}\right)$

Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

c $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\arcsin\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$

$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{\sqrt{2}}{2}\right)$

$= \left(\frac{\pi}{3}\right) - \left(\frac{\pi}{4}\right)$

$= \left(\frac{\pi}{12}\right)$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 8

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

The curve C has equation $y = \frac{2}{\sqrt{2x^2 + 9}}$. The region R is bounded by C, the coordinate axes and the lines $x = -1$ and $x = 3$.

a Find the area of R.

The region R is rotated through $360°$ about the x-axis.

b Find the volume of the solid generated.

Solution:

a Area of $R = \int_{-1}^{3} y \, dx = \int_{-1}^{3} \frac{2}{\sqrt{2x^2+9}} \, dx$

$$= \int_{-1}^{3} \frac{2}{\sqrt{2\left(x^2+\frac{9}{2}\right)}} \, dx$$

Curve is always above x-axis

$$= \sqrt{2}\left[\operatorname{arsinh} \frac{x}{\left(\frac{3}{\sqrt{2}}\right)}\right]_{-1}^{3}$$

$$= \sqrt{2}\left[\operatorname{arsinh}\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$= \sqrt{2}\left[\operatorname{arsinh}\sqrt{2} - \operatorname{arsinh}\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$= 2.27 \text{ (3 s.f.)}$$

b Volume $= \pi \int_{-1}^{3} y^2 \, dx = \pi \int_{-1}^{3} \frac{4}{2x^2+9} \, dx$

$$= 2\pi \int_{-1}^{3} \frac{1}{x^2+\left(\frac{9}{2}\right)} \, dx$$

$$= 2\pi \left[\frac{1}{\left(\frac{3}{\sqrt{2}}\right)} \arctan \frac{x}{\left(\frac{3}{\sqrt{2}}\right)}\right]_{-1}^{3}$$

$$= \left(\frac{2\sqrt{2}\pi}{3}\right)\left[\arctan\left(\frac{\sqrt{2}x}{3}\right)\right]_{-1}^{3}$$

$$= \left(\frac{2\sqrt{2}\pi}{3}\right)\left[\arctan\left(\sqrt{2}\right) - \arctan\left(-\frac{\sqrt{2}}{3}\right)\right]$$

$$= 1.32\pi \text{ (3 s.f.)} = 4.13 \text{ (3 s.f.)}$$

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Integration
Exercise C, Question 9

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

A circle C has centre the origin and radius r.

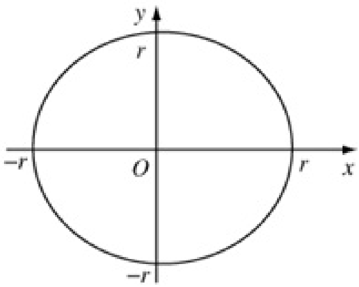
a Show that the area of C can be written as $4\int_0^r \sqrt{r^2 - x^2}\,dx$.

b Hence show that the area of C is πr^2.

Solution:

a Cartesian equation of circle is $x^2 + y^2 = r^2$.

Area of C can be written as $4\int_0^r y\,dx = 4\int_0^r \sqrt{r^2 - x^2}\,dx$



b Use substitution $x = r\sin\theta$, so $dx = r\cos\theta\,d\theta$,

$$4\int_0^r \sqrt{r^2 - x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\,d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2r^2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta$$

$$= 2r^2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2r^2\left(\frac{\pi}{2}\right)$$

$$= \pi r^2$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 10

Question:

a Use the substitution $x=\frac{2}{3}\tan\theta$ to find $\int\frac{x^2}{9x^2+4}\,dx$.

b Use the substitution $x=\sinh^2 u$ to find $\int\sqrt{\frac{x}{x+1}}\,dx,\ x>0$.

Solution:

a With $x=\frac{2}{3}\tan\theta$ and $dx=\frac{2}{3}\sec^2\theta\,d\theta$,

$$9x^2+4=9\left(\frac{4}{9}\tan^2\theta\right)+4=4\tan^2\theta+4=4\left(\tan^2\theta+1\right)=4\sec^2\theta$$

and $\frac{x^2}{9x^2+4}=\frac{\frac{4}{9}\tan^2\theta}{4\sec^2\theta}=\frac{\tan^2\theta}{9\sec^2\theta}$

so $$\int\frac{x^2}{9x^2+4}\,dx=\int\frac{\tan^2\theta}{9\sec^2\theta}\times\frac{2}{3}\sec^2\theta\,d\theta$$
$$=\frac{2}{27}\int\tan^2\theta\,d\theta$$
$$=\frac{2}{27}\int\left(\sec^2\theta-1\right)d\theta$$
$$=\frac{2}{27}(\tan\theta-\theta)+C$$
$$=\frac{2}{27}\left(\frac{3x}{2}-\arctan\frac{3x}{2}\right)+C$$
$$=\frac{x}{9}-\frac{2}{27}\arctan\frac{3x}{2}+C$$

b With $x=\sinh^2 u$ and $dx=2\sinh u\cosh u\,du$,

and $\frac{x}{x+1}=\frac{\sinh^2 u}{\sinh^2 u+1}=\frac{\sinh^2 u}{\cosh^2 u}$

$$\int\sqrt{\frac{x}{x+1}}\,dx=\int\frac{\sinh u}{\cosh u}2\sinh u\cosh u\,du$$
$$=\int 2\sinh^2 u\,du$$
$$=\int\left(\cosh 2u-1\right)du$$
$$=\frac{\sinh 2u}{2}-u+C$$
$$=\sinh u\cosh u-\text{arsinh}\left(\sqrt{x}\right)+C$$
$$=\sqrt{x}\sqrt{1+x}-\text{arsinh}\left(\sqrt{x}\right)+C$$

$\boxed{\sinh u=\sqrt{x}\text{ and }\cosh u=\sqrt{1+\sinh^2 u}}$

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Integration
Exercise C, Question 11

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

By splitting up each integral into two separate integrals, or otherwise, find

a $\int \frac{x-2}{\sqrt{x^2-4}}\,dx$

b $\int \frac{2x-1}{\sqrt{2-x^2}}\,dx$

c $\int \frac{2+3x}{1+3x^2}\,dx$.

Solution:

a
$$\int \frac{x-2}{\sqrt{x^2-4}}\,dx = \int \frac{x}{\sqrt{x^2-4}}\,dx - \int \frac{2}{\sqrt{x^2-4}}\,dx$$
$$= \sqrt{x^2-4} - 2\,\text{arcosh}\left(\frac{x}{2}\right) + C$$

b
$$\int \frac{2x-1}{\sqrt{2-x^2}}\,dx = \int \frac{2x}{\sqrt{2-x^2}}\,dx - \int \frac{1}{\sqrt{2-x^2}}\,dx$$
$$= -2\sqrt{2-x^2} - \arcsin\left(\frac{x}{\sqrt{2}}\right) + C$$

c
$$\int \frac{2+3x}{1+3x^2}\,dx = \int \frac{2}{1+3x^2}\,dx + \int \frac{3x}{1+3x^2}\,dx$$
$$= \frac{2}{3}\int \frac{1}{\left(\frac{1}{3}+x^2\right)}\,dx + \frac{1}{2}\int \frac{6x}{1+3x^2}\,dx$$
$$= \frac{2\sqrt{3}}{3}\arctan\left(\sqrt{3}x\right) + \frac{1}{2}\ln\left(1+3x^2\right) + C$$

$$\boxed{a = \frac{1}{\sqrt{3}} \text{ in } \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 12

Question:

Unless a substitution is given or asked for, use the standard results 7 to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Use the method of partial fractions to find $\int \frac{x^2+4x+10}{x^3+5x}\,\mathrm{d}x, x>0$.

Solution:

Setting up the model $\frac{x^2+4x+10}{x(x^2+5)}=\frac{A}{x}+\frac{Bx+C}{x^2+5}$

$\Rightarrow x^2+4x+10 \equiv A(x^2+5)+(Bx+C)x$

$x=0 \quad \Rightarrow 10=5A \quad \Rightarrow \quad A=2$

Coefficient of $x \quad \Rightarrow \quad 4=C$

Coefficient of $x^2 \quad \Rightarrow \quad 1=A+B \quad \Rightarrow \quad B=-1$

So $\int \frac{x^2+4x+10}{x^3+5x}\,\mathrm{d}x=\int\left(\frac{2}{x}+\frac{-x+4}{x^2+5}\right)\mathrm{d}x$

$=\int\left(\frac{2}{x}+\frac{4}{x^2+5}-\frac{1}{2}\frac{2x}{x^2+5}\right)\mathrm{d}x$

$=2\ln x+\frac{4}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)-\frac{1}{2}\ln\left(x^2+5\right)+C$

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Integration
Exercise C, Question 13

Question:

Unless a substitution is given or asked for, use the standard results 7 to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

Show that $\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \frac{1}{4}(\pi + 2\ln 2)$.

Solution:

Setting up the model $\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$$\Rightarrow 2 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$x=-1 \Rightarrow 2 = 2A \Rightarrow A = 1$

Coefficient of $x^2 \Rightarrow 0 = A + B \Rightarrow B = -1$

Coefficient of $x \Rightarrow 0 = B + C \Rightarrow C = 1$

So $\int_0^1 \frac{2}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{1}{(x+1)}dx + \int_0^1 \frac{1-x}{(x^2+1)}dx$

$$= \int_0^1 \frac{1}{(x+1)}dx + \int_0^1 \frac{1}{(x^2+1)}dx - \int_0^1 \frac{x}{(x^2+1)}dx$$

$$= \left[\ln(1+x)\right]_0^1 + \left[\arctan x\right]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \ln 2 + \arctan 1 - \frac{1}{2}\ln 2$$

$$= \frac{\pi}{4} + \frac{1}{2}\ln 2$$

$$= \frac{1}{4}(\pi + 2\ln 2)$$

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Integration
Exercise C, Question 14

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

By using the substitution $u = x^2$ evaluate $\int_2^3 \frac{2x}{\sqrt{x^4 - 1}} \, dx$.

Solution:

With $u = x^2$ and $du = 2x \, dx$,

$$\int_2^3 \frac{2x}{\sqrt{x^4 - 1}} \, dx = \int_4^9 \frac{du}{\sqrt{u^2 - 1}}$$

$$= [ar\cosh u]_4^9$$

$$= ar\cosh 9 - \cosh 4$$

$$= 0.824 \quad (3 \text{ s.f.})$$

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Integration
Exercise C, Question 15

Question:

By using the substitution $x = \frac{1}{2}\sin\theta$, show that $\int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \frac{1}{192}(2\pi - 3\sqrt{3})$.

Solution:

With $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$

$1-4x^2 = 1-\sin^2\theta = \cos^2\theta$ and so $\frac{x^2}{\sqrt{1-4x^2}} = \frac{\sin^2\theta}{4\cos\theta}$

$$\text{So } \int_0^{\frac{1}{4}} \frac{x^2}{\sqrt{1-4x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{4\cos\theta} \times \frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$$

$$= \frac{1}{16}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{16}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{16}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{192}\left(2\pi - 3\sqrt{3}\right)$$

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Integration
Exercise C, Question 16

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

a Use the substitution $x = 2\cosh u$ to show that
$\int \sqrt{x^2 - 4}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2 - 4} - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$.

b Find the area enclosed between the hyperbola with equation $\frac{x^2}{4} - \frac{y^2}{9} = 1$ and the line $x = 4$.

Solution:

a Using $x = 2\cosh u$, $\mathrm{d}x = 2\sinh u \, \mathrm{d}u$

$$\int \sqrt{x^2-4}\,\mathrm{d}x = \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u \, \mathrm{d}u$$

$$= 4\int \sinh^2 u \, \mathrm{d}u$$

$$= 2\int (\cosh 2u - 1)\,\mathrm{d}u$$

$$= 2\left\{\frac{\sinh 2u}{2} - u\right\} + C$$

$$= 2\sinh u \cosh u - 2u + C$$

$$= 2\left(\sqrt{\left(\frac{x}{2}\right)^2 - 1}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$

$$= 2\left(\frac{\sqrt{x^2-4}}{2}\right)\left(\frac{x}{2}\right) - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$

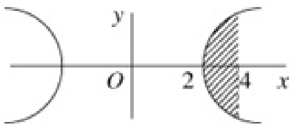
$$= \frac{1}{2}x\sqrt{x^2-4} - 2\operatorname{arcosh}\left(\frac{x}{2}\right) + C$$

$\cosh u = \frac{x}{2}$ and

$\sinh u = \sqrt{\cosh^2 u - 1}$

b $\text{Area} = 2\int_2^4 y \, \mathrm{d}x$

Rearranging $\frac{x^2}{4} - \frac{y^2}{9} = 1$ gives $9x^2 - 4y^2 = 36$



$4y^2 = 9x^2 - 36$

$= 9(x^2 - 4)$

So $y = \frac{3}{2}\sqrt{x^2-4}$, taking the $+$ve value, representing the part of curve in first quadrant

$$\text{Area} = 3\int_2^4 \sqrt{x^2-4}\,\mathrm{d}x = \left[\frac{3}{2}x\sqrt{x^2-4} - 6\operatorname{arcosh}\left(\frac{x}{2}\right)\right]_2^4$$

Using result from **a**

$$= \left[6\sqrt{12} - 6\operatorname{arcosh} 2\right] - \left[0 - 6\operatorname{arcosh} 1\right]$$

$$= 12.9 \text{ (3 s.f.)}$$

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Integration
Exercise C, Question 17

Question:

Unless a substitution is given or asked for, use the standard results **7** to **14**. Give numerical answers to 3 significant figures, unless otherwise stated.

a Show that $\int \frac{1}{2\cosh x - \sinh x}\,dx$ can be written as $\int \frac{2e^x}{e^{2x}+3}\,dx$.

b Hence, by using the substitution $u = e^x$, find $\int \frac{1}{2\cosh x - \sinh x}\,dx$.

Solution:

a $2\cosh x - \sinh x = 2\left(\frac{e^x + e^{-x}}{2}\right) - \left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + 3e^{-x}}{2}$

So $\int \frac{1}{2\cosh x - \sinh x}\,dx = \int \frac{2}{e^x + 3e^{-x}}\,dx$

$= \int \frac{2e^x}{e^{2x}+3}\,dx$

Multiplying numerator and denominator by e^x.

b Using the substitution $u = e^x$, $du = e^x\,dx$ and

$$\int \frac{2e^x}{e^{2x}+3}\,dx = 2\int \frac{du}{u^2+3}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{e^x}{\sqrt{3}}\right) + C$$

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Integration
Exercise C, Question 18

Question:

Using the substitution $u = \frac{2}{3}\sinh x$, evaluate $\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x + 9}}\,dx$.

Solution:

With $u = \frac{2}{3}\sinh x, du = \frac{2}{3}\cosh x dx$ or $\cosh x dx = \frac{3}{2}du$

$4\sinh^2 x + 9 = 4\left(\frac{3u}{2}\right)^2 + 9 = 9u^2 + 9 = 9(u^2 + 1)$

so $\int_0^1 \frac{\cosh x}{\sqrt{4\sinh^2 x + 9}}\,dx = \int_0^{\frac{2}{3}\sinh 1} \frac{1}{3\sqrt{u^2+1}} \times \frac{3}{2}\,du$

$= \frac{1}{2}\operatorname{arsinh}(u)$ between the given limits

$= \frac{1}{2}\operatorname{arsinh}\left(\frac{2}{3}\sinh 1\right)$

$= 0.360$ (3 s.f.)

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 19

Question:

a Find $\int \frac{\mathrm{d}x}{a^2 - x^2}$ $|x| < a$, by using

i partial fractions,

ii the substitution $x = a \tanh \theta$.

b Deduce the logarithmic form of $\text{artanh}\left(\frac{x}{a}\right)$.

Solution:

a i Using partial fractions $\frac{1}{a^2 - x^2} = \frac{1}{2a}\left\{\frac{1}{a-x} + \frac{1}{a+x}\right\}$

So $\int \frac{\mathrm{d}x}{a^2 - x^2} = \frac{1}{2a}\int\left\{\frac{1}{a-x} + \frac{1}{a+x}\right\}\mathrm{d}x$

$$= \frac{1}{2a}\left[-\ln|a-x| + \ln|a+x|\right] + C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

ii Using the substitution $x = a \tanh \theta, \mathrm{d}x = a\,\text{sech}^2\theta\ \mathrm{d}\theta$

$$\int \frac{\mathrm{d}x}{a^2 - x^2} = \int \frac{a\,\text{sech}^2\theta}{a^2\,\text{sech}^2\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{a}\theta + D$$

$$= \frac{1}{a}\,\text{artanh}\left(\frac{x}{a}\right) + D$$

b Using the result in **a** $\text{artanh}\left(\frac{x}{a}\right) = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + \text{constant}$

At $x = 0, 0 = 0 + \text{constant}, \Rightarrow \text{constant} = 0$ and so $\text{artanh}\left(\frac{x}{a}\right) = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right|$

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Integration
Exercise C, Question 20

Question:

Using the substitution $x = \sec\theta$, find

a $\int \frac{1}{x\sqrt{x^2-1}}\,dx$

b $\int \frac{\sqrt{x^2-1}}{x}\,dx$.

Solution:

With $x = \sec\theta$,

a
$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-1}}\,dx &= \int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta\\ &= \int 1\,d\theta\\ &= \theta + C\\ &= \operatorname{arcsec} x + C\end{aligned}$$

b
$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$$
$$\begin{aligned}&= \int \tan^2\theta\,d\theta\\ &= \int \left(\sec^2\theta-1\right)d\theta\\ &= \tan\theta - \theta + C\\ &= \sqrt{\sec^2\theta-1} - \theta + C\\ &= \sqrt{x^2-1} - \operatorname{arcsec} x + C\end{aligned}$$

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Integration
Exercise D, Question 1

Question:

Find the following.

a $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx$

b $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx$

c $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$

d $\int \frac{1}{\sqrt{x(x-2)}}\,dx$

e $\int \frac{1}{2x^2+4x+7}\,dx$

f $\int \frac{1}{\sqrt{-4x^2-12x}}\,dx$

g $\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx$

h $\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx$

Solution:

a $5-4x-x^2=-(x^2+4x-5)=-\{(x+2)^2-9\}=9-(x+2)^2$

So $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx=\int \frac{1}{\sqrt{9-(x+2)^2}}\,dx$

Let $u=(x+2)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{5-4x-x^2}}\,dx=\int \frac{1}{\sqrt{9-u^2}}\,du$

$=\arcsin\left(\frac{u}{3}\right)+C$

$=\arcsin\left(\frac{x+2}{3}\right)+C$

b $x^2-4x-12=\{(x-2)^2-16\}$

So $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx=\int \frac{1}{\sqrt{(x-2)^2-16}}\,dx$

Let $u=(x-2)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{x^2-4x-12}}\,dx=\int \frac{1}{\sqrt{u^2-16}}\,dx$

$=\operatorname{arcosh}\left(\frac{u}{4}\right)+C$

$=\operatorname{arcosh}\left(\frac{x-2}{4}\right)+C$

c $x^2+6x+10=\{(x+3)^2+1\}$

So $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx=\int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$

Let $u=(x+3)$, so $du=dx$.

