Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Find, in the form $y = \mathrm{f}(x)$, the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{4}{x}y = 6x - 5, \quad x > 0.$$

Solution:

The integrating factor is

$$e^{\int \frac{4}{x}\mathrm{d}x} = e^{4\ln x} = e^{\ln x^4} = x^4$$

If the differential equation has the form $\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q$, the integrating factor is $e^{\int P \mathrm{d}x}$.

For any function $\mathrm{f}(x)$, $e^{\ln \mathrm{f}(x)} = \mathrm{f}(x)$.

Multiply the equation throughout by x^4

$$x^4 \frac{\mathrm{d}y}{\mathrm{d}x} + 4x^3 y = 6x^5 - 5x^4$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4 y) = 6x^5 - 5x^4$$

$$x^4 y = \int (6x^5 - 5x^4)\,\mathrm{d}x = x^6 - x^5 + C$$

$$y = x^2 - x + \frac{C}{x^4}$$

It is important that you remember to add the constant of integration. When you divide by x^4, the constant becomes a function of x and its omission would be a significant error.

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Exercise A, Question 2

Question:

Solve the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = x^2, \quad x > 0,$$

giving your answer for y in terms of x.

Solution:

The interating factor is

$$e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$$

For all n, $n \ln x = \ln x^n$, so for $n = -1$, $-\ln x = \ln x^{-1} = \ln \frac{1}{x}$.

Multiply the equation throughout by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = x$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = x$$

The product rule for differentiating, in this case $\frac{d}{dx}\left(y \times \frac{1}{x}\right) = \frac{dy}{dx} \times \frac{1}{x} + y \times -\frac{1}{x^2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to x.

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$y = \frac{x^3}{2} + Cx$$

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Exercise A, Question 3

Question:

Find the general solution of the differential equation

$$(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}, \quad x > 0,$$

giving your answer in the form $y = f(x)$.

Solution:

$$(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}$$

$$\frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x(x+1)}$$

If the equation is in the form $R\frac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(x+1)$, before finding the integrating factor.

The integrating factor is

$$e^{\int \frac{2}{x+1}dx} = e^{2\ln(x+1)} = e^{\ln(x+1)^2} = (x+1)^2$$

Multiply throughout by $(x+1)^2$

$$(x+1)^2\frac{dy}{dx} + 2(x+1)y = \frac{x+1}{x}$$

To integrate $\frac{x+1}{x}$, write $\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$.

$$\frac{d}{dx}\left((x+1)^2 y\right) = 1 + \frac{1}{x}$$

$$(x+1)^2 y = \int\left(1 + \frac{1}{x}\right)dx = x + \ln x + C$$

$$y = \frac{x + \ln x + C}{(x+1)^2}$$

You divide throughout by $(x+1)^2$ to obtain the equation in the form $y = f(x)$. This is required by the wording of the question.

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Exercise A, Question 4

Question:

Obtain the solution of

$$\frac{dy}{dx} + y\tan x = e^{2x}\cos x,\ 0 \leqslant x < \frac{\pi}{2},$$

for which $y = 2$ at $x = 0$, giving your answer in the form $y = f(x)$.

Solution:

The integrating factor is $e^{\int \tan x\,dx}$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\ln\cos x = \ln\frac{1}{\cos x} = \ln\sec x$$

In C4 you learnt that $\int \frac{f'(x)}{f(x)}dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $\int \frac{-\sin x}{\cos x}dx = \ln\cos x$.

Hence

$$e^{\int \tan x\,dx} = e^{\ln\sec x} = \sec x$$

Multiply the differential equation throughout by $\sec x$

$$\sec x\frac{dy}{dx} + y\sec x\tan x = e^{2x}\sec x\cos x = e^{2x}$$

$\sec x\cos x = \frac{1}{\cos x} \times \cos x = 1$

$$\frac{d}{dx}(y\sec x) = e^{2x}$$

$$y\sec x = \int e^{2x}\,dx = \frac{e^{2x}}{2} + C$$

Multiply throughout by $\cos x$

$$y = \left(\frac{e^{2x}}{2} + C\right)\cos x$$

$y = 2$ at $x = 0$

$$2 = \tfrac{1}{2} + C \Rightarrow C = \tfrac{3}{2}$$

The condition $y = 2$ at $x = 0$ enables you to evaluate the constant of integration and find the particular solution of the differential equation for these values.

$$y = \tfrac{1}{2}(e^{2x} + 3)\cos x$$

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Exercise A, Question 5

Question:

Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y \cot 2x = \sin x, \quad 0 < x < \frac{\pi}{2},$$

giving your answer in the form $y = f(x)$.