Then $\int \frac{1}{\sqrt{x^2+6x+10}}\,dx=\int \frac{1}{\sqrt{u^2+1}}\,du$

$=\operatorname{arsinh}(u)+C$

$=\operatorname{arsinh}(x+3)+C$

d $x(x-2)=x^2-2x=\left\{(x-1)^2-1\right\}$

So $\displaystyle\int \frac{1}{\sqrt{x(x-2)}}\,dx=\int \frac{1}{\sqrt{(x-1)^2-1}}\,dx$

Let $u=(x-1)$, so $du=dx$.

Then $\displaystyle\int \frac{1}{\sqrt{x(x-2)}}\,dx=\int \frac{1}{\sqrt{u^2-1}}\,du$

$=\text{arcosh}\,(u)+C$

$=\text{arcosh}\,(x-1)+C$

e $2x^2+4x+7=2\left(x^2+2x+\frac{7}{2}\right)=2\left\{(x+1)^2+\frac{5}{2}\right\}$

Let $u=(x+1)$, so $du=dx$.

Then $\displaystyle\int \frac{1}{2x^2+4x+7}\,dx=\frac{1}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2}\,du$

$=\frac{1}{2}\left\{\frac{\sqrt{2}}{\sqrt{5}}\arctan\left(\frac{\sqrt{2}u}{\sqrt{5}}\right)\right\}+C$

$=\frac{\sqrt{10}}{10}\arctan\left(\frac{\sqrt{2}(x+1)}{\sqrt{5}}\right)+C$

f $-4x^2-12x=-4\left(x^2+3x\right)=-4\left\{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}\right\}=4\left\{\frac{9}{4}-\left(x+\frac{3}{2}\right)^2\right\}$

So $\displaystyle\int \frac{1}{\sqrt{-4x^2-12x}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{3}{2}\right)^2}}\,dx$

Let $u=\left(x+\frac{3}{2}\right)$, so $du=dx$.

Then $\displaystyle\int \frac{1}{\sqrt{-4x^2-12x}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-u^2}}\,du$

$=\frac{1}{2}\arcsin\left(\frac{2u}{3}\right)+C$

$=\frac{1}{2}\arcsin\left(\frac{2x+3}{3}\right)+C$

g $14-12x-2x^2 = -2\left(x^2+6x-7\right)$

$$= -2\left((x+3)^2-16\right)$$

$$= 2\left(16-(x+3)^2\right)$$

So $\displaystyle\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-(x+3)^2}}\,dx$

Let $u = x+3$, so $du = dx$

Then $\displaystyle\int \frac{1}{\sqrt{14-12x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{4^2-u^2}}\,du$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{u}{4}\right)+C$$

$$= \frac{1}{\sqrt{2}}\arcsin\left(\frac{x+3}{4}\right)+C$$

h $9x^2-8x+1 = 9\left(x^2-\frac{8}{9}x+\frac{1}{9}\right) = 9\left\{\left(x-\frac{4}{9}\right)^2-\frac{7}{81}\right\}$

So $\displaystyle\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{\left(x-\frac{4}{9}\right)^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,dx$

Let $u = \left(x-\frac{4}{9}\right)$, so $du = dx$.

Then $\displaystyle\int \frac{1}{\sqrt{9x^2-8x+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2-\left(\frac{\sqrt{7}}{9}\right)^2}}\,du$

$$= \frac{1}{3}\operatorname{arcosh}\left(\frac{9u}{\sqrt{7}}\right)+C$$

$$= \frac{1}{3}\operatorname{arcosh}\left(\frac{9x-4}{\sqrt{7}}\right)+C$$

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Integration
Exercise D, Question 2

Question:

Find

a $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx$

b $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx$.

Solution:

a $4x^2-12x+10=4\left(x^2-3x+\frac{5}{2}\right)=4\left\{\left(x-\frac{3}{2}\right)^2+\frac{1}{4}\right\}$

So $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{2}\right)$, so $du=dx$.

Then $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+10}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{u^2+\left(\frac{1}{2}\right)^2}}\,du$

$=\frac{1}{2}\text{arsinh}\,(2u)+C$

$=\frac{1}{2}\text{arsinh}\,(2x-3)+C$

b $4x^2-12x+4=4\left(x^2-3x+1\right)=4\left\{\left(x-\frac{3}{2}\right)^2-\frac{5}{4}\right\}$

So $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{2}\right)$, so $du=dx$.

Then $\displaystyle\int \frac{1}{\sqrt{4x^2-12x+4}}\,dx=\frac{1}{2}\int \frac{1}{\sqrt{u^2-\left(\frac{\sqrt{5}}{2}\right)^2}}\,du$

$=\frac{1}{2}\text{arcosh}\left(\frac{2u}{\sqrt{5}}\right)+C$

$=\frac{1}{2}\text{arcosh}\left(\frac{2x-3}{\sqrt{5}}\right)+C$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration

Exercise D, Question 3

Question:

Evaluate the following, giving answers to 3 significant figures.

a $\displaystyle\int_1^3 \frac{1}{\sqrt{x^2+2x+5}}\,dx$

b $\displaystyle\int_1^3 \frac{1}{x^2+x+1}\,dx$

c $\displaystyle\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx$

Solution:

a $x^2+2x+5=(x+1)^2+4$

So $\int_0^1 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_0^1 \frac{1}{\sqrt{(x+1)^2+4}}\,dx$

Let $u=(x+1)$, so $du=dx$.

$$\text{Then } \int_0^1 \frac{1}{\sqrt{x^2+2x+5}}\,dx=\int_1^2 \frac{1}{\sqrt{u^2+2^2}}\,du$$
$$=\left[\operatorname{arsinh}\left(\frac{u}{2}\right)\right]_1^2$$
$$=\left[\operatorname{arsinh}1-\operatorname{arsinh}\left(\frac{1}{2}\right)\right]$$
$$=0.400\ (3\text{ s.f.})$$

b $\int_1^3 \frac{1}{x^2+x+1}\,dx=\int_1^3 \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

Let $u=\left(x+\frac{1}{2}\right)$, so $du=dx$.

$$\text{Then } \int_1^3 \frac{1}{x^2+x+1}\,dx=\int_{\frac{3}{2}}^{\frac{7}{2}} \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$
$$=\left[\frac{2}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right)\right]_{\frac{3}{2}}^{\frac{7}{2}}$$
$$=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{7}{\sqrt{3}}\right)-\arctan\left(\sqrt{3}\right)\right]$$
$$=0.325\ (3\text{ s.f.})$$

c $2+3x-2x^2=-2\left(x^2-\frac{3}{2}x-1\right)=-2\left\{\left(x-\frac{3}{4}\right)^2-\frac{25}{16}\right\}=2\left\{\frac{25}{16}-\left(x-\frac{3}{4}\right)^2\right\}$

So $\int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx=\frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}\,dx$

Let $u=\left(x-\frac{3}{4}\right)$, so $du=dx$.

$$\text{Then } \int_0^1 \frac{1}{\sqrt{2+3x-2x^2}}\,dx=\frac{1}{\sqrt{2}}\int_{-\frac{3}{4}}^{\frac{1}{4}} \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2-u^2}}\,du$$
$$=\frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{4u}{5}\right)\right]_{\frac{-3}{4}}^{\frac{1}{4}}$$
$$=\frac{1}{\sqrt{2}}\left[\arcsin\left(\frac{1}{5}\right)-\arcsin\left(\frac{-3}{5}\right)\right]$$
$$=0.597\ (3\text{ s.f.})$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 4

Question:

Evaluate

a $\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx$, giving your answer as a single natural logarithm,

b $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx$, giving your answer in the form $k\pi$.

Solution:

a $x^2-2x+2=(x-1)^2+1$

So $\int_1^3 \frac{1}{\sqrt{x^2-2x+2}}\,dx = \int_1^3 \frac{1}{\sqrt{(x-1)^2+1}}\,dx$

$$= \left[\text{arsinh}\,(x-1)\right]_1^3$$

$$= \text{arsinh}\,2$$

$$= \ln\left\{2+\sqrt{5}\right\}$$

$\boxed{ar\sinh x = \ln\left\{x+\sqrt{x^2+1}\right\}}$

b $1+6x-3x^2=-3\left(x^2-2x-\frac{1}{3}\right)=-3\left\{(x-1)^2-\frac{4}{3}\right\}=3\left[\frac{4}{3}-(x-1)^2\right]$

So $\int_1^2 \frac{1}{\sqrt{1+6x-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int_1^2 \frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}\right)^2-(x-1)^2}}\,dx$

$$= \frac{1}{\sqrt{3}}\left[\arcsin\left(\frac{\sqrt{3}(x-1)}{2}\right)\right]_1^2$$

$$= \frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{3\sqrt{3}}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 5

Question:

Show that $\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}}\,dx = \frac{1}{\sqrt{3}}\ln(2+\sqrt{3})$.

Solution:

$$3x^2-6x+7 = 3\left(x^2-2x+\frac{7}{3}\right) = 3\left\{(x-1)^2+\frac{4}{3}\right\}$$

So $\int \frac{1}{\sqrt{3x^2-6x+7}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{(x-1)^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,dx$

Let $u=(x-1)$, so $du = dx$.

Then $\int_1^3 \frac{1}{\sqrt{3x^2-6x+7}} = \frac{1}{\sqrt{3}}\int_0^2 \frac{1}{\sqrt{u^2+\left(\frac{2}{\sqrt{3}}\right)^2}}\,du$

$$= \frac{1}{\sqrt{3}}\left[\operatorname{arsinh}\left(\frac{\sqrt{3}u}{2}\right)\right]_0^2$$

$$= \frac{1}{\sqrt{3}}\operatorname{arsinh}\sqrt{3}$$

$$= \frac{1}{\sqrt{3}}\ln\left\{\sqrt{3}+\sqrt{3+1}\right\} \qquad \operatorname{arsinh}x = \ln\left\{x+\sqrt{x^2+1}\right\}$$

$$= \frac{1}{\sqrt{3}}\ln\left\{2+\sqrt{3}\right\}$$

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Integration
Exercise D, Question 6

Question:

Using a suitable hyperbolic or trigonometric substitution find

a $\int \frac{1}{\sqrt{x^2+4x+5}}\,dx$

b $\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx$.

Solution:

a $x^2+4x+5=(x+2)^2+1$

So let $(x+2)=\sinh u$, then $dx=\cosh u\,du$ and $(x+2)^2+1=\sinh^2 u+1=\cosh^2 u$

Then $$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx=\int \frac{1}{\cosh u}\cosh u\,du$$
$$=\int 1\,du$$
$$=u+C$$
$$=\operatorname{arsinh}(x+2)+C$$

b $-x^2+4x+5=-(x^2-4x-5)=-\{(x-2)^2-9\}=9-(x-2)^2$

So let $(x-2)=3\sin\theta$, then $dx=3\cos\theta\,d\theta$

and $9-(x-2)^2=9(1-\sin^2\theta)=9\cos^2\theta$

Then $$\int \frac{1}{\sqrt{-x^2+4x+5}}\,dx=\int \frac{1}{3\cos\theta}3\cos\theta\,d\theta$$
$$=\int 1\,d\theta$$
$$=\theta+C$$
$$=\arcsin\left(\frac{x-2}{3}\right)+C$$

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Integration
Exercise D, Question 7

Question:

Using the substitution $x = \frac{1}{5}(\sqrt{3}\tan\theta - 1)$, obtain $\int_{-0.2}^{0} \frac{1}{25x^2+10x+4}\,dx$, giving your answer in terms of π.

Solution:

Using the substitution $x = \frac{1}{5}\left(\sqrt{3}\tan\theta - 1\right)$, $dx = \frac{\sqrt{3}}{5}\sec^2\theta\,d\theta$ and

$$\begin{aligned}25x^2+10x+4 &= \left(3\tan^2\theta - 2\sqrt{3}\tan\theta + 1\right) + 2\left(\sqrt{3}\tan\theta - 1\right) + 4 \\ &= 3\tan^2\theta + 3 \\ &= 3\left(\tan^2\theta + 1\right) = 3\sec^2\theta\end{aligned}$$

Then
$$\begin{aligned}\int_{-0.2}^{0} \frac{1}{25x^2+10x+4}\,dx &= \frac{\sqrt{3}}{5}\int_0^{\frac{\pi}{6}} \frac{1}{3\sec^2\theta}\sec^2\theta\,d\theta \\ &= \frac{\sqrt{3}}{15}\int_0^{\frac{\pi}{6}} 1\,d\theta \\ &= \frac{\pi\sqrt{3}}{90}\end{aligned}$$

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Integration
Exercise D, Question 8

Question:

Evaluate $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx$, giving your answer in the form $\ln(a+b\sqrt{c})$, where a, b and c are integers to be found.

Solution:

$(x-2)(x+4)=x^2+2x-8=(x+1)^2-9$

So $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx=\int_3^4 \frac{1}{\sqrt{(x+1)^2-3^2}}\,dx$

Let $u=(x+1)$, so $du=dx$.

Then $\int_3^4 \frac{1}{\sqrt{(x-2)(x+4)}}\,dx=\int_4^5 \frac{1}{\sqrt{u^2-3^2}}\,du$

$$=\left[\operatorname{arcosh}\left(\frac{u}{3}\right)\right]_4^5$$

$$=\operatorname{arcosh}\left(\frac{5}{3}\right)-\operatorname{arcosh}\left(\frac{4}{3}\right)$$

$$=\ln\left\{\left(\frac{5}{3}\right)+\sqrt{\frac{25}{9}-1}\right\}-\ln\left\{\left(\frac{4}{3}\right)+\sqrt{\frac{16}{9}-1}\right\}$$

$\boxed{ar\cosh x=\ln\left\{x+\sqrt{x^2-1}\right\}}$

$$=\ln 3-\ln\left\{\frac{4+\sqrt{7}}{3}\right\}$$

$$=\ln\left(\frac{9}{4+\sqrt{7}}\right)$$

$\boxed{\ln a-\ln b=\ln\left(\frac{a}{b}\right)}$

$$=\ln\left(\frac{9\left(4-\sqrt{7}\right)}{9}\right)$$

Rationalising the denominator

$$=\ln\left(4-\sqrt{7}\right)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise D, Question 9

Question:

Using the substitution $x = 1 + \sinh\theta$, show that

$$\int \frac{x}{\left(x^2 - 2x + 2\right)^{\frac{3}{2}}}\,dx = \frac{x-1}{\sqrt{x^2 - 2x + 2}} + C.$$

Solution:

Using the substitution $x = 1 + \sinh\theta$, $dx = \cosh\theta\, d\theta$ and

$$x^2 - 2x + 2 = \left(\sinh^2\theta + 2\sinh\theta + 1\right) - 2\left(\sinh\theta + 1\right) + 2 = \sinh^2\theta + 1 = \cosh^2\theta$$

So
$$\int \frac{1}{\left(x^2 - 2x + 2\right)^{\frac{3}{2}}}\,dx = \int \frac{1}{\cosh^3\theta}\cdot\cosh\theta\, d\theta$$

$$= \int \operatorname{sech}^2\theta\, d\theta$$

$$= \tanh\theta + C$$

$$= \frac{x-1}{\sqrt{x^2 - 2x + 2}} + C$$

$$\boxed{\begin{aligned}\sinh\theta &= x - 1\\ \cosh\theta = \sqrt{1 + \sinh^2\theta} &= \sqrt{2 - 2x + x^2}\end{aligned}}$$

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Integration
Exercise D, Question 10

Question:

Use the substitution $x = 2\sin\theta - 1$ to find $\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$.

Solution:

Using the substitution $x = 2\sin\theta - 1$, $dx = 2\cos\theta\, d\theta$

$$\begin{aligned} \text{and } 3-2x-x^2 &= 3-2(2\sin\theta-1)-(4\sin^2\theta-4\sin\theta+1) \\ &= 4-4\sin^2\theta \\ &= 4\cos^2\theta \end{aligned}$$

$$\begin{aligned} \text{So } \int \frac{x}{\sqrt{3-2x-x^2}}\,dx &= \int \frac{2\sin\theta-1}{2\cos\theta}\cdot 2\cos\theta\, d\theta \\ &= \int (2\sin\theta-1)\,d\theta \\ &= -2\cos\theta-\theta+C \\ &= -2\sqrt{1-\left(\frac{x+1}{2}\right)^2}-\theta+C \\ &= -\sqrt{3-2x-x^2}-\arcsin\left(\frac{x+1}{2}\right)+C \end{aligned}$$

$$\boxed{\begin{aligned} &\cos\theta = \sqrt{1-\sin^2\theta} \\ &\text{and } \sin\theta = \frac{x+1}{2} \end{aligned}}$$

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Integration
Exercise E, Question 1

Question:

a Show that $\int \text{arsinh}x \, dx = x\text{arsinh}x - \sqrt{1+x^2} + C$.

b Evaluate $\int_0^1 \text{arsinh}x \, dx$, giving your answer to 3 significant figures.

c Using the substitution $u = 2x+1$ and the result in **a**, or otherwise,

find $\int \text{arsinh}\,(2x+1)\, dx$.

Solution:

a $I = \int 1 \cdot \text{arsinh}x \, dx$

Let $u = \text{arsinh}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{x^2+1}} \quad v = x$

So $I = x\text{arsinh}x - \int \frac{x}{\sqrt{x^2+1}} dx$ ← Using integration by parts

$= x\text{arsinh}x - \sqrt{x^2+1} + C$ ← Using $\int f^n(x) f'(x) dx = \frac{1}{n+1} f^{n+1}(x) + C, n \neq -1$

b $\int_0^1 \text{arsinh}x = \left[x\text{arsinh}x - \sqrt{x^2+1} \right]_0^1$

$= \left[\text{arsinh}1 - \sqrt{2} \right] - [-1]$

$= 0.467$ (3 s.f.)

c Let $u = 2x+1$, so $du = dx$

Then $\int ar\sinh(2x+1)\, dx = \frac{1}{2} \int ar\sinh u \, dx$

$= \frac{1}{2} ar\sinh u - \sqrt{1+u^2} + C$ using a

$= \frac{1}{2}(2x+1)\sinh(2x+1) - \sqrt{4x^2+4x+2} + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 2

Question:

Show that $\int \arctan 3x \, \mathrm{d}x = x \arctan 3x - \frac{1}{6}\ln(1+9x^2) + C$.

Solution:

Let $u = \arctan 3x \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$

So $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{3}{1+(3x)^2} \quad v = x$

Using $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$, where $u = \arctan t$ and $t = 3x$

Then $\int \arctan 3x \, \mathrm{d}x = x \arctan 3x - \int \frac{3x}{1+9x^2} \, \mathrm{d}x$

$$= x \arctan 3x - \frac{1}{6}\int \frac{18x}{1+9x^2} \, \mathrm{d}x$$

$$= x \arctan 3x - \frac{1}{6}\ln\left(1+9x^2\right) + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 3

Question:

a Show that $\int \text{arcosh} x \, dx = x \text{arcosh} x - \sqrt{x^2 - 1} + C$.

b Hence show that $\int_1^2 \text{arcosh} x = \ln(7 + 4\sqrt{3}) - \sqrt{3}$.

Solution:

a Let $u = \text{arcosh} x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{x^2 - 1}} \quad v = x$

So $\int \text{arcosh} x \, dx = x \text{arcosh} x - \int \frac{x}{\sqrt{x^2 - 1}} dx$

$= x \text{arcosh} x - \sqrt{x^2 - 1} + C$

b Using limits

$\int_1^2 \text{arcosh} x = \left[2\text{arcosh} 2 - \sqrt{3} \right] - \left[\text{arcosh} 1 \right] = \left[2\text{arcosh} 2 - \sqrt{3} \right]$ $\boxed{\text{as } \text{arcosh} 1 = 0}$

As $\text{arcosh} x = \ln \left\{ x + \sqrt{x^2 - 1} \right\}$

$\int_1^2 \text{arcosh} x = \left[2\ln \left\{ 2 + \sqrt{3} \right\} - \sqrt{3} \right]$

$= \left[\ln \left\{ 2 + \sqrt{3} \right\}^2 - \sqrt{3} \right]$

$= \ln \left(7 + 4\sqrt{3} \right) - \sqrt{3}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 4

Question:

a Show that $\int \arctan x \, dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$.

b Hence show that $\int_{-1}^{\sqrt{3}} \arctan x \, dx = \frac{(4\sqrt{3}-3)\pi}{12} - \frac{1}{2}\ln 2$.

The curve C has equation $y = 2\arctan x$. The region R is enclosed by C, the y-axis, the line $y = \pi$ and the line $x = 3$.

c Find the area of R, giving your answer to 3 significant figures.

Solution:

a $I = \int 1 \times \arctan x \, dx$

Let $u = \arctan x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{1+x^2} \quad v = x$

So $I = x\arctan x - \int \frac{x}{1+x^2} dx$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

Using integration by parts

Using $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + C$

b $\int_{-1}^{\sqrt{3}} \arctan x = \left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]_{-1}^{\sqrt{3}}$

$= \left[\sqrt{3}\arctan\sqrt{3} - \frac{1}{2}\ln 4\right] - \left[-\arctan(-1) - \frac{1}{2}\ln 2\right]$

$= \frac{\sqrt{3}\pi}{3} - \ln 2 + \left(-\frac{\pi}{4}\right) + \frac{1}{2}\ln 2$

$= \frac{(4\sqrt{3}-3)\pi}{12} - \frac{1}{2}\ln 2$

c Area of R = area of rectangle $- \int_0^3 2\arctan x \, dx$

$= 3\pi - 2\left[x\arctan x - \frac{1}{2}\ln(1+x^2)\right]_0^3$ Using *a*

$= 3\pi - 6\arctan 3 + \ln 10$

$= 4.23$ (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 5

Question:

Evaluate

a $\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx$

b $\int_0^1 x \arctan x \, dx$ giving your answers in terms of π.

Solution:

a Let $u = \arcsin x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad v = x$

Then $\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \, dx = [x \arcsin x]_0^{\frac{\sqrt{2}}{2}} - \int_0^{\frac{\sqrt{2}}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$

$$= \left[x \arcsin x + \sqrt{1-x^2} \right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \left[\frac{\sqrt{2}}{2} \frac{\pi}{4} + \sqrt{\frac{1}{2}} \right] - [+1]$$

$$= \frac{\sqrt{2}}{8}\pi - 1 + \frac{\sqrt{2}}{2} = 0.262 \text{ (3 s.f.)}$$

b Let $u = \arctan x \quad \frac{dv}{dx} = x$

So $\frac{du}{dx} = \frac{1}{1+x^2} \quad v = \frac{x^2}{2}$

Then $\int_0^1 x \arctan x \, dx = \left[\frac{x^2}{2} \arctan x \right]_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} \, dx$

$$= \left[\frac{1}{2} \arctan 1 \right]_0^1 - \frac{1}{2} \int_0^1 \frac{1+x^2-1}{1+x^2} \, dx$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} [x - \arctan x]_0^1$$

$$= \left[\frac{\pi}{8} \right] - \frac{1}{2} \left[1 - \frac{\pi}{4} \right]$$

$$= \frac{\pi - 2}{4}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 6

Question:

Using the result that if $y = \text{arcsec}x$, then $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$, show that

$$\int \text{arcsec}x\,dx = x\,\text{arcsec}x - \ln\{x + \sqrt{x^2-1}\} + C.$$

Solution:

Let $u = \text{arcsec}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{x\sqrt{x^2-1}} \quad v = x$

and
$$\int \text{arcsec}x\,dx = x\,\text{arcsec}x - \int \frac{x}{x\sqrt{x^2-1}}\,dx$$
$$= x\,\text{arcsec}x - \text{arcosh}x + C$$
$$= x\,\text{arcsec}x - \ln\left\{x + \sqrt{x^2-1}\right\} + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise E, Question 7

Question:

a Show that $\int \text{arsinh}(2x+1)\,dx = X\text{arsinh}(2x+1) - \int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$.

b Find $\int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$, using the substitution $2x+1 = \sin\text{H}u$, and hence find $\int \arcsin(2x+1)\,dx$.

Solution:

a Let $u = \text{arsinh}(2x+1)$ $\quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{2}{\sqrt{(2x+1)^2+1}}$ $\quad v = x$

Then $\int \text{arsinh}(2x+1)\,dx = x\text{arsinh}(2x+1) - \int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx$

b Let $2x+1 = \sinh u$ then $2\,dx = \cosh u\,du$

$$\text{So } \int \frac{2x}{\sqrt{(2x+1)^2+1}}\,dx = \frac{1}{2}\int \frac{(\sinh u - 1)}{\cosh u}\cosh u\,du$$

$$= \frac{1}{2}\left[\int \sinh u\,du - u\right]$$

$$= \frac{1}{2}\left[\cosh u - u\right] + C$$

$$= \frac{1}{2}\left\{\sqrt{1+(2x+1)^2} - \text{arsinh}(2x+1)\right\} + C$$

$$\int \text{arsinh}(2x+1)\,dx = x\text{arsinh}(2x+1) + \frac{1}{2}\text{arsinh}(2x+1) - \frac{1}{2}\sqrt{1+(2x+1)^2} + C$$ Using **a** and **b**.

$$= \frac{1}{2}(2x+1)\text{arsinh}(2x+1) - \frac{1}{2}\sqrt{1+(2x+1)^2} + C$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 1

Question:

Given that $I_n = \int x^n e^{\frac{x}{2}} dx$,

a show that $I_n = 2x^n e^{\frac{x}{2}} - 2nI_{n-1}$, $n \geq 1$.

b Hence find $\int x^3 e^{\frac{x}{2}} dx$.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = e^{\frac{x}{2}}$

so $\frac{du}{dx} = nx^{n-1}, v = 2e^{\frac{x}{2}}$

So $I_n = 2x^n e^{\frac{x}{2}} - \int 2nx^{n-1} e^{\frac{x}{2}} dx$

$= 2x^n e^{\frac{x}{2}} - 2n \int x^{n-1} e^{\frac{x}{2}} dx$

$= 2x^n e^{\frac{x}{2}} - 2nI_{n-1}$ *

b $I_3 = 2x^3 e^{\frac{x}{2}} - 6I_2$

$= 2x^3 e^{\frac{x}{2}} - 6\left(2x^2 e^{\frac{x}{2}} - 4I_1\right)$

Substituting $n = 3, 2$ and 1 respectively in *

$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 24\left(2xe^{\frac{x}{2}} - 2I_0\right)$, where $I_0 = \int e^{\frac{x}{2}} dx = 2e^{\frac{x}{2}} + C$

$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 48I_0$

So $\int x^3 e^{\frac{x}{2}} dx = 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + C$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 2

Question:

Given that $I_n = \int_1^e x(\ln x)^n \, dx, n \in N$,

a show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}, n \in N$.

b Hence show that $\int_1^e x(\ln x)^4 \, dx = \frac{e^2 - 3}{4}$.

Solution:

a Let $u = (\ln x)^n$ and $\frac{dv}{dx} = x$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = \frac{x^2}{2}$

Integration by parts:

$$\int_1^e x(\ln x)^n \, dx = \left[\frac{x^2(\ln x)^n}{2}\right]_1^e - \int_1^e \frac{nx^2(\ln x)^{n-1}}{2x} \, dx$$

$$= \left[\frac{e^2}{2} - 0\right] - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$$

So $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ *

b $\int_1^e x(\ln x)^4 \, dx = I_4$

Substituting $n = 4, 3, 2$ and 1 respectively in the reduction formula *

$$I_4 = \frac{e^2}{2} - \frac{4}{2} I_3$$

$$= \frac{e^2}{2} - 2\left(\frac{e^2}{2} - \frac{3}{2} I_2\right)$$

$$= \frac{e^2}{2} - e^2 + 3\left(\frac{e^2}{2} - \frac{2}{2} I_1\right)$$

$$= \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - 3\left(\frac{e^2}{2} - \frac{1}{2} I_0\right), \text{ where } I_0 = \int_1^e x \, dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$$

$$\text{So } \int_1^e x(\ln x)^4 \, dx = \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - \frac{3e^2}{2} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$$

$$= \frac{e^2}{4} - \frac{3}{4} = \frac{e^2 - 3}{4}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 3

Question:

In Example 21, you saw that, if $I_n = \int_0^1 x^n \sqrt{1-x}\,dx$, then $I_n = \frac{2n}{2n+3} I_{n-1}, n \geq 1$.

Use this reduction formula to evaluate $\int_0^1 (x+1)(x+2)\sqrt{1-x}\,dx$

Solution:

$$\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \int_0^1 \left[\left(x^2+3x+2\right)\sqrt{1-x}\right] dx$$

$$= \int_0^1 \left[x^2\sqrt{1-x}\right] dx + \int_0^1 \left[3x\sqrt{1-x}\right] dx + \int_0^1 \left[2\sqrt{1-x}\right] dx$$

$$= I_2 + 3I_1 + 2I_0$$

Now $I_0 = \int_0^1 \sqrt{1-x}\,dx = \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$

$I_1 = \frac{2}{5} I_0 = \left(\frac{2}{5}\right)\left(\frac{2}{3}\right) = \frac{4}{15}$ ← Using the given formula with $n = 1$

$I_2 = \frac{4}{7} I_1 = \left(\frac{4}{7}\right)\left(\frac{4}{15}\right) = \frac{16}{105}$ ← Using the given formula with $n = 2$

So $\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \frac{16}{105} + 3\left(\frac{4}{15}\right) + 2\left(\frac{2}{3}\right)$

$$= \frac{16 + 12(7) + 4(35)}{105}$$

$$= \frac{240}{105} = \frac{16}{7}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 4

Question:

Given that $I_n = \int x^n e^{-x}\, dx$, where n is a positive integer,

a show that $I_n = -x^n e^{-x} + nI_{n-1}$, $n \geq 1$.

b Find $\int x^3 e^{-x}\, dx$.

c Evaluate $\int_0^1 x^4 e^{-x} dx$, giving your answer in terms of e.

Solution:

a Using integration by parts with $u = x^n$ and $\frac{dv}{dx} = e^{-x}$

so $\frac{du}{dx} = nx^{n-1}$ and $v = -e^{-x}$

$\int x^n e^{-x}\, dx = -x^n e^{-x} - \int -nx^{n-1} e^{-x}\, dx$, so $I_n = -x^n e^{-x} + nI_{n-1}$

b Repeatedly using the reduction formula to find I_3

$$\begin{aligned} I_3 &= -x^3 e^{-x} + 3I_2 \\ &= -x^3 e^{-x} + 3\left(-x^2 e^{-x} + 2I_1\right) \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6I_1 \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6\left(-xe^{-x} + I_0\right) \end{aligned}$$

But $I_0 = \int e^{-x}\, dx = -e^{-x} + C$

So $I_3 = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + K$

c $I_4 = -x^4 e^{-x} + 4I_3$

$= -x^4 e^{-x} + 4\left(-x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + C\right)$ Using the result from **b**

So $\int_0^1 x^4 e^{-x}\, dx = \left[-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^1$

$= \left[-65e^{-1}\right] - \left[-24\right]$

$= 24 - 65e^{-1}$ or $\frac{24e - 65}{e}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 5

Question:

$I_n = \int \tanh^n x \, dx$,

a By writing $\tanh^n x = \tanh^{n-2} x \tanh^2 x$, show that for $n \geq 2$,

$I_n = I_{n-2} - \frac{1}{n-1} \tanh^{n-1} x$.

b Find $\int \tanh^5 x \, dx$.

c Show that $\int_0^{\ln 2} \tanh^4 x \, dx = \ln 2 - \frac{84}{125}$.

Solution:

a $I_n = \int \tanh^n x \,dx = \int \tanh^{n-2} x \tanh^2 x \,dx$

$= \int \tanh^{n-2} x\left(1-\text{sech}^2 x\right)dx$

$= \int \tanh^{n-2} x - \int \tanh^{n-2} \text{sech}^2 x \,dx$

So $I_n = I_{n-2} - \frac{1}{n-1}\tanh^{n-1} x, \quad n \neq 1$

Using $1-\tan\text{h}^2 x = \sec\text{h}^2 x$

b $\int \tanh^5 x \,dx = I_5 = I_3 - \frac{1}{4}\tanh^4 x$

$= \left(I_1 - \frac{1}{2}\tanh^2 x\right) - \frac{1}{4}\tanh^4 x$

$= \int \tanh x \,dx - \frac{1}{2}\tanh^2 x - \frac{1}{4}\tanh^4 x$

$= \ln\cosh x - \frac{1}{2}\tanh^2 x - \frac{1}{4}\tanh^4 x + C$

c As $\int \tanh^n x \,dx = \int \tanh^{n-2} x dx - \frac{1}{n-1}\tanh^{n-1} x$, it follows that

$$\int_0^{\ln 2} \tanh^n x \,dx = \int_0^{\ln 2} \tanh^{n-2} x \,dx - \left[\frac{1}{n-1}\tanh^{n-1} x\right]_0^{\ln 2} \quad *$$

Now $\tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2-\frac{1}{2}}{2+\frac{1}{2}} = \frac{3}{5}$

Reminder: $e^{-\ln a} = e^{\ln a^{-1}} = a^{-1}$

So $\int_0^{\ln 2} \tanh^4 x \,dx = \int_0^{\ln 2} \tanh^2 x \,dx - \frac{1}{3}\times\left(\frac{3}{5}\right)^3$

Using $*$ with $n=4$ and $\tanh(\ln 2) = \frac{3}{5}$

$= \left[\int_0^{\ln 2} \tanh^0 x \,dx - 1\times\left(\frac{3}{5}\right)\right] - \frac{1}{3}\times\frac{27}{125}$

Using $*$ with $n=2$ and $\tanh(\ln 2) = \frac{3}{5}$

$= \ln 2 - \frac{3}{5} - \frac{9}{125}$

$= \ln 2 - \frac{84}{125}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 6

Question:

Given that $\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$ (derived in Example 23)

a find $\int \tan^4 x \, dx$.

b Evaluate $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$.

c Show that $\int_0^{\frac{\pi}{3}} \tan^6 x \, dx = \frac{9\sqrt{3}}{5} - \frac{\pi}{3}$.

Solution:

a $\int \tan^4 x \, dx = \frac{1}{3}\tan^3 x - \int \tan^2 x \, dx$

$= \frac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x dx \right)$

$= \frac{1}{3}\tan^3 x - \tan x + \int 1 \, dx$

$= \frac{1}{3}\tan^3 x - \tan x + x + C$

b $\int_0^{\frac{\pi}{4}} \tan^n x \, dx = \left[\frac{1}{n-1}\tan^{n-1} x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x dx$

Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$, then $I_n = \frac{1}{n-1} - I_{n-2}$

$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left(\frac{1}{2} - I_1 \right) = \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x \, dx = \frac{1}{4} - \frac{1}{2} + [\ln \sec x]_0^{\frac{\pi}{4}}$

$= -\frac{1}{4} + \left(\ln \sqrt{2} - \ln 1 \right)$

So $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx = \ln \sqrt{2} - \frac{1}{4}$

c Defining $J_n = \int_0^{\frac{\pi}{3}} \tan^n x \, dx$,

$J_n = \left[\frac{1}{n-1}\tan^{n-1} x \right]_0^{\frac{\pi}{3}} - J_{n-2} = \frac{\left(\sqrt{3}\right)^{n-1}}{n-1} - J_{n-2}$

So $J_6 = \frac{\left(\sqrt{3}\right)^5}{5} - J_4 = \frac{\left(\sqrt{3}\right)^5}{5} - \left(\frac{\left(\sqrt{3}\right)^3}{3} - J_2 \right) = \frac{\left(\sqrt{3}\right)^5}{5} - \frac{\left(\sqrt{3}\right)^3}{3} + \left(\frac{\sqrt{3}}{1} - J_0 \right)$

As $J_0 = \int_0^{\frac{\pi}{3}} 1 \, dx = \frac{\pi}{3}$, $\int_0^{\frac{\pi}{3}} \tan^6 x \, dx = \frac{9\sqrt{3}}{5} - \frac{3\sqrt{3}}{3} + \sqrt{3} - \frac{\pi}{3} = \frac{9\sqrt{3}}{5} - \frac{\pi}{3}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 7

Question:

Given that $I_n = \int_1^a (\ln x)^n \, dx$, where $a > 1$ is a constant,

a show that, for $n \geq 1, I_n = a(\ln a)^n - nI_{n-1}$.

b Find the exact value of $\int_1^2 (\ln x)^3 \, dx$.

c Show that $\int_1^e (\ln x)^6 \, dx = 5(53e - 144)$.