Solution:

The integrating factor is $e^{\int 2 \cot 2x \, dx}$

$$\int 2 \cot 2x \, dx = \int \frac{2 \cos 2x}{\sin 2x} dx = \ln \sin 2x$$

Hence

$$e^{\int 2 \cot 2x \, dx} = e^{\ln \sin 2x} = \sin 2x$$

Multiply the differential equation throughout by $\sin 2x$

$$\sin 2x \frac{dy}{dx} + 2y \cos 2x = \sin x \sin 2x$$

Using the identity $\sin 2x = 2 \sin x \cos x$.

$$\frac{d}{dx}(y \sin 2x) = 2 \sin^2 x \cos x$$

$$y \sin 2x = \frac{2 \sin^3 x}{3} + C$$

As $\frac{d}{dx}(\sin^3 x) = 3 \sin^2 x \cos x$, then $\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3}$. It saves time to find integrals of this type by inspection. However, you can use the substitution $\sin x = s$ if you find inspection difficult.

$$y = \frac{2 \sin^3 x}{3 \sin 2x} + \frac{C}{\sin 2x}$$

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Exercise A, Question 6

Question:

Solve the differential equation

$$(1+x)\frac{dy}{dx} - xy = xe^{-x},$$

given that $y = 1$ at $x = 0$.

Solution:

$$(1+x)\frac{dy}{dx} - xy = xe^{-x}$$

$$\frac{dy}{dx} - \frac{xy}{1+x} = \frac{xe^{-x}}{1+x} \quad ①$$

If the equation is in the form $R\frac{dy}{dx} + Sy = T$, you must begin by dividing throughout by R, in this case $(1+x)$ before finding the integrating factor.

The integrating factor is $e^{\int -\frac{x}{1+x}dx}$

$$\frac{x}{1+x} = \frac{1+x-1}{1+x} = 1 - \frac{1}{1+x}$$

To integrate an expression in which the degree of the numerator is greater or equal to the degree of the denominator, you must transform the expression into one with a proper fraction. This can be done using partial fractions, long division or, as here, using decomposition.

Hence

$$\int \frac{x}{1+x}dx = x - \ln(1+x)$$

and the integrating factor is

$$e^{-x+\ln(1+x)} = e^{-x}e^{\ln(1+x)} = e^{-x}(1+x)$$

Multiplying ① throughout by $(1+x)e^{-x}$

$$(1+x)e^{-x}\frac{dy}{dx} - xe^{-x}y = xe^{-2x}$$

$$\frac{d}{dx}\left(y(1+x)e^{-x}\right) = xe^{-2x}$$

$$y(1+x)e^{-x} = \int xe^{-2x}dx$$

You integrate xe^{-2x} using integration by parts.

$$= -\frac{xe^{-2x}}{2} + \int \frac{e^{-2x}}{2}dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$$

$$\frac{e^{-2x}}{e^{-x}} = e^{-2x-(-x)} = e^{-2x+x} = e^{-x}$$

$$y = -\frac{xe^{-2x}}{2(1+x)e^{-x}} - \frac{e^{-2x}}{4(1+x)e^{-x}} + \frac{C}{(1+x)e^{-x}}$$

$$= -\frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)} + \frac{Ce^{x}}{(1+x)}$$

$y = 1$ at $x = 0$

$$1 = 0 - \tfrac{1}{4} + C \Rightarrow C = \tfrac{5}{4}$$

$$y = \frac{5e^{x}}{4(1+x)} - \frac{xe^{-x}}{2(1+x)} - \frac{e^{-x}}{4(1+x)}$$

This expression could be put over a common denominator but, other than requiring that y is expressed in terms of x, the question asks for no particular form and this is an acceptable answer.