Solution:

a $I_n = \int_1^a (\ln x)^n \, dx = \int_1^a 1(\ln x)^n \, dx$

Let $u = (\ln x)^n$ and $\frac{dv}{dx} = 1$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = x$

Integration by parts:

$$\int_1^a (\ln x)^n \, dx = \left[x(\ln x)^n\right]_1^a - \int_1^a \frac{n(\ln x)^{n-1}}{x} x \, dx$$

$$= \left[a(\ln a)^n - 0\right] - n\int_1^a (\ln x)^{n-1} \, dx$$

So $I_n = a(\ln a)^n - nI_{n-1}$

b Putting $a = 2, I_n = \int_1^2 (\ln x)^n \, dx = 2(\ln 2)^n - nI_{n-1}$

$$I_3 = \int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 3I_2$$

$$= 2(\ln 2)^3 - 3\left\{2(\ln 2)^2 - 2I_1\right\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6\left\{2(\ln 2) - I_0\right\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6I_0$$

As $I_0 = \int_1^2 1 \, dx = [x]_1^2 = 1$,

$$\int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6$$

c Putting $a = e, I_n = \int_1^e (\ln x)^n \, dx = e(\ln e)^n - nI_{n-1} = e - nI_{n-1}$

$$I_6 = \int_1^e (\ln x)^6 \, dx = e - 6I_5$$

$$= e - 6(e - 5I_4)$$

$$= e - 6e + 30(e - 4I_3)$$

$$= e - 6e + 30e - 120(e - 3I_2)$$

$$= e - 6e + 30e - 120e + 360(e - 2I_1)$$

$$= e - 6e + 30e - 120e + 360e - 720(e - I_0)$$

As $I_0 = \int_1^e 1 \, dx = [x]_1^e = e - 1$,

$$\int_1^e (\ln x)^6 \, dx = e - 6e + 30e - 120e + 360e - 720e + 720(e - 1)$$

$$= 265e - 720$$

$$= 5(53e - 144)$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration

Exercise F, Question 8

Question:

Using the results given in Example 22, evaluate

a $\int_0^{\frac{\pi}{2}} \sin^7 x \, dx$

b $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

c $\int_0^1 x^5 \sqrt{1-x^2} \, dx$, using the substitution $x = \sin\theta$

d $\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt$, using a suitable substitution.

Solution:

a $I_7 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$

b $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \left(1-\sin^2 x\right)^2 dx = \int_0^{\frac{\pi}{2}} \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) dx$

$= I_2 - 2I_4 + I_6$

$I_2 = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}; \quad I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}; \quad I_6 = \frac{5}{6} I_4 = \frac{5\pi}{32}$

So $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx = \frac{\pi}{4} - \frac{3\pi}{8} + \frac{5\pi}{32} = \frac{\pi}{32}$

c Using $x = \sin\theta$, $\int_0^1 x^5 \sqrt{1-x^2} \, dx = \int_0^{\frac{\pi}{2}} \sin^5\theta \cos\theta \left(\cos\theta \, d\theta\right)$

$= \int_0^{\frac{\pi}{2}} \sin^5 x \left(1 - \sin^2 x\right) dx = I_5 - I_7$

$I_5 = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$ and $I_7 = \frac{16}{35}$ from **a**

So $\int_0^1 x^5 \sqrt{1-x^2} \, dx = \frac{8}{15} - \frac{16}{35} = \frac{56-48}{105} = \frac{8}{105}$

d Using $x = 3t$, $\int_0^{\frac{\pi}{6}} \sin^8 3t \, dt = \int_0^{\frac{\pi}{2}} \sin^8 x \left(\frac{1}{3} dx\right) = \frac{1}{3} I_8$

$= \frac{1}{3} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35\pi}{768}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 9

Question:

Given that $I_n = \int \frac{\sin^{2n} x}{\cos x} \, dx$,

a write down a similar expression for I_{n+1} and hence show that $I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$.

b Find $\int \frac{\sin^4 x}{\cos x} \, dx$ and hence show that $\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos x} \, dx = \ln\left(1+\sqrt{2}\right) - \frac{7\sqrt{2}}{12}$.

Solution:

a $I_{n+1} = \int \frac{\sin^{2n+2} x}{\cos x}\,dx$

So $I_n - I_{n+1} = \int \frac{\sin^{2n} x - \sin^{2n+2} x}{\cos x}\,dx$

$= \int \frac{\sin^{2n} x\left(1-\sin^2 x\right)}{\cos x}\,dx$

$= \int \sin^{2n} x \cos x\,dx$ $\boxed{\text{as } 1-\sin^2 x = \cos^2 x}$

So $I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$

or $I_{n+1} = I_n - \frac{\sin^{2n+1} x}{2n+1}$ $*$ $\boxed{[+C \text{ not necessary at this stage}]}$

b i $\int \frac{\sin^4 x}{\cos x}\,dx = I_2$

Substituting $n = 1$ in $*$ gives $I_2 = I_1 - \frac{\sin^3 x}{3}$

$= \left(I_0 - \frac{\sin x}{1}\right) - \frac{\sin^3 x}{3}$ using $n = 0$ in $*$

$I_0 = \int \frac{1}{\cos x}\,dx = \int \sec x\,dx = \ln\left|(\sec x + \tan x)\right| + C$

So $\int \frac{\sin^4 x}{\cos x}\,dx = \ln\left|(\sec x + \tan x)\right| - \sin x - \frac{\sin^3 x}{3} + C$

Applying the given limits gives

$$\int_0^{\frac{\pi}{4}} \frac{\sin^4 x}{\cos x}\,dx = \left[\ln\left|(\sec x + \tan x)\right| - \sin x - \frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{4}}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{\sqrt{2}}{2} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{7\sqrt{2}}{12}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 10

Question:

a Given that $I_n = \int_0^1 x(1-x^3)^n \, dx$, show that $I_n = \frac{3n}{3n+2} I_{n-1}, n \geq 1$.

b Use your reduction formula to evaluate I_4.

Hint: After integrating by parts, write x^4 as $x(1-(1-x^3))$

Solution:

a Let $u = (1-x^3)^n$ and $\frac{dv}{dx} = x$, so $\frac{du}{dx} = n(1-x^3)^{n-1}(-3x^2), v = \frac{x^2}{2}$

Integration by parts gives

$$\int_0^1 x(1-x^3)^n \, dx = \left[\frac{x^2}{2}(1-x^3)^n\right]_0^1 - \int_0^1 -3nx^2(1-x^3)^{n-1}\frac{x^2}{2} \, dx$$

$$= [0-0] + \frac{3n}{2}\int_0^1 x^4(1-x^3)^{n-1} \, dx \quad \text{providing } n \geq 0$$

Writing $x^4 = x.x^3 = x\{1-(1-x^3)\}$ and $I_n = \int_0^1 x(1-x^3)^n \, dx$

we have $I_n = \frac{3n}{2}\int_0^1 x\{1-(1-x^3)\}(1-x^3)^{n-1} \, dx$

$$= \frac{3n}{2}\int_0^1 x(1-x^3)^{n-1} \, dx - \frac{3n}{2}\int_0^1 x(1-x^3)^n \, dx$$

$$= \frac{3n}{2} I_{n-1} - \frac{3n}{2} I_n$$

$\Rightarrow (3n+2)I_n = 3nI_{n-1}$, so $I_n = \frac{3n}{3n+2} I_{n-1}, n \geq 1$

b $I_4 = \frac{12}{14} I_3 = \frac{12}{14} \times \frac{9}{11} I_2 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} I_1 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} I_0 = \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \int_0^1 x \, dx$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \times \frac{1}{2} = \frac{243}{1540}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 11

Question:

Given that $I_n = \int_0^a (a^2 - x^2)^n \, dx$, where a is a positive constant,

a show that, for $n > 0$, $I_n = \frac{2na^2}{2n+1} I_{n-1}$.

b Use the reduction formula to evaluate

i $\int_0^1 (1-x^2)^4 \, dx$

ii $\int_0^3 (9-x^2)^3 \, dx$

iii $\int_0^2 \sqrt{4-x^2} \, dx$.

c Check your answer to part **b iii** by using another method.

Solution:

a Integrating by parts with $u = (a^2 - x^2)^n$ and $\frac{dv}{dx} = 1$

$\frac{du}{dx} = -2nx(a^2 - x^2)^{n-1} \quad v = x$

So $\int_0^a (a^2 - x^2)^n \, dx = \left[x(a^2 - x^2)^n\right]_0^a - \int_0^a x\left\{-2nx(a^2 - x^2)^{n-1}\right\} dx$

$= [0 - 0] + 2n\int_0^a x^2(a^2 - x^2)^{n-1} \, dx = 2n\int_0^a x^2(a^2 - x^2)^{n-1} \, dx$ (if $n > 0$)

Writing x^2 as $\{a^2 - (a^2 - x^2)\}$ and defining $I_n = \int_0^a (a^2 - x^2)^n \, dx$,

we have

$I_n = 2n\int_0^a \left\{a^2(a^2 - x^2)^{n-1} - (a^2 - x^2)^n\right\} dx$

$= 2na^2 I_{n-1} - 2nI_n$

So $(2n+1)I_n = 2na^2 I_{n-1}$

b i With $a = 1$, $I_n = \int_0^1 (1 - x^2)^n \, dx$ and $I_n = \frac{2n}{2n+1} I_{n-1}$

So $I_4 = \frac{8}{9} I_3 = \frac{8}{9} \times \frac{6}{7} I_2 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} I_1 = \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{128}{315}$

$$\boxed{I_0 = \int_0^a dx = a}$$

ii With $a = 3$, $I_n = \int_0^3 (9 - x^2)^n \, dx$ and $I_n = \frac{18n}{2n+1} I_{n-1}$

So $I_3 = \frac{54}{7} I_2 = \frac{54}{7} \times \frac{36}{5} I_1 = \frac{54}{7} \times \frac{36}{5} \times \frac{18}{3} I_0 = \frac{54}{7} \times \frac{36}{5} \times \frac{18}{3} \times 3 = \frac{34\,992}{35}$

iii With $a = 2$, $I_n = \int_0^2 (4 - x^2)^n \, dx$ and $I_n = \frac{8n}{2n+1} I_{n-1}$

So $I_{\frac{1}{2}} = \frac{4}{2} I_{o\frac{1}{2}} = 2\int_0^2 \frac{dx}{\sqrt{4 - x^2}} = 2\left[\arcsin\left(\frac{x}{2}\right)\right]_0^2 = 2\arcsin 1 = 2 \times \frac{\pi}{2} = \pi$

c Using the substitution $x = 2\sin\theta$,

$\int_0^2 (4 - x^2)^{\frac{1}{2}} \, dx = \int_0^{\frac{\pi}{2}} (2\cos\theta)(2\cos\theta \, d\theta)$

$= 2\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$

$= [2\theta + \sin 2\theta]_0^{\frac{\pi}{2}} = \pi$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 12

Question:

Given that $I_n = \int_0^4 x^n \sqrt{4-x}\,dx$,

a establish the reduction formula $I_n = \frac{8n}{2n+3} I_{n-1}, n \geq 1$.

b Evaluate $\int_0^4 x^3 \sqrt{4-x}$, giving your answer correct to 3 significant figures.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \sqrt{4-x}$

$\frac{du}{dx} = nx^{n-1}, \quad v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$

So $\int_0^4 x^n \sqrt{4-x}\,dx = \left[-\frac{2}{3}x^n(4-x)^{\frac{3}{2}}\right]_0^4 - \int_0^4 -\frac{2}{3}nx^{n-1}(4-x)^{\frac{3}{2}}\,dx$

$$= [0-0] + \frac{2}{3}n\int_0^4 x^{n-1}(4-x)^{\frac{3}{2}}\,dx \ (n > 0)$$

$$= \frac{2}{3}n\int_0^4 x^{n-1}\left\{(4-x)\sqrt{4-x}\right\}dx$$

You need to write $(4-x)^{\frac{3}{2}}$ as $(4-x)\sqrt{4-x}$

$$= \frac{2}{3}n\int_0^4 x^{n-1}4\sqrt{4-x}\,dx + \frac{2}{3}n\int_0^4 x^{n-1}\left\{-x\sqrt{4-x}\right\}dx$$

$$= \frac{8}{3}n\int_0^4 x^{n-1}\sqrt{4-x}\,dx - \frac{2}{3}n\int_0^4 x^n\sqrt{4-x}\,dx$$

So $I_n = \frac{8}{3}nI_{n-1} - \frac{2}{3}nI_n$

$\Rightarrow (2n+3)I_n = 8nI_{n-1} \leq I_n = \frac{8n}{2n+3}I_{n-1}, n \geq 1$

b $\int_0^4 x^3\sqrt{4-x}\,dx = I_3 = \frac{24}{9}I_2 = \frac{24}{9}\times\frac{16}{7}I_1 = \frac{24}{9}\times\frac{16}{7}\times\frac{8}{5}I_0 = \frac{1024}{105}I_0$

As $I_0 = \int_0^4 \sqrt{4-x}\,dx = \left[-\frac{2}{3}(4-x)^{\frac{3}{2}}\right]_0^4 = \left[0 - \left\{-\frac{2}{3}(4)^{\frac{3}{2}}\right\}\right] = \frac{16}{3}$,

$\int_0^4 x^3\sqrt{4-x}\,dx = \frac{1024}{105}\times\frac{16}{3} = 52.0 \text{ (3 s.f.)}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 13

Question:

Given that $I_n = \int \cos^n x \, dx$,

a establish, for $n \geq 2$, the reduction formula $nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$.

Defining $J_n = \int_0^{2\pi} \cos^n x \, dx$,

b write down a reduction formula relating J_n and J_{n-2}, for $n \geq 2$.

c Hence evaluate

i J_4

ii J_8.

d Show that if n is odd, J_n is always equal to zero.

Solution:

a $I_n = \int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$

Integrating by parts with $u = \cos^{n-1} x$ and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = (n-1)\cos^{n-2} x(-\sin x), \quad v = \sin x$

So $I_n = \int \cos^n x \, dx = \cos^{n-1} x \sin x - \int -(n-1)\cos^{n-2} x \sin^2 x \, dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1 - \cos^2 x)dx$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x \, dx$

Giving $I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$

b It follows that $n\int_0^{2\pi} \cos^n x dx = \left[\cos^{n-1} x \sin x\right]_0^{2\pi} + (n-1)\int_0^{2\pi} \cos^{n-2} x \, dx$

So $nJ_n = (n-1)J_{n-2}$, as $\left[\cos^{n-1} x \sin x\right]_0^{2\pi} = 0$

c i $J_4 = \int_0^{2\pi} \cos^4 x dx = \frac{3}{4}J_2 = \frac{3}{4} \times \frac{1}{2}J_0 = \frac{3}{8}\int_0^{2\pi} 1 \, dx = \frac{3}{8} \times 2\pi = \frac{3\pi}{4}$

ii $J_8 = \int_0^{2\pi} \cos^8 x \, dx = \frac{7}{8}J_6 = \frac{7}{8} \times \frac{5}{6}J_4 = \frac{35}{48}J_4 = \frac{35}{48} \times \frac{3\pi}{4} = \frac{35\pi}{64}$ $\boxed{\text{using } \mathbf{c\ i}}$

d If n is odd, J_n always reduces to a multiple of J_1,

but $J_1 = \int_0^{2\pi} \cos x dx = \left[\sin x\right]_0^{2\pi} = 0$.

(You could also consider the graphical representation.)

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 14

Question:

Given $I_n = \int_0^1 x^n \sqrt{(1-x^2)}\,dx, n \geq 0$,

a show that $(n+2)I_n = (n-1)I_{n-2}, n \geq 2$.

b Hence evaluate $\int_0^1 x^7 \sqrt{(1-x^2)}\,dx$.

Hint: Write $x^n\sqrt{1-x^2}$ as $x^{n-1}\left\{x\sqrt{1-x^2}\right\}$ before integrating by parts.

Solution:

a Integrating by parts with $u = x^{n-1}$ and $\frac{dv}{dx} = x\sqrt{1-x^2}$

Using the hint.

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$

$$\text{So } I_n = \int_0^1 x^{n-1}\left\{x\sqrt{1-x^2}\right\}dx = \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 + \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}\,dx \text{ as } \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 = 0$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} \left\{x^{n-2}\sqrt{1-x^2} - x^n\sqrt{1-x^2}\right\}dx$$

$$\text{So } I_n = \frac{(n-1)}{3}I_{n-2} - \frac{(n-1)}{3}I_n$$

$$\Rightarrow \{3+(n-1)\}I_n = (n-1)I_{n-2}$$

$$\Rightarrow (n+)I_n = (n-1)I_{n-2} \quad *$$

b Using $*$ $\quad I_7 = \frac{6}{9}I_5 = \frac{6}{9}\times\frac{4}{7}I_3 = \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}I_1 = \frac{48}{315}\int_0^1 x\sqrt{1-x^2}\,dx$

$$= \frac{48}{315}\left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{48}{315}\left[\frac{1}{3}\right] = \frac{16}{315}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 15

Question:

Given $I_n = \int x^n \cosh x \, dx$

a show that for $n \geq 2, I_n = x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2}$

b Find $I_4 = \int x^4 \cosh x \, dx$.

c Evaluate $\int_0^1 x^3 \cosh x$, giving your answer in terms of e.

Solution:

a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \cosh x$

$\frac{du}{dx} = nx^{n-1}, \quad v = \sinh x$

So $\int x^n \cosh x\,dx = x^n \sinh x - \int nx^{n-1} \sinh x\,dx$

Integrating by parts again with $u = x^{n-1}$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = \cosh x$

So $I_n = x^n \sinh x - n\left\{x^{n-1} \cosh x - \int (n-1)x^{n-2} \cosh x\,dx\right\}$

$= x^n \sinh x - nx^{n-1} \cosh x + n(n-1)I_{n-2}, \quad n \geq 2 \quad *$

b $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2,$ ← Substituting $n = 4$ in $*$

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x + 2I_0\}$

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x\} + 24\int \cosh x\,dx$

$= x^4 \sinh x - 4x^3 \cosh x + 12\{x^2 \sinh x - 2x \cosh x\} + 24 \sinh x + C$

$= (x^4 + 12x^2 + 24)\sinh x - (4x^3 + 24x)\cosh x + C$

c $\int_0^1 x^3 \cosh x\,dx = \left[x^3 \sinh x - 3x^2 \cosh x\right]_0^1 + 6\int_0^1 x \cosh x\,dx$ Using a

$= \{\sinh 1 - 3\cosh 1\} + 6\left\{\left[x \sinh x\right]_0^1 - \int_0^1 1 \sinh x\,dx\right\}$ Integrating by parts

$= \{\sinh 1 - 3\cosh 1\} + 6\{\sinh 1 - [\cosh 1 - 1]\}$

$= 7\sinh 1 - 9\cosh 1 + 6$

$= 7\left(\frac{e^1 - e^{-1}}{2}\right) - 9\left(\frac{e^1 + e^{-1}}{2}\right) + 6$

$= 6 - e - 8e^{-1}$ or $\frac{6e - e^2 - 8}{e}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 16

Question:

Given that $I_n = \int \frac{\sin nx}{\sin x} \, dx, n > 0$,

a write down a similar expression for I_{n-2}, and hence show that

$$I_n - I_{n-2} = \frac{2\sin(n-1)x}{n-1}.$$

b Find

i $\int \frac{\sin 4x}{\sin x} \, dx$

ii the exact value of $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x} \, dx$.

Solution:

a $I_{n-2} = \int \frac{\sin(n-2)x}{\sin x}\,dx$

So $I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x}\,dx$

$$= \int \frac{2\cos\left\{\frac{n+(n-2)}{2}\right\}x\sin\left\{\frac{n-(n-2)}{2}\right\}x}{\sin x}\,dx$$

Using Edexcel formula booklet

$$= \int \frac{2\cos(n-1)x\sin x}{\sin x}\,dx$$

$$= \int 2\cos(n-1)x\,dx$$

$$= \frac{2\sin(n-1)x}{n-1}, n \geq 2$$

It is not necessary to have $+C$.

b i $\int \frac{\sin 4x}{\sin x}\,dx = I_4$

Using a with $n = 4$: $I_4 = I_2 + \frac{2\sin 3x}{3}$

$I_2 = \int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx$

$$= \int 2\cos x\,dx + \frac{2\sin 3x}{3}$$

$$= 2\sin x + \frac{2\sin 3x}{3} + C$$

ii Using a with $n = 5$: $I_5 = I_3 + \frac{2\sin 4x}{4}$

$$= \left\{I_1 + \frac{2\sin 2x}{2}\right\} + \frac{2\sin 4x}{4}$$

$$= \int 1\,dx + \sin 2x + \frac{\sin 4x}{2}$$

$$= x + \sin 2x + \frac{\sin 4x}{2}$$

It follows that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x}\,dx = \left[x + \sin 2x + \frac{\sin 4x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$$= \left[\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right] - \left[\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right]$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi - 3\sqrt{3}}{6}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 17

Question:

Given that $I_n = \int \sinh^n x \, dx, n \in N$,

a derive the reduction formula $nI_n = \sinh^{n-1} x \cosh x - (n-1) I_{n-2}, n \geq 2$.

b Hence

i evaluate $\int_0^{\ln 3} \sinh^5 x \, dx$,

ii show that $\int_0^{\text{arsinh} 1} \sinh^4 x \, dx = \frac{1}{8}\{3 \ln(1+\sqrt{2}) - \sqrt{2}\}$.

Solution:

a $I_n = \int \sinh^n x\,dx = \int \sinh^{n-1} x \sinh x\,dx$

Integrating by parts with $u = \sinh^{n-1} x$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)\sinh^{n-2} x \cosh x, \quad v = \cosh x$

So $I_n = \int \sinh^n x\,dx = \sinh^{n-1} x \cosh x - \int (n-1)\sinh^{n-2} x \cosh^2 x\,dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\left(1+\sinh^2 x\right)dx$

$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x\,dx - (n-1)\int \sinh^n x\,dx$

Giving $I_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \sinh^{n-1} x \cosh x - (n-1)I, \quad n \geq 2 \qquad *$

b i $I_5 = \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{5}I_3$ ← using * with $n = 5$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{5}\left\{\frac{1}{3}\sinh^2 x \cosh x - \frac{2}{3}I_1\right\}$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{15}\sinh^2 x \cosh x + \frac{8}{15}\int \sinh x\,dx$

$= \frac{1}{5}\sinh^4 x \cosh x - \frac{4}{15}\sinh^2 x \cosh x + \frac{8}{15}\cosh x + C$

When $x = \ln 3$, $\sinh x = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$, $\cosh x = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

When $x = 0$, $\sinh x = 0$, $\cosh x = 1$

Applying the limits 0 and ln 3 to the result in **b**

$$\int_0^{\ln 3} \sinh^5 x\,dx = \left[\frac{1}{5}\left(\frac{4}{3}\right)^4\left(\frac{5}{3}\right) - \frac{4}{15}\left(\frac{4}{3}\right)^2\left(\frac{5}{3}\right) + \frac{8}{15}\left(\frac{5}{3}\right)\right] - \left[0 + 0 + \frac{8}{15}\right]$$

$$= \frac{752}{1215} = 0.619 \text{ (3 s.f.)}$$

ii $\int \sinh^4 x\,dx = I_4 = \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4}I_2$ ← Using * with $n = 4$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{4}\left\{\frac{1}{2}\sinh x \cosh x - \frac{1}{2}I_0\right\}$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}\int 1\,dx$

$= \frac{1}{4}\sinh^3 x \cosh x - \frac{3}{8}\sinh x \cosh x + \frac{3}{8}x + C$

When $x = \text{arsinh}1 \quad \sinh x = 1, \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{2}$

When $x = 0 \quad \sinh x = 0 \quad \cosh x = 1$

Applying the limits 0 and arsinh 1 gives

$$\int_0^{\text{arsinh}1} \sinh^4 x \, dx = \frac{1}{4}(1)^3(\sqrt{2}) - \frac{3}{8}(1)(\sqrt{2}) + \frac{3}{8}\text{arsinh}1$$

$$= \frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{8} + \frac{3}{8}\ln\left(1+\sqrt{1^2+1}\right)$$

$$= -\frac{\sqrt{2}}{8} + \frac{3}{8}\ln\left(1+\sqrt{2}\right)$$

$$= \frac{1}{8}\left\{3\ln\left(1+\sqrt{2}\right) - \sqrt{2}\right\}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 1

Question:

Find the length of the arc of the curve with equation $y=\frac{1}{3}x^{\frac{3}{2}}$, from the origin to the point with x-coordinate 12.

Solution:

$y=\frac{1}{3}x^{\frac{3}{2}}$, so $\frac{dy}{dx}=\frac{1}{2}x^{\frac{1}{2}}$

$$\text{Arc length } = \int_0^{12}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{12}\sqrt{1+\frac{x}{4}}\,dx$$

$$=\frac{1}{2}\int_0^{12}\sqrt{4+x}\,dx$$

$$=\frac{1}{2}\left[\frac{2}{3}(4+x)^{\frac{3}{2}}\right]_0^{12}$$

$$=\frac{1}{3}\left[16^{\frac{3}{2}}-4^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}[64-8]$$

$$=\frac{56}{3} \text{ or } 18\frac{2}{3}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 2

Question:

The curve C has equation $y = \ln \cos x$. Find the length of the arc of C between the points with x-coordinates 0 and $\frac{\pi}{3}$.

Solution:

$y = \ln \cos x$, so $\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x$

$$\text{Arc length} = \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\frac{\pi}{3}} \sec x\, dx$$

$$= \left[\ln\left(\sec x + \tan x\right)\right]_0^{\frac{\pi}{3}}$$

$$= \ln\left(2 + \sqrt{3}\right)$$

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Integration
Exercise G, Question 3

Question:

Find the length of the arc on the catenary, with equation $y = 2\cosh\left(\frac{x}{2}\right)$, between the points with x-coordinates 0 and $\ln 4$.

Solution:

$y = 2\cosh\left(\frac{x}{2}\right)$, so $\frac{dy}{dx} = \sinh\left(\frac{x}{2}\right)$

$$\text{arc length} = \int_0^{\ln 4} \sqrt{1+\sinh^2\left(\frac{x}{2}\right)}\,dx$$

$$= \int_0^{\ln 4} \cosh\left(\frac{x}{2}\right)dx$$

$$= \left[2\sinh\left(\frac{x}{2}\right)\right]_0^{\ln 4}$$

$$= 2\frac{e^{\frac{\ln 4}{2}} - e^{-\frac{\ln 4}{2}}}{2}$$

$$= e^{\ln 2} - e^{-\ln 2}$$

As $\ln 4 = \ln 2^2 = 2\ln 2$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

As $e^{\ln k} = k$; $e^{-\ln k} = e^{\ln k^{-1}} = k^{-1}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 4

Question:

Find the length of the arc of the curve with equation $y^2 = \frac{4}{9}x^3$, from the origin to the point $(3, 2\sqrt{3})$.

Solution:

$y^2 = \frac{4}{9}x^3$, so $2y\frac{dy}{dx} = \frac{4}{3}x^2 \Rightarrow \frac{dy}{dx} = \frac{2x^2}{3y} = \pm\frac{x^2}{x^{\frac{3}{2}}} = \pm\sqrt{x}$

The arc in question is above the x-axis.

$$\text{arc length} = \int_0^3 \sqrt{1+x}\,dx$$
$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3$$
$$= \frac{2}{3}[8-1] = 4\frac{2}{3}$$

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Integration
Exercise G, Question 5

Question:

The curve C has equation $y = \frac{1}{2}\sinh^2 2x$. Find the length of the arc on C from the origin to the point whose x-coordinate is 1, giving your answer to 3 significant figures.

Solution:

$y = \frac{1}{2}\sinh^2 2x$, so $\frac{dy}{dx} = 2\sinh 2x \cosh 2x = \sinh 4x$

So arc length $= \int_0^1 \sqrt{1 + \sinh^2 4x}\,dx$

$= \int_0^1 \cosh 4x\,dx$

$= \frac{1}{4}[\sinh 4x]_0^1$

$= \frac{1}{4}\sinh 4 = 6.82 \quad (3\text{ s.f.})$

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Integration
Exercise G, Question 6

Question:

The curve C has equation $y = \frac{1}{4}(2x^2 - \ln x)$, $x > 0$. The points A and B on C have x-coordinates 1 and 2 respectively. Show that the length of the arc from A to B is $\frac{1}{4}(6 + \ln 2)$.

Solution:

$y = \frac{1}{4}\left(2x^2 - \ln x\right)$, so $\frac{dy}{dx} = x - \frac{1}{4x}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$\begin{aligned}\text{So arc length } &= \int_1^2 \left(x + \frac{1}{4x}\right) dx \\ &= \left[\frac{x^2}{2} + \frac{1}{4}\ln x\right]_1^2 \\ &= \left[2 + \frac{1}{4}\ln 2\right] - \left[\frac{1}{2}\right] \\ &= \frac{1}{4}(6 + \ln 2)\end{aligned}$$

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Integration
Exercise G, Question 7

Question:

Find the length of the arc on the curve $y = 2\text{arcosh}\left(\frac{x}{2}\right)$, from the point at which the curve crosses the x-axis to the point with x-coordinate $\frac{5}{2}$. Compare your answer with that in Example 25 and explain the relationship.

Solution:

$y = 2\text{arcosh}\left(\frac{x}{2}\right)$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \times \frac{1}{2}\frac{1}{\sqrt{\left(\frac{x}{2}\right)^2 - 1}} = \frac{2}{\sqrt{x^2 - 4}}$

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{4}{x^2 - 4} = \frac{x^2}{x^2 - 4}$$

The curve crosses the x-axis at $x = 2$,

So arc length $= \int_2^{\frac{5}{2}} x\left(x^2 - 4\right)^{-\frac{1}{2}} \, \mathrm{d}x$

$= \left[\sqrt{x^2 - 4}\right]_2^{2.5}$

$= 1.5$

Eliminating t from the two equations in Example 25, you find that the Cartesian equation is $\frac{x}{2} = \cosh\left(\frac{y}{2}\right)$. For $t \geq 1$, the curve is $y = 2\text{arcosh}\left(\frac{x}{2}\right)$. The limits in both questions correspond, and so they are essentially the same question.

[For $0 < t < 1$, the reflection of $y = 2\text{arcosh}\left(\frac{x}{2}\right)$ in the x-axis is generated.]

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Integration
Exercise G, Question 8

Question:

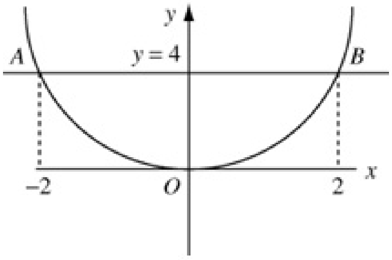
The line $y=4$ intersects the parabola with equation $y=x^2$ at the points A and B. Find the length of the arc of the parabola from A to B.

Solution:

The line $y=4$ intersects the parabola with equation $y=x^2$ where $x=-2$ and $x=+2$.

Using symmetry arc length $= 2\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\int_0^2 \sqrt{1+4x^2}\,dx$$



Using the substitution $2x = \sinh u$, so that $2dx = \cosh u\,du$,

$$\text{arc length} = \int_0^{\text{arsinh}4} \sqrt{1+\sinh^2 u}\cosh u\,du$$

$$= \int_0^{\text{arsinh}4} \cosh^2 u\,du$$

$$= \int_0^{\text{arsinh}4} \frac{(1+\cosh 2u)}{2}du$$

$$= \frac{1}{2}\left[u+\frac{1}{2}\sinh 2u\right]_0^{\text{arsinh}4}$$

$$= \frac{1}{2}\left[u+\sinh u\cosh u\right]_0^{\text{arsinh}4}$$

$$= \frac{1}{2}\text{arsinh}4+\frac{1}{2}\left(4\sqrt{1+16}\right)$$ Using $\cosh u = \sqrt{1+\sinh^2 u}$ and $\sinh u = 4$

$$= \frac{1}{2}\text{arsinh}4+2\sqrt{17}$$

$$= \frac{1}{2}\ln\left(4+\sqrt{17}\right)+2\sqrt{17}$$ Using $\text{arsinh}x = \ln\left\{x+\sqrt{(1+x^2)}\right\}$

$$= 9.29 \text{ (3 s.f.)}$$

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Integration
Exercise G, Question 9

Question:

The circle C has parametric equations $x = r\cos\theta, y = r\sin\theta$. Use the formula for arc length on page 79 for to show that the length of the circumference is $2\pi r$.

Solution:

As $x = r\cos\theta, y = r\sin\theta, \frac{dx}{d\theta} = -r\sin\theta, \frac{dy}{d\theta} = r\cos\theta$

So $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$

The circumference of the circle $= 4\int_0^{\frac{\pi}{2}} r\,d\theta$ Using symmetry.

$= 4r[\theta]_0^{\frac{\pi}{2}}$

$= 2\pi r$

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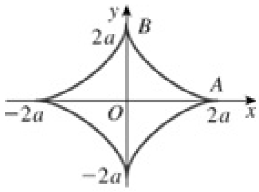
Integration
Exercise G, Question 10

Question:

The diagram shows the astroid, with parametric equations
$x = 2a\cos^3 t, y = 2a\sin^3 t, 0 \le t < 2\pi$.
Find the length of the arc of the curve AB, and hence find the total length of the curve.



Solution:

$x = 2a\cos^3 t, y = 2a\sin^3 t$, so $\frac{dx}{dt} = -6a\cos^2 t\sin t, \frac{dy}{dt} = 6a\sin^2 t\cos t$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t) = 36a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)$$
$$= 36a^2\sin^2 t\cos^2 t$$

At A, $t = 0$, at B, $t = \frac{\pi}{2}$,

so arc length $AB = \int_0^{\frac{\pi}{2}} 6a\sin t\cos t\,dt$

$$= 3a\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$
$$= \frac{3}{2}a\left[-\cos 2t\right]_0^{\frac{\pi}{2}}$$
$$= \frac{3}{2}a\left[1-(-1)\right]$$
$$= 3a$$

Total length of curve $= 4\times 3a = 12a$ (symmetry)

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Integration
Exercise G, Question 11

Question:

Calculate the length of the arc on the curve with parametric equations $x = \tanh u$, $y = \text{sech}\, u$, between the points with parameters $u = 0$ and $u = 1$.

Solution:

$x = \tanh u, y = \text{sech}\, u$, so $\frac{dx}{du} = \text{sech}^2 u, \frac{dy}{du} = -\text{sech}\, u \tanh u$,

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = \text{sech}^4 u + \text{sech}^2 u \tanh^2 u = \text{sech}^2 u\left(\text{sech}^2 u + \tanh^2 u\right) = \text{sech}^2 u$$

So arc length $= \int_0^1 \text{sech}\, u\, du$ ← See Example 7.

$$= \int_0^1 \frac{2}{e^u + e^{-u}}\, du$$

$$= \int_0^1 \frac{2e^u}{\left(e^u\right)^2 + 1}\, du$$

$$= 2\left[\arctan\left(e^u\right)\right]_0^1$$

$$= 2\arctan(e) - \frac{\pi}{2} \quad \text{or } 0.866 \quad (3 \text{ s.f.})$$

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Integration
Exercise G, Question 12

Question:

The cycloid has parametric equations $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$. Find the length of the arc from $\theta = 0$ to $\theta = \pi$.

Solution:

As $x = a(\theta + \sin\theta), y = a(1 - \cos\theta), \frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$$
$$= a^2(2 + 2\cos\theta)$$
$$= 4a^2\cos^2\left(\frac{\theta}{2}\right)$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = \left(\frac{\theta}{2}\right)$

So arc length $= 2a\int_0^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta$
$$= 4a\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi}$$
$$= 4a$$

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Integration
Exercise G, Question 13

Question:

Show that the length of the arc, between the points with parameters $t = 0$ and $t = \frac{\pi}{3}$ on the curve defined by the equations $x = t + \sin t, y = 1 - \cos t$, is 2.

Solution:

$x = t + \sin t, y = 1 - \cos t$

$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = \sin t$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left\{\left(1 + 2\cos t + \cos^2 t\right) + \left(\sin^2 t\right)\right\}$

$= 2\left(1 + \cos t\right) = 4\cos^2\left(\frac{t}{2}\right)$

Using $s = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

arc length $= \int_0^{\frac{\pi}{3}} \sqrt{4\cos^2\left(\frac{t}{2}\right)}\, dt$

$= 2\int_0^{\frac{\pi}{3}} \cos\left(\frac{t}{2}\right) dt$

$= 4\left[\sin\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}$

$= 2$

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Integration
Exercise G, Question 14

Question:

Find the length of the arc of the curve given by the equations $x = e^t \cos t, y = e^t \sin t$, between the points with parameters $t = 0$ and $t = \frac{\pi}{4}$.

Solution:

$x = e^t \cos t, y = e^t \sin t$

$\frac{dx}{dt} = e^t(\cos t - \sin t), \frac{dy}{dt} = e^t(\sin t + \cos t)$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t)^2\left\{(\cos^2 t - 2\sin t\cos t + \sin^2 t) + (\sin^2 t + 2\sin t\cos t + \cos^2 t)\right\},$

$= 2(e^t)^2(\sin^2 t + \cos^2 t)$

$= 2(e^t)^2$

$$\text{arc length} = \int_0^{\frac{\pi}{4}} \sqrt{2(e^t)^2}\, dt$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} e^t\, dt$$

$$= \sqrt{2}\left[e^t\right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}\left[e^{\frac{\pi}{4}} - 1\right] \text{ or } 1.69 \quad (3 \text{ s.f.})$$

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Integration
Exercise G, Question 15

Question:

a Denoting the length of one complete wave of the sine curve with equation $y = \sqrt{3}\sin x$ by L, show that $L = 4\int_0^{\frac{\pi}{2}} \sqrt{1+3\cos^2 x}\,dx$.

b The ellipse has parametric equations $x = \cos t, y = 2\sin t$. Show that the length of its circumference is equal to that of the wave in **a**.

Solution:

a $y = \sqrt{3}\sin x$, so $\frac{dy}{dx} = \sqrt{3}\cos x$

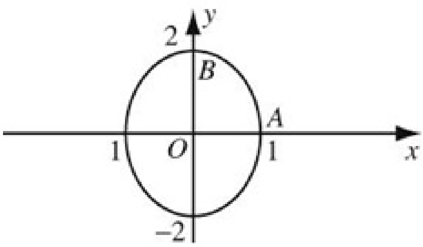
Using the symmetry of the sine curve $s = 4\int_0^{\frac{\pi}{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{1+3\cos^2 x}\,dx$$

b $x = \cos t, y = 2\sin t$

$\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = 2\cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + 4\cos^2 t$$
$$= 1 - \cos^2 t + 4\cos^2 t$$
$$= 1 + 3\cos^2 t$$



From the diagram, at A, $t = 0$,

at B, $t = \frac{\pi}{2}$,

so using the symmetry of the ellipse, the length of the circumference is

$4\int_0^{\frac{\pi}{2}} \sqrt{1+3\cos^2 t}\,dt$, equal to that of the sine curve in **a**

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Integration
Exercise H, Question 1

Question:

a The section of the line $y=\frac{3}{4}x$ between points with x-coordinates 4 and 8 is rotated completely about the x-axis. Use integration to find the area of the surface generated.

b The same section of line is rotated completely about the y-axis. Show that the area of the surface generated is 60π.

Solution:

a $y=\frac{3}{4}x \Rightarrow \frac{dy}{dx}=\frac{3}{4} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2=\frac{25}{16}$

Surface area $=\int_4^8 2\pi\left(\frac{3}{4}x\right)\left(\frac{5}{4}\right)dx$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$=\frac{15}{8}\pi\int_4^8 x\,dx$

$=\frac{15}{8}\pi\left[\frac{x^2}{2}\right]_4^8=45\pi$

b Rotating about the y-axis:

From the work in **a** $1+\left(\frac{dx}{dy}\right)^2=1+\frac{16}{9}=\frac{25}{9}$

As integration is w.r.t. y, the integrand must be in terms of y
The limits for y are 3 (when $x=4$) and 6 (when $x=8$),

so area of surface is $\int_3^6 2\pi\left(\frac{4}{3}y\right)\left(\frac{5}{3}\right)dy$,

$=\frac{40}{9}\pi\left[\frac{y^2}{2}\right]_3^6$

$=\frac{40\times 27}{9\times 2}\pi=60\pi$

Although it is quicker to use $\int_4^8 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, here $\int_{y_1}^{y_2} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ is used to give an example of its use.

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Integration
Exercise H, Question 2

Question:

The arc of the curve $y = x^3$, between the origin and the point (1, 1), is rotated through 4 right-angles about the x-axis. Find the area of the surface generated.

Solution:

$y = x^3$ so $\frac{dy}{dx} = 3x^2$

Using $\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx$

$$= \frac{2\pi}{36} \int_0^1 36x^3 \sqrt{1 + 9x^4}\, dx$$

$$= \frac{2\pi}{36} \left[\frac{2}{3} (1 + 9x^4)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{\pi}{27} \left[10\sqrt{10} - 1 \right] \quad (3.56, 3 \text{ s.f.})$$

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Integration
Exercise H, Question 3

Question:

The arc of the curve $y = \frac{1}{2}x^2$, between the origin and the point (2, 2), is rotated through 4 right-angles about the y-axis. Find the area of the surface generated.

Solution:

$y = \frac{1}{2}x^2$, so $\frac{dy}{dx} = x$

Using $\int_{x_1}^{x_2} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$,

the area of the surface is $\int_0^2 2\pi x\sqrt{1+x^2}\ dx$

$$= \pi\int_0^2 2x\sqrt{1+x^2}\ dx$$

$$= \pi\left[\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{2\pi}{3}\left[5\sqrt{5}-1\right]$$

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Integration
Exercise H, Question 4

Question:

The points A and B, in the first quadrant, on the curve $y^2 = 16x$ have x-coordinates 5 and 12 respectively. Find, in terms π, the area of the surface generated when the arc AB is rotated completely about the x-axis.

Solution:

$y^2 = 16x$ so $2y\dfrac{dy}{dx} = 16 \Rightarrow \dfrac{dy}{dx} = \dfrac{8}{y}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{64}{y^2} = 1 + \frac{4}{x} = \frac{x+4}{x}$$

Using $\displaystyle\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\displaystyle\int_5^{12} 2\pi\, 4\sqrt{x}\sqrt{\frac{4+x}{x}}\, dx$

$$= 8\pi\int_5^{12} \sqrt{4+x}\, dx$$

$$= 8\pi\left[\frac{2}{3}(4+x)^{\frac{3}{2}}\right]_5^{12}$$

$$= \frac{16\pi}{3}[37]$$

$$= \frac{592\pi}{x}$$

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Integration
Exercise H, Question 5

Question:

The curve C has equation $y = \cosh x$. The arc s on C, has end points (0, 1) and (1, cosh 1).

a Find the area of the surface generated when s is rotated completely about the x-axis.

b Show that the area of the surface generated when s is rotated completely about the y-axis is $2\pi\left(\frac{e-1}{e}\right)$.

Solution:

$y = \cosh x$, so $\frac{dy}{dx} = \sinh x$

$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$

a Using $\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi \cosh^2 x \, dx$

$$= \pi \int_0^1 (\cosh 2x + 1) dx$$
$$= \pi \left[\frac{\sinh 2x}{2} + x\right]_0^1$$
$$= \pi [\sinh x \cosh x + x]_0^1$$
$$= \pi [\sinh 1 \cosh 1 + 1]$$
$$= 8.84 \text{ (3 s.f.)}$$

b Using $\int_{x_1}^{x_2} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi x \cosh x \, dx$

$$= 2\pi \left\{ [x \sinh x]_0^1 - \int_0^1 \sinh x \, dx \right\}$$

Using integration by parts

$$= 2\pi \left\{ \sinh 1 - [\cosh x]_0^1 \right\}$$
$$= 2\pi \{ \sinh 1 - \cosh 1 + 1 \}$$
$$= 2\pi \left\{ \frac{1}{2}\left(e - \frac{1}{e} - e - \frac{1}{e} \right) + 1 \right\}$$
$$= 2\pi \left(1 - \frac{1}{e} \right)$$
$$= 2\pi \left(\frac{e - 1}{e} \right)$$

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Integration
Exercise H, Question 6

Question:

The curve C has equation $y = \frac{1}{2x} + \frac{x^3}{6}$.

a Show that $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$.

The arc of the curve between points with x-coordinates 1 and 3 is rotated completely about the x-axis.

b Find the area of the surface generated.

Solution:

a $y = \frac{1}{2x} + \frac{x^3}{6}$, so $\frac{dy}{dx} = -\frac{1}{2x^2} + \frac{x^2}{2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2$$

So $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$

b Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\pi \int_1^3 \left(\frac{1}{2x} + \frac{x^3}{6}\right)\left(x^2 + \frac{1}{x^2}\right) dx$

$$= \pi \int_1^3 \left(\frac{2x}{3} + \frac{x^5}{6} + \frac{1}{2x^3}\right) dx$$

$$= \pi \left[\frac{x^2}{3} + \frac{x^6}{36} - \frac{1}{4x^2}\right]_1^3$$

$$= 23\frac{1}{9}\pi = 72.6 \text{ (3 s.f.)}$$

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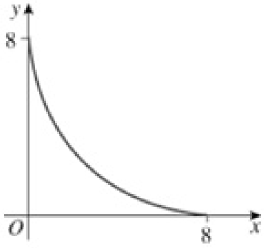
Integration
Exercise H, Question 7

Question:

The diagram shows part of the curve with equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$. Find the area of the surface generated when this arc is rotated completely about the y-axis.



Solution:

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, so $\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

So $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}} = \frac{4}{x^{\frac{2}{3}}}$

Using $\int_{x_1}^{x_2} 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $2\pi \int_0^8 x\left(\frac{2}{x^{\frac{1}{3}}}\right) dx$

$$= 2\pi \int_0^8 2x^{\frac{2}{3}}\, dx$$

$$= 2\pi \left[\frac{6}{5}x^{\frac{5}{3}}\right]_0^8$$

$$= \frac{12\pi}{5}[32]$$

$$= \frac{384\pi}{5} = 241\ (3\text{ s.f.})$$

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Integration
Exercise H, Question 8

Question:

a The arc of the circle with equation $x^2+y^2=R^2$, between the points $(-R,0)$ and $(R,0)$, is rotated through 2π radians about the x-axis. Use integration to find the surface area of the sphere S formed.

b The axis of a cylinder C of radius R is the x-axis. Show that the areas of the surface of S and C, contained between planes with equations $x=a$ and $x=b$, where $a<b<R$, are equal.

Solution:

a $x^2+y^2=R^2$, so $2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{y}$

So $1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{y^2}=\frac{x^2+y^2}{y^2}=\frac{R^2}{y^2}$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface of the sphere is $2\pi\int_{-R}^{R} y\left(\frac{R}{y}\right)dx$

$$= 4\pi\int_0^R R\,dx$$

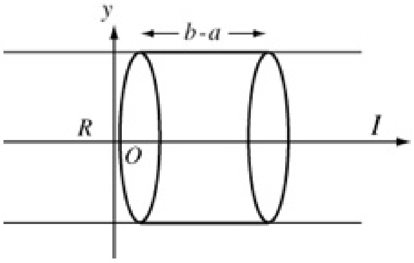
Using the symmetry

$$= 4\pi R[x]_0^R$$
$$= 4\pi R^2$$

b The required area is $2\pi\int_a^b y\left(\frac{R}{y}\right)dx$

see diagram

$$= 2\pi\int_a^b R\,dx$$
$$= 2\pi R(b-a)$$



This is the same area as that of a cylinder of radius R and height $(b-a)$.

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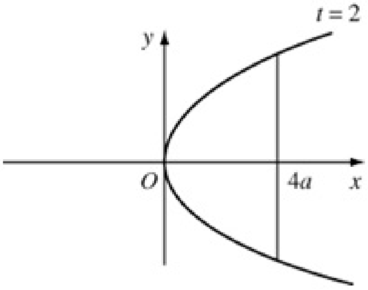
Integration
Exercise H, Question 9

Question:

The finite arc of the parabola with parametric equations $x = at^2, y = 2at$, where a is a positive constant, cut off by the line $x = 4a$, is rotated through $180°$ about the x-axis.
Show that the area of the surface generated is $\frac{8}{3}\pi a^2(5\sqrt{5}-1)$.

Solution:



$x = at^2, y = 2at$, so $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4a^2t^2 + 4a^2 = 4a^2(1+t^2)$

$x = 4a$ when $t = \pm 2$ (See diagram.)

A rotation of π radians gives a surface which would be found by rotating the section $y \geq 0$, i.e. $t = 0$ to $t = 2$ through 2π radians.

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi\int_0^2 4a^2t\sqrt{1+t^2}\, dt$

$$= 8\pi a^2\left[\frac{1}{3}(1+t^2)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{8}{3}\pi a^2\left[5^{\frac{3}{2}} - 1\right]$$

$$= \frac{8}{3}\pi a^2\left(5\sqrt{5}-1\right)$$

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Integration
Exercise H, Question 10

Question:

The arc, in the first quadrant, of the curve with parametric equations $x = \text{sech}\, t, y = \tanh t$, between the points where $t = 0$ and $t = \ln 2$, is rotated completely about the x-axis. Show that the area of the surface generated is $\frac{2\pi}{5}$.

Solution:

$x = \text{sech}\, t, y = \tanh t$, so $\frac{dx}{dt} = -\text{sech}\, t \tanh t, \frac{dy}{dt} = \text{sech}^2 t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \text{sech}^2 t \tanh^2 t + \text{sech}^4 t = \text{sech}^2 t\left(\tanh^2 t + \text{sech}^2 t\right) = \text{sech}^2 t$$

Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_0^{\ln 2} \tanh t \,\text{sech}\, t\, dt$

$$= 2\pi\left[-\text{sech}\, t\right]_0^{\ln 2}$$

$$= 2\pi\left[-\frac{2}{e^t + e^{-t}}\right]_0^{\ln 2}$$

$$= \frac{2\pi}{5}\left[\frac{-2}{2.5} + 1\right]$$

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Integration
Exercise H, Question 11

Question:

The arc of the curve given by $x = 3t^2, y = 2t^3$, from $t = 0$ and $t = 2$, is completely rotated about the y-axis.

a Show that the area of the surface generated can be expressed as $36\pi\int_0^2 t^3\sqrt{1+t^2}\,dt$.

b Using integration by parts, find the exact value of this area.

Solution:

a $x = 3t^2, y = 2t^3$, so $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2\left(t^2+1\right)$

Using $\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$,

the area of the surface is $2\pi\int_0^2 3t^2 \times 6t\sqrt{1+t^2}\,dt$

$= 36\pi\int_0^2 t^3\sqrt{1+t^2}\,dt$

b Let $u = t^2, \frac{dv}{dt} = t\sqrt{1+t^2}$

So $\frac{du}{dt} = 2t, v = \frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}$

$$36\pi\int_0^2 t^2\left(t\sqrt{1+t^2}\right)dt = 36\pi\left\{\left[\frac{1}{3}t^2\left(1+t^2\right)^{\frac{3}{2}}\right]_0^2 - \int_0^2 \frac{2}{3}t\left(1+t^2\right)^{\frac{3}{2}}dt\right\}$$

$$= 12\pi\left[t^2\left(1+t^2\right)^{\frac{3}{2}} - \frac{2}{5}\left(1+t^2\right)^{\frac{5}{2}}\right]_0^2$$

$$= 12\pi\left[4\left(5\sqrt{5}\right) - \frac{2}{5}\left(25\sqrt{5}\right) + \frac{2}{5}\right]$$

$$= 12\pi\left[10\sqrt{5} + \frac{2}{5}\right]$$

$$= \frac{24\pi}{5}\left[25\sqrt{5}+1\right]$$

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Integration
Exercise H, Question 12

Question:

The arc of the curve with parametric equations $x=t^2, y=t-\frac{1}{3}t^3$, between the points where $t=0$ and $t=1$, is rotated through $360°$ about the x-axis. Calculate the area of the surface generated.

Solution:

$x=t^2, y=t-\frac{1}{3}t^3$, so $\frac{dx}{dt}=2t, \frac{dy}{dt}=1-t^2$

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4t^2+1-2t^2+t^4=\left(1+t^2\right)^2$

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$,

the area of the surface is $2\pi\int_0^1\left(t-\frac{1}{3}t^3\right)\left(1+t^2\right)dt$

$=2\pi\int_0^1\left(t+\frac{2}{3}t^3-\frac{1}{3}t^5\right)dt$

$=2\pi\left[\frac{t^2}{2}+\frac{t^4}{6}-\frac{t^6}{18}\right]_0^1$

$=\frac{11\pi}{9}$

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Integration

Exercise H, Question 13

Question:

The astroid C has parametric equations $x = a\cos^3 t, y = a\sin^3 t$, where a is a positive constant. The arc of C, between $t = \frac{\pi}{6}$ and $t = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis. Find the area of the surface of revolution formed.

Solution:

$x = a\cos^3 t, y = a\sin^3 t$, so $\frac{dx}{dt} = -3a\cos^2 t\sin t$, $\frac{dy}{dt} = 3a\sin^2 t\cos t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right)$$

$$= 9a^2\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)$$

$$= 9a^2\sin^2 t\cos^2 t$$

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} a\sin^3 t\left(3a\sin t\cos t\right)dt$

$$= 6\pi a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin^4 t\cos t\, dt$$

$$= 6\pi a^2\left[\frac{1}{5}\sin^5 t\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{6\pi a^2}{5}\left[1 - \frac{1}{32}\right]$$

$$= \frac{93\pi a^2}{80}$$

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Integration
Exercise H, Question 14

Question:

The part of the curve $y = \mathrm{e}^x$, between $(0, 1)$ and $(\ln 2, 2)$, is rotated completely about the x-axis. Show that the area of the surface generated is $\pi(\operatorname{arsinh} 2 - \operatorname{arsinh} 1 + 2\sqrt{5} - \sqrt{2})$.

Solution:

$y = \mathrm{e}^x, \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$

Using $\displaystyle\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$,

the area of the surface is $\displaystyle 2\pi \int_0^{\ln 2} \mathrm{e}^x \sqrt{1 + \mathrm{e}^{2x}}\,\mathrm{d}x$

Make the substitution $\mathrm{e}^x = \sinh u$, so $\mathrm{e}^x \mathrm{d}x = \cosh u \,\mathrm{d}u$

Limits: when $x = \ln 2, u = \operatorname{arsinh} \mathrm{e}^{\ln 2} = \operatorname{arsinh} 2$

when $x = 0, u = \operatorname{arsinh} \mathrm{e}^0 = \operatorname{arsinh} 1$

Then the area of the surface is $\displaystyle 2\pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2} \cosh^2 u \,\mathrm{d}u$

$$= \pi \int_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2} (1 + \cosh 2u)\,\mathrm{d}u$$

$$= \pi \left[u + \frac{\sinh 2u}{2}\right]_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2}$$

$$= \pi \left[u + \sinh u \cosh u\right]_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2}$$

$\boxed{\cosh u = \sqrt{1 + \sinh^2 u}}$

$$= \pi \left[\operatorname{arsinh} 2 + 2\sqrt{5} - \left(\operatorname{arsinh} 1 + (1)\left(\sqrt{2}\right)\right)\right]_{\operatorname{arsinh} 1}^{\operatorname{arsinh} 2}$$

$$= \pi \left(\operatorname{arsinh} 2 - \operatorname{arsinh} 1 + 2\sqrt{5} - \sqrt{2}\right)$$

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Integration
Exercise I, Question 1

Question:

Show that the volume of the solid generated when the finite region enclosed by the curve with equation $y = \tanh x$, the line $x = 1$ and the x-axis is rotated through 2π radians about the x-axis is $\frac{2\pi}{1+e^2}$. **[E]**

Solution:

$$\text{Volume} = \pi\int_0^1 y^2\,dx = \pi\int_0^1 \tanh^2 x\,dx$$
$$= \pi\int_0^1 \left(1-\operatorname{sech}^2 x\right)dx$$
$$= \pi\left[x-\tanh x\right]_0^1$$
$$= \pi\left(1-\tanh 1\right)$$
$$= \pi\left(1-\frac{e^2-1}{e^2+1}\right)$$
$$= \frac{2\pi}{1+e^2}$$

$$\boxed{\tanh 1 = \frac{e-e^{-1}}{e+e^{-1}} = \frac{e^2-1}{e^2+1}}$$

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Integration
Exercise I, Question 2

Question:

$4x^2+4x+17 \equiv (ax+b)^2+c, a>0$.

a Find the values of a, b and c.

b Find the exact value of $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,dx$ **[E]**

Solution:

$4x^2+4x+17 \equiv (ax+b)^2+c, \quad a>0$

a $4x^2+4x+17 \equiv (2x+b)^2+c \quad a=2$

$\equiv 4x^2+4bx+b^2+c$

Comparing coefficient of x: $b=1$

Comparing constant term: $17=1+c \Rightarrow c=16$

b Using **a**, $\int \frac{1}{4x^2+4x+17}\,dx = \int \frac{1}{(2x+1)^2+16}\,dx$

Let $2x+1=4\tan\theta$, then $2dx=4\sec^2\theta d\theta$

and $\int \frac{1}{(2x+1)^2+16}\,dx = \int \frac{2\sec^2\theta}{16\tan^2\theta+16}\,d\theta$

$= \int \frac{2\sec^2\theta}{16\sec^2\theta}\,d\theta$

$= \frac{1}{8}\theta + C$

$= \frac{1}{8}\arctan\left(\frac{2x+1}{4}\right)+C$

So $\int_{-0.5}^{1.5} \frac{1}{4x^2+4x+17}\,dx = \frac{1}{8}[\arctan 1 - \arctan 0]$

$= \frac{\pi}{32}$

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Integration
Exercise I, Question 3

Question:

Find the following.

a $\int \sinh 4x \cosh 6x \, dx$

b $\int \frac{\text{sech } x \tanh x}{1 + 2\text{sech } x} dx$

c $\int e^x \sinh x \, dx$

Solution:

a Using the definitions of $\sinh 4x$ and $\cosh 6x$

$$\int \sinh 4x \cosh 6x \, dx = \int \left(\frac{e^{4x} - e^{-4x}}{2}\right)\left(\frac{e^{6x} + e^{-6x}}{2}\right) dx$$

$$= \frac{1}{4}\int \left(e^{10x} + e^{-2x} - e^{2x} - e^{-10x}\right) dx$$

You could use hyperbolic identities to split up into a difference of two sinhs.

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10} + \frac{e^{-2x}}{-2} - \frac{e^{2x}}{2} - \frac{e^{-10x}}{-10}\right\} + C$$

$$= \frac{1}{4}\left\{\frac{e^{10x}}{10} + \frac{e^{-10x}}{10} - \frac{e^{2x}}{2} - \frac{e^{-2x}}{2}\right\} + C$$

$$= \frac{1}{20}\cosh 10x - \frac{1}{4}\cosh 2x + C$$

as $\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

b $\int \frac{\text{sech}x \tanh x}{1 + 2\text{sech}x} dx = -\frac{1}{2}\int \frac{-2\text{sech}x \tanh x}{1 + 2\text{sech}x} dx = -\frac{1}{2}\ln\left(1 + 2\text{sech}x\right) + C$

c You cannot use by parts for $\int e^x \sinh x dx$

Using the definition of $\sinh x$

$$\int e^x \sinh x dx = \int e^x \left(\frac{e^x - e^{-x}}{2}\right) dx$$

$$= \frac{1}{2}\int \left(e^{2x} - 1\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{2}e^{2x} - x\right) + C$$

$$= \frac{1}{4}e^{2x} - \frac{1}{2}x + C$$

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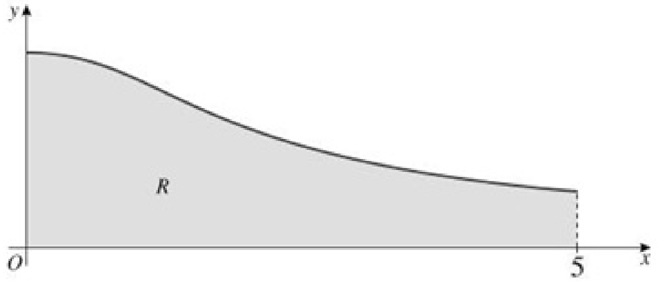
Integration
Exercise I, Question 4

Question:



The diagram shows the cross-section R of an artificial ski slope. The slope is modelled by the curve with equation

$$y = \frac{10}{\sqrt{(4x^2+9)}}, 0 \le x \le 5.$$

Given that 1 unit on each axis represents 10 metres, use integration to calculate the area R. Show your method clearly and give your answer to 2 significant figures. **[E]**

Solution:

$$\text{Area under curve} = \int_0^5 y\,dx = \int_0^5 \frac{10}{\sqrt{4x^2+9}}\,dx$$

$$= 5\int_0^5 \frac{1}{\sqrt{x^2+\frac{9}{4}}}\,dx$$

$$= 5\left[\operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^5$$

$$= 5\operatorname{arsinh}\left(\frac{10}{3}\right)(\text{sq. units})$$

Using $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right)$

'Real' area $= 5\operatorname{arsinh}\left(\frac{10}{3}\right) \times 100 \text{ m}^2 = 960$ (2 s.f.)

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Integration
Exercise I, Question 5

Question:

a Find $\int \frac{1+2x}{1+4x^2}\,dx$.

b Find the exact value of

$\int_0^{0.5} \frac{1+2x}{1+4x^2}\,dx$.

Solution:

a
$$\int \frac{1+2x}{1+4x^2}\,dx = \int \frac{1}{1+4x^2}\,dx + \int \frac{2x}{1+4x^2}\,dx$$
$$= \int \frac{1}{4(\frac{1}{4}+x^2)}\,dx + \frac{1}{4}\int \frac{8x}{1+4x^2}\,dx$$
$$= \frac{1}{2}\arctan 2x + \frac{1}{4}\ln(1+4x^2) + C$$

b
$$\int_0^{0.5} \frac{1+2x}{1+4x^2}\,dx = \frac{1}{2}\arctan 1 + \frac{1}{4}\ln 2$$

Using the result from **a**

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Integration
Exercise I, Question 6

Question:

A rope is hung from points two points on the same horizontal level. The curve formed by the rope is modelled by the equation

$y = 4\cosh\left(\frac{x}{4}\right), -20 \le x \le 20,$

Find the length of the rope, giving your answer to 3 significant figures.

Solution:

$y = 4\cosh\left(\frac{x}{4}\right)$, so $\frac{dy}{dx} = \frac{4}{4}\sinh\left(\frac{x}{4}\right) = \sinh\left(\frac{x}{4}\right)$

$$\text{arc length} = \int_{-20}^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^{20} \sqrt{1 + \sinh^2\left(\frac{x}{4}\right)}\, dx$$

Using the symmetry of the catenary

$$= 2\int_0^{20} \cosh\left(\frac{x}{4}\right) dx$$

$$= 2\left[4\sinh\left(\frac{x}{4}\right)\right]_0^{20}$$

$$= 8\sinh 5$$

$$= 594 \text{ (3 s.f.)}$$

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Integration
Exercise I, Question 7

Question:

Show that $\int_0^{\frac{1}{2}} \text{artanh}x\,dx = \frac{1}{4}\ln\left(\frac{a}{b}\right)$, where a and b are positive integers to be found.

Solution:

Let $u = \text{artanh}x \quad \frac{dv}{dx} = 1$

So $\frac{du}{dx} = \frac{1}{1-x^2} \quad v = x$

Then $\int_0^{\frac{1}{2}} \text{artanh}\ x\,dx = [x\ \text{artanh}\ x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2}\,dx$

$= [x\ \text{artanh}\ x]_0^{\frac{1}{2}} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{-2x}{1-x^2}\,dx$

$= \left[x\ \text{artanh}\ x + \frac{1}{2}\ln(1-x^2)\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\text{artanh}\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$

$= \frac{1}{2}\left\{\frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)\right\} + \frac{1}{2}\ln\left(\frac{3}{4}\right)$ $\quad$ $\boxed{\text{Using } \text{artanh}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$

$= \frac{1}{4}\ln 3 + \frac{1}{2}\ln\left(\frac{3}{4}\right)$

$= \frac{1}{4}\left\{\ln 3 + 2\ln\left(\frac{3}{4}\right)^2\right\}$

$= \frac{1}{4}\left\{\ln 3 + \ln\left(\frac{9}{16}\right)\right\}$

$= \frac{1}{4}\ln\left(\frac{27}{16}\right)^2$ so $a = 27$ and $b = 16$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 8

Question:

Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx$,

a find the values of

i I_0 and

ii I_1,

b show, by using integration by parts twice, that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1) I_{n-2}, n \geq 2$.

c Hence show that $\int_0^{\frac{\pi}{2}} x^3 \cos x \, dx = \frac{1}{8}(\pi^3 - 24\pi + 48)$.

d Evaluate $\int_0^{\frac{\pi}{2}} x^4 \cos x dx$, leaving your answer in terms of π.

Solution:

a **i** $I_0 = \int_0^{\frac{\pi}{2}} \cos x \mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = 1$

ii $I_1 = \int_0^{\frac{\pi}{2}} x \cos x \mathrm{d}x = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, \mathrm{d}xc$ — Using integration by parts

$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + [0 - 1] = \frac{\pi}{2} - 1$

b Integrating by parts with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}, \quad v = \sin x$

So $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, \mathrm{d}x = \left[x^n \sin x \right]_0^{\frac{\pi}{2}} - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, \mathrm{d}x$

$= \left(\frac{\pi}{2} \right)^n - n \int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, \mathrm{d}x \quad *$

Integrating by parts on $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, \mathrm{d}x$ with $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2}, \quad v = -\cos x$

gives $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, \mathrm{d}x = \left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, \mathrm{d}x$

$= (n-1) I_{n-2} \quad \text{as} \quad \left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} = 0$

Substituting in $*$

$I_n = \left(\frac{\pi}{2} \right)^n - n(n-1) I_{n-2}$

c $\int_0^{\frac{\pi}{2}} x^3 \cos x \, \mathrm{d}x = I_3 = \left(\frac{\pi}{2} \right)^3 - 3(2) I_1$

$= \left(\frac{\pi}{2} \right)^3 - 6 \left(\frac{\pi}{2} - 1 \right)$ — Using **a ii**

$= \frac{\pi^3}{8} - 3\pi + 6$

$= \frac{1}{8} (\pi^3 - 24\pi + 48)$

d $\int_0^{\frac{\pi}{2}} x^4 \cos x \, \mathrm{d}x = I_4 = \left(\frac{\pi}{2} \right)^4 - 4(3) I_2$

$= \left(\frac{\pi}{2} \right)^4 - 12 \left\{ \left(\frac{\pi}{2} \right)^2 - 2(1) I_0 \right\}$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$ — as $I_0 = 1$ from **a i**

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 9

Question:

a Find $\int \frac{dx}{\sqrt{x^2 - 2x + 10}}$.

b Find $\int \frac{dx}{x^2 - 2x + 10}$.

c By using the substitution $x = \sin\theta$, show that $\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{(1-x^2)}} = \frac{(4\pi - 7\sqrt{3})}{64}$ **[E]**

Solution:

a $x^2-2x+10=(x-1)^2+9$

So $\int \frac{dx}{\sqrt{x^2-2x+10}}=\int \frac{dx}{\sqrt{(x-1)^2+9}}$

Let $x-1=3\sinh u$, then $dx=3\cosh u\,du$

so $\int \frac{dx}{\sqrt{x^2-2x+10}}=\int \frac{3\cosh u}{3\cosh u}\,du$

$=u+C$

$=\text{arsinh}\left(\frac{x-1}{3}\right)+C$

b $\int \frac{dx}{x^2-2x+10}=\int \frac{dx}{(x-1)^2+9}$

Let $x-1=3\tan\theta$, then $dx=3\sec^2\theta\,d\theta$

so $\int \frac{dx}{x^2-2x+10}=\int \frac{3\sec^2\theta}{9\tan^2\theta+9}\,d\theta$

$=\int \frac{3\sec^2\theta}{9\sec^2\theta}\,d\theta$

$=\frac{1}{3}\theta+C$

$=\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)+C$

c Using the substitution $x=\sin\theta$, so $dx=\cos\theta\,d\theta$

$$\int_0^{\frac{1}{2}} \frac{x^4\,dx}{\sqrt{(1-x^2)}}=\int_0^{\frac{\pi}{6}} \frac{\sin^4\theta\cos\theta\,d\theta}{\cos\theta}$$

$$=\int_0^{\frac{\pi}{6}} \sin^4\theta\,d\theta$$

$\boxed{\sin^4\theta=(\sin^2\theta)^2=\frac{1}{4}(1-\cos2\theta)^2}$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}(1-2\cos2\theta+\cos^2 2\theta)\,d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{6}}\left(1-2\cos2\theta+\frac{1+\cos4\theta}{2}\right)d\theta$$

$$=\frac{1}{4}\left[\frac{3\theta}{2}-\sin2\theta+\frac{\sin4\theta}{8}\right]_0^{\frac{\pi}{6}}$$

$$=\frac{1}{4}\left(\frac{\pi}{4}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{16}\right)$$

$$=\frac{(4\pi-7\sqrt{3})}{64}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 10

Question:

Given that $I_n = \int_0^1 x^n (1-x)^{\frac{1}{3}}\,dx, n \geq 0$,

a show that $I_n = \frac{3n}{3n+4} I_{n-1}, n \geq 1$

b Hence find the exact value of $\int_0^1 (x+1)(1-x)^{\frac{1}{3}}$. **[E]**

Solution:

a Using integration by parts on I_n, with $u = x^n$ and $\frac{dv}{dx} = (1-x)^{\frac{1}{3}}$

so $\frac{du}{dx} = nx^{n-1}$ and $v = -\frac{3}{4}(1-x)^{\frac{4}{3}}$

$$I_n = -\frac{3}{4}\left[x^n (1-x)^{\frac{4}{3}}\right]_0^8 + \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,dx$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,dx$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)(1-x)^{\frac{1}{3}}\,dx$$

$$= 6n\int_0^8 x^{n-1}(1-x)^{\frac{1}{3}}\,dx - \frac{3n}{4}\int_0^8 x^n (1-x)^{\frac{1}{3}}\,dx$$

$$\Rightarrow 4I_n = 6nI_{n-1} - \frac{3n}{4} I_n \Rightarrow I_n = \frac{24n}{3n+4} I_{n-1}$$

b $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,dx = \int_0^1 (1+x^2)(1-x)^{\frac{1}{3}}\,dx = I_0 - I_2$

$I_0 = \int_0^1 (1+x)^{\frac{1}{3}}\,dx = \left[-\frac{3}{4}(1-x)^{\frac{4}{3}}\right]_0^1 = \frac{3}{4}$

Using **a** $I_2 = \frac{3}{5} I_1 = \frac{3}{5}\left(\frac{3}{7} I_0\right)\left(= \frac{27}{140}\right)$

So $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,dx = \frac{3}{4} - \frac{27}{140} = \frac{78}{140} = \frac{39}{70}$

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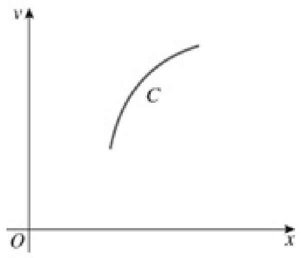
Integration
Exercise I, Question 11

Question:



The curve C has parametric equations
$x = t - \ln t$,
$y = 4\sqrt{t}, 1 \le t \le 4.$
a Show that the length of C is $3 + \ln 4$.
The curve is rotated through 2π radians about the x-axis.
b Find the exact area of the curved surface generated. **[E]**

Solution:

$x = t - \ln t$, so $\frac{dx}{dt} = 1 - \frac{1}{t}$

$y = 4\sqrt{t}$, so $\frac{dy}{dt} = \frac{2}{\sqrt{t}}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t} = 1 + \frac{2}{t} + \frac{1}{t^2} = \left(1 + \frac{1}{t}\right)^2$$

a Arc length $= \int_1^4 \sqrt{\left(1 + \frac{1}{t}\right)^2}\, dt = \int_1^4 \left(1 + \frac{1}{t}\right) dt = [t + \ln t]_1^4 = (4 + \ln 4) - 1 = 3 + \ln 4$

b Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi \int_1^4 4\sqrt{t}\left(1 + \frac{1}{t}\right) dt$

$$= 8\pi \int_1^4 \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) dt$$

$$= 8\pi \left[\frac{2}{3} t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right]_1^4$$

$$= 8\pi \left[\left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{160\pi}{3}$$

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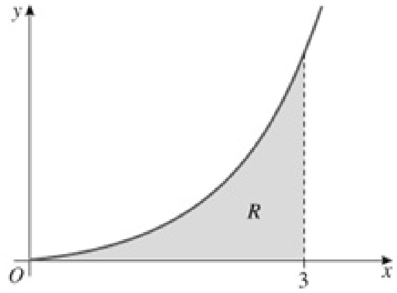
Integration
Exercise I, Question 12

Question:



Above is a sketch of part of the curve with equation
$y = x^2 \text{arsinh} x$.
The region R, shown shaded, is bounded by the curve, the x-axis and the line $x = 3$.
Show that the area of R is
$9 \ln(3 + \sqrt{10}) - \frac{1}{9}(2 + 7\sqrt{10})$. **[E]**

Solution:

$$\text{Area} = \int_0^3 y\,dx = \int_0^3 x^2 \operatorname{arsinh} x\,dx$$

Using integration by parts on I_n, with $u = \operatorname{arsinh} x$ and $\frac{dv}{dx} = x^2$

so $\frac{du}{dx} = \frac{1}{\sqrt{1+x^2}}$ and $v = \frac{x^3}{3}$

$$\int x^2 \operatorname{arsinh} x\,dx = \frac{1}{3}x^3 \operatorname{arsinh} x - \frac{1}{3}\int \frac{x^3}{\sqrt{1+x^2}}\,dx$$

Let $x = \sinh u$ so $dx = \cosh u\,du$

$$\int_0^3 x^2 \operatorname{arsinh} x\,dx = 9\operatorname{arsinh}3 - \frac{1}{3}\int_0^{\operatorname{arsinh}3} \frac{\sinh^3 u}{\cosh u}\cosh u\,du$$

$$= 9\operatorname{arsinh}3 - \frac{1}{3}\int_0^{\operatorname{arsinh}3} \sinh^3 u\,du$$

$$= 9\operatorname{arsinh}3 - \frac{1}{3}\int_0^{\operatorname{arsinh}3} \sinh u\left(\cosh^2 u - 1\right)$$

$$= 9\operatorname{arsinh}3 - \frac{1}{3}\left[\frac{1}{3}\cosh^3 u - \cosh u\right]_0^{\operatorname{arsinh}3}$$

You could use integration by parts with $u = x^2$ and $\frac{du}{dx} = \frac{x}{\sqrt{1+x^2}}$

When $x = 3, \sinh u = 3$ so $\cosh u = \sqrt{1+\sinh^2 u} = \sqrt{10}$

$\operatorname{arsinh} x = \ln\left\{x + \sqrt{1+x^2}\right\}$

$$\text{So } \int_0^3 x^2 \operatorname{arsinh} x\,dx = 9\ln\left\{3+\sqrt{10}\right\} - \frac{1}{3}\left[\frac{10}{3}\sqrt{10} - \sqrt{10} - \left(\frac{1}{3} - 1\right)\right]$$

$$= 9\ln\left\{3+\sqrt{10}\right\} - \frac{1}{9}\left[7\sqrt{10} + 2\right]$$

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Integration
Exercise I, Question 13

Question:

a Use the substitution $u = x^2$ to find $\int_0^1 \frac{x}{1+x^4}\,dx$

b Find

i $\int \frac{1}{\sqrt{4x-x^2}}\,dx$

ii $\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$.

Hence, or otherwise, evaluate

iii $\int_3^4 \frac{5-2x}{\sqrt{4x-x^2}}\,dx$.

Solution:

a Using $x^2 = u$ '$2x\,\mathrm{d}x$' becomes '$\mathrm{d}u$'

So $\int_0^1 \frac{x}{1+x^4}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{\mathrm{d}u}{1+u^2}$

$= \frac{1}{2}[\arctan u]_0^1$

$= \frac{\pi}{8}$

b **i** $4x - x^2 = -(x^2 - 4x) = -\left[(x-2)^2 - 4\right]$

$= 4 - (x-2)^2$

$\int \frac{1}{\sqrt{4x-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4-(x-2)^2}}\,\mathrm{d}x$

$= \arcsin\left(\frac{x-2}{2}\right) + C$

Using $\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

ii $\int \frac{4-2x}{\sqrt{4x-x^2}}\,\mathrm{d}x$

$= 2\left(4x - x^2\right)^{\frac{1}{2}} + C$

Notice that $\frac{\mathrm{d}}{\mathrm{d}x}\left(4x - x^2\right) = 4 - 2x$

iii $\int_3^4 \frac{5-2x}{\sqrt{4x-x^2}}\,\mathrm{d}x = \int_3^4 \left\{\frac{1}{\sqrt{4x-x^2}} + \frac{4-2x}{\sqrt{4x-x^2}}\right\}\mathrm{d}x$

$= \int_3^4 \frac{1}{\sqrt{4x-x^2}}\,\mathrm{d}x + \int_3^4 \frac{4-2x}{\sqrt{4x-x^2}}\,\mathrm{d}x$

$= \left[\arcsin\left(\frac{x-2}{2}\right) + 2\left(4x - x^2\right)^{\frac{1}{2}}\right]_3^4$

Using **i** and **ii**

$= \left(\frac{\pi}{2}\right) - \left(\frac{\pi}{6} + 2\sqrt{3}\right) = \frac{\pi}{3} - 2\sqrt{3}$

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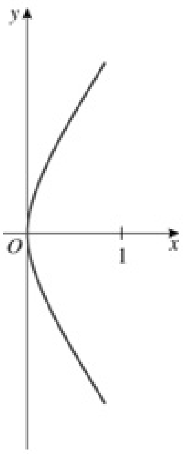
Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 14

Question:

The curve C shown in the diagram has equation $y^2 = 4x, 0 \leq x \leq 1$.
The part of the curve in the first quadrant is rotated through 2π radians about the x-axis.



a Show that the surface area of the solid generated is given by $4\pi \int_0^1 \sqrt{(1+x)}\,dx$.

b Find the exact value of this surface area.

c Show also that the length of the curve C, between the points $(1, -2)$ and $(1, 2)$, is given by $2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\,dx$.

d Use the substitution $x = \sinh^2\theta$ to show that the exact value of this length is $2[\sqrt{2} + \ln(1+\sqrt{2})]$. **[E]**

Solution:

$y = 2\sqrt{x}$ represents the section of curve for $x \geq 0, y \geq 0$, so $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$

a Using $2\pi \int_{x_1}^{x_2} y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\, dx$

$$\text{area of surface} = 2\pi \int_0^1 2\sqrt{x}\sqrt{1+\frac{1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{x}\sqrt{\frac{x+1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{1+x}\, dx$$

b $4\pi \int_0^1 \sqrt{1+x}\, dx = 4\pi \left[\dfrac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1$

$$= \frac{8\pi}{3}\left(2\sqrt{2}-1\right)$$

c Using the symmetry of the parabola, arc length is $2\times$ the length of arc from origin to (1, 2)

$$\text{so arc length} = 2\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}dx$$

d Using $x = \sinh^2\theta, dx = 2\sinh\theta\cosh\theta\, d\theta$

$$2\int\sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\int\sqrt{\left(\frac{\sinh^2\theta+1}{\sinh^2\theta}\right)}2\sinh\theta\cosh\theta\, d\theta$$
$$= 4\int \cosh^2\theta\, d\theta$$
$$= 2\int (1+\cosh 2\theta)\, d\theta$$
$$= 2\left(\theta + \frac{\sinh 2\theta}{2}\right) + C$$
$$= 2(\theta + \sinh\theta\cosh\theta) + C$$
$$= 2\left\{\operatorname{arsinh}\sqrt{x} + \sqrt{x}\sqrt{1+x}\right\} + C$$

$$\text{So arc length} = 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\left(\operatorname{arsinh} 1 + \sqrt{2}\right)$$
$$= 2\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right] \longleftarrow \boxed{\operatorname{arsinh} x = \ln\left\{x+\sqrt{1+x^2}\right\}}$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 15

Question:

a Show that $\int x \operatorname{arcosh} x \, dx = \frac{1}{4}(2x^2 - 1)\operatorname{arcosh} x - \frac{1}{4}x\sqrt{x^2-1} + C$

b Hence, using the substitution $x = u^2$, find $\int \operatorname{arcosh}(\sqrt{x})dx$.

Solution:

a Using integration by parts with $u = \operatorname{arcosh} x$ and $\frac{dv}{dx} = x$,

$\frac{du}{dx} = \frac{1}{\sqrt{x^2-1}}$ and $v = \frac{x^2}{2}$

So $\int x \operatorname{arcosh} x dx = \frac{x^2}{2}\operatorname{arcosh} x - \int \frac{x^2}{2\sqrt{x^2-1}} dx$ *

You could use integration by parts with $u = x$ and $\frac{dy}{dx} = \frac{x}{\sqrt{x^2-1}}$

Substitute $x = \cosh u$ in $\int \frac{x^2}{\sqrt{x^2-1}} dx$ gives

$$\int \frac{x^2}{\sqrt{x^2-1}} dx = \int \frac{\cosh^2 u}{\sinh u}\sinh u \, du$$

$$= \int \cosh^2 u \, du$$

$$= \frac{1}{2}\int (1 + \cosh 2u) du$$

$$= \frac{1}{2}[u + \sinh u \cosh u] + C$$

$$= \frac{1}{2}\left[\operatorname{arcosh} x + x\sqrt{x^2-1}\right] + C$$

So $\int x \operatorname{arcosh} x dx = \frac{x^2}{2}\operatorname{arcosh} x - \frac{1}{4}\left[\operatorname{arcosh} x + x\sqrt{x^2-1}\right] + C$ from *

$$= \frac{1}{4}(2x^2 - 1)\operatorname{arcosh} x - \frac{1}{4}x\sqrt{x^2-1} + C$$

b Let $x = u^2$, so $dx = 2u \, du$,

then $\int \operatorname{arcosh}(\sqrt{x}) dx = 2\int u \operatorname{arcosh} u \, du$

$$= \frac{1}{2}(2u^2 - 1)\operatorname{arcosh} u - \frac{1}{2}u\sqrt{u^2-1} + C$$

Using **a**

$$= \frac{1}{2}(2x-1)\operatorname{arcosh}\sqrt{x} - \frac{1}{2}\sqrt{x}\sqrt{x-1} + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 16

Question:

Given that $I_n = \int \frac{\sin(2n+1)x}{\sin x} \mathrm{d}x$,

a show that $I_n - I_{n-1} = \frac{\sin 2nx}{n}$.

b Hence find I_5.

c Show that, for all positive integers n, $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} \mathrm{d}x$ always has the same value, which should be found.

Solution:

a
$$I_n - I_{n-1} = \int \frac{\left[\sin(2n+1)x - \sin(2n-1)x\right]}{\sin x} \mathrm{d}x$$
$$= \int \frac{2\cos 2nx \sin x}{\sin x} \mathrm{d}x$$
$$\boxed{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$
$$= \int 2\cos 2nx \, \mathrm{d}x$$
$$= \frac{\sin 2nx}{n}$$

b $I_5 - I_4 = \frac{\sin 10x}{5}, I_4 - I_3 = \frac{\sin 8x}{4}, I_3 - I_2 = \frac{\sin 6x}{3}, I_2 - I_1 = \frac{\sin 4x}{2}$

$I_1 - I_0 = \sin 2x$

Adding: $I_5 = \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + I_0$

where $I_0 = \int 1 \, \mathrm{d}x = x + C$

$$= \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + x + C$$

c
$$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} \mathrm{d}x - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} \mathrm{d}x = \left[\frac{\sin 2nx}{n}\right]_0^{\frac{\pi}{2}} = \frac{\sin(n\pi)}{n}$$

So, if n is any a positive integer $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} \mathrm{d}x - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} \mathrm{d}x = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} \mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x} \mathrm{d}x = \ldots = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x} \mathrm{d}x = \frac{\pi}{2}$$

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Integration
Exercise I, Question 17

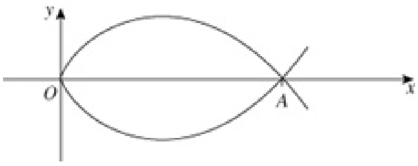
Question:

The diagram shows part of the graph of the curve with equation $y^2 = \frac{1}{3}x(x-1)^2$.

a Show that the length of the loop is $\frac{4\sqrt{3}}{3}$.

The arc OA (in boys) is rotated completely about the x-axis.

b Find the area of the surface generated.



Solution:

a The point A on the curve has coordinates $(1, 0)$.

Using symmetry, the length of the loop is $2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

As $y^2 = \frac{1}{3}x(x-1)^2 = \frac{1}{3}(x^3 - 2x^2 + x)$

$$2y\frac{dy}{dx} = \frac{1}{3}(3x^2 - 4x + 1) = \frac{1}{3}(3x-1)(x-1)$$

So $\frac{dy}{dx} = \frac{\frac{1}{3}(3x-1)(x-1)}{\pm 2\sqrt{\frac{x}{3}}(x-1)} = \pm\frac{1}{2\sqrt{3}}\frac{(3x-1)}{\sqrt{x}}$

and $1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{9x^2-6x+1}{12x} = \frac{9x^2+6x+1}{12x} = \frac{(3x+1)^2}{12x}$

Therefore, arc length $= 2\int_0^1 \frac{3x+1}{2\sqrt{3}\sqrt{x}}\,dx$

$$= \frac{1}{\sqrt{3}}\int_0^1 \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)dx$$

$$= \frac{1}{\sqrt{3}}\left[2x^{\frac{3}{2}} + 2\sqrt{x}\right]_0^1$$

$$= \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

b Using $2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ for area of surface generated about the x-axis

Area of surface $= 2\pi\int_0^1 \frac{1}{\sqrt{3}}\sqrt{x}(1-x)\frac{(3x+1)}{\sqrt{12x}}\,dx$

$$= \frac{\pi}{3}\int_0^1 (1-x)(3x+1)\,dx$$

$$= \frac{\pi}{3}\int_0^1 (1+2x-3x^2)\,dx$$

$$= \frac{\pi}{3}\left[x + x^2 - x^3\right]_0^1$$

$$= \frac{\pi}{3}$$

Note: y is +ve for OA, so you need to take $y = -\frac{\sqrt{x}(x-1)}{\sqrt{3}} = \frac{\sqrt{x}(1-x)}{\sqrt{3}}$

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Integration
Exercise I, Question 18

Question:

a Find $\int \frac{1}{\sinh x + 2\cosh x}\,dx$.

b Show that $\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = 9(\sqrt{2}-1) + 2\text{arsinh}\,1$. **[E]**

Solution:

a Using the exponential forms

$$\int \frac{1}{\sinh x + 2\cosh x}\,dx = \int \frac{1}{\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x + e^{-x}}{2}\right)}\,dx$$
$$= \int \frac{2}{3e^x + e^{-x}}\,dx$$
$$= \int \frac{2e^x}{3e^{2x}+1}\,dx$$

Using the substitution $u = e^x$, then $\frac{du}{dx} = e^x$ so '$e^x\,dx$' can be replaced by 'du',

So $$\int \frac{1}{\sinh x + 2\cosh x}\,dx = \int \frac{2}{3u^2+1}\,du$$
$$= \frac{2}{3}\int \frac{1}{u^2+\frac{1}{3}}\,du$$
$$= \frac{2}{3}\left(\sqrt{3}\right)\arctan\left(\sqrt{3}u\right) + C$$
$$= \frac{2}{\sqrt{3}}\arctan\left(\sqrt{3}e^x\right) + C$$

b $x^2 - 2x + 10 = (x-1)^2 + 9$

So let $x - 1 = 3\sinh u$, then $dx = 3\cosh u\,du$

and $$\int \frac{3x-1}{\sqrt{x^2-2x+10}}\,dx = \int \frac{9\sinh u + 2}{\sqrt{9\sinh^2 u + 9}}\,3\cosh u\,du$$
$$= \int \frac{9\sinh u + 2}{3\cosh u}\,3\cosh u\,du$$
$$= 9\cosh u + 2u + C$$
$$= 9\sqrt{1+\left(\frac{x-1}{3}\right)^2} + 2\text{arsinh}\left(\frac{x-1}{3}\right) + C$$

So $$\int_1^4 \frac{3x-1}{\sqrt{x^2-2x+10}} = \left[9\sqrt{2} + 2\text{arsinh}\,1\right] - [9]$$
$$= 9\left(\sqrt{2}-1\right) + 2\text{arsinh}\,1$$

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Integration
Exercise I, Question 19

Question:

Given that $I_n = \int \sec^n x \, dx$;

a by writing $\sec^n x = \sec^{n-2} x \sec^2 x$, show that, for $n \geq 2$,
$(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$.

b Find I_5.

c Hence show that $\int_0^{\frac{\pi}{4}} \sec^5 x \, dx = \frac{1}{8}\{7\sqrt{2} + 3\ln(1+\sqrt{2})\}$

Solution:

a $\int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$

Let $u = \sec^{n-2} x$ and $\frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = (n-2)\sec^{n-3} x(\sec x \tan x) = (n-2)\sec^{n-2} x \tan x$ and $v = \tan x$

Integrating by parts

$$\int \sec^n x dx = I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x \, dx + (n-2)\int \sec^{n-2} x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$$

So $(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}, n \geq 2, \quad *$

b $\int \sec^5 x \, dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}I_3$ ← Substituting $n = 5$ in $*$

$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}I_1\right)$ ← Substituting $n = 3$ in $*$

But $I_1 = \int \sec x \, dx = \ln|\sec x + \tan x| + C$ ← On Edexcel formula sheet

So $\int \sec^5 x \, dx = I_5 = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

c $\int_0^{\frac{\pi}{4}} \sec^5 x dx = \frac{1}{4}(\sqrt{2})^3 + \frac{3}{8}(\sqrt{2}) + \frac{3}{8}\ln(\sqrt{2}+1)$

$= \frac{1}{8}\{7\sqrt{2} + 3\ln(\sqrt{2}+1)\}$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Integration

Exercise I, Question 20

Question:

a Show by using a suitable substitution for x, that

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$$

b Hence show that the area of the region enclosed by the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

Solution:

a Let $x = a\sin\theta$, then $\frac{\mathrm{d}x}{\mathrm{d}\theta}=a\cos\theta$

$$\text{So } \int \sqrt{a^2-x^2}\,\mathrm{d}x = \int a^2\cos^2\theta\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\,\mathrm{d}\theta$$

$$= \frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$

$$= \frac{a^2}{2}(\theta+\sin\theta\cos\theta)+C$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right)+\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right)+C$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$$

b Area enclosed by the ellipse $=4\times$ area enclosed by arc in first quadrant and the positive coordinate axes (symmetry)

$$=4\int_0^a y\,\mathrm{d}x$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

+ve square root required

$$\text{So area} = 4\frac{b}{a}\left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}\right]_0^a$$

from a

$$=2ab\arcsin 1$$

$$=\pi ab$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 21

Question:

a Show by using a suitable substitution for x, that

$$\int \sqrt{a^2 - x^2}\,dx = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Hence show that the area of the region enclosed by the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

Solution:

a Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$

So $\int \sqrt{a^2 - x^2}\,dx = \int a^2\cos^2\theta\,d\theta$

$$= \frac{a^2}{2}\int (1 + \cos 2\theta)\,d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \left(\frac{x}{a}\right)^2}\right) + C$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

b Area enclosed by the ellipse $= 4\times$ area enclosed by arc in first quadrant (symmetry)

$= 4\int_0^a y\,dx$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$ $\boxed{(+\text{ve square root required})}$

So area $= 4\frac{b}{a}\left[\frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2}\right]_0^a$ $\boxed{\text{from a}}$

$= 2ab\arcsin 1$

$= \pi ab$