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Exercise A, Question 7

Question:

a By using the substitution $y = \frac{1}{2}(u - x)$, or otherwise, find the general solution of the differential equation

$$\frac{dy}{dx} = x + 2y.$$

Given that $y = 2$ at $x = 0$,

b express y in terms of x.

Solution:

a $y = \frac{1}{2}u - \frac{1}{2}x$

Differentiate throughout with respect to x.

$$\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx} - \frac{1}{2}$$

$$\frac{dy}{dx} = x + 2y$$

$y = \frac{1}{2}(u - x) \Rightarrow 2y = u - x$

transforms to

$$\frac{1}{2}\frac{du}{dx} - \frac{1}{2} = x + u - x = u$$

$$\frac{du}{dx} - 1 = 2u$$

$$\frac{du}{dx} = 2u + 1$$

This is a separable equation. You learnt how to solve separable equations in C4.

$$\int \frac{1}{2u + 1}\, du = \int 1\, dx$$

Separating the variables.

$$\frac{1}{2}\ln(2u + 1) = x + A$$

$$\ln(2u + 1) = 2x + B$$

Twice one arbitrary constant A is another arbitrary constant, $B = 2A$.

$$e^{\ln(2u+1)} = e^{2x+B} = e^B e^{2x} = C\,e^{2x}$$

e to an arbitrary constant is another arbitrary constant. Here $C = e^B$.

$$2u + 1 = 4y + 2x + 1 = C\,e^{2x}$$

$$y = \frac{C\,e^{2x} - 2x - 1}{4}$$

This is the general solution of the original differential equation.

b $y = 2$ at $x = 0$

$$2 = \frac{C - 1}{4} \Rightarrow 8 = C - 1 \Rightarrow C = 9$$

$$y = \frac{9e^{2x} - 2x - 1}{4}$$

This is the particular solution of the original differential equation for which $y = 2$ at $x = 0$.

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Exercise A, Question 8

Question:

a Find the general solution of the differential equation

$$t\frac{dv}{dt} - v = t, \quad t > 0$$

and hence show that the solution can be written in the form $v = t(\ln t + c)$, where c is an arbitrary constant.

b This differential equation is used to model the motion of a particle which has speed $v\,\text{m s}^{-1}$ at time t seconds. When $t = 2$ the speed of the particle is $3\,\text{m s}^{-1}$. Find, to 3 significant figures, the speed of the particle when $t = 4$.

Solution:

a $t\frac{dv}{dt} - v = t$

Divide throughout by t

$$\frac{dv}{dt} - \frac{v}{t} = 1 \quad ①$$

The integrating factor is

$$e^{\int -\frac{1}{t}dt} = e^{-\ln t} = e^{\ln\frac{1}{t}} = \frac{1}{t}$$

Multiply ① throughout by $\frac{1}{t}$

$$\frac{1}{t}\frac{dv}{dt} - \frac{v}{t^2} = \frac{1}{t}$$

$$\frac{d}{dt}\left(\frac{v}{t}\right) = \frac{1}{t}$$

$$\frac{v}{t} = \int \frac{1}{t}\,dt = \ln t + c$$

$v = t(\ln t + c)$, as required

The product rule for differentiating, in this case $\frac{d}{dt}(v \times t^{-1}) = \frac{dv}{dt} \times t^{-1} + v \times (-1)t^{-2}$, enables you to write the differential equation as an exact equation, where one side is the exact derivative of a product and the other side can be integrated with respect to t.

b $v = 3$ when $t = 2$

$$3 = 2(\ln 2 + c) = 2\ln 2 + 2c \Rightarrow c = 1.5 - \ln 2$$

$$v = t(\ln t + 1.5 - \ln 2)$$

When $t = 4$

$$v = 4(\ln 4 + 1.5 - \ln 2)$$
$$\approx 8.77$$

Use your calculator to evaluate this expression.

The speed of the particle when $t = 4$ is $8.77\,\text{m s}^{-1}$ (3 s.f.).

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Exercise A, Question 9

Question:

a Use the substitution $y = vx$ to transform the equation

$$\frac{dy}{dx} = \frac{(4x + y)(x + y)}{x^2} \quad x > 0, \qquad ①$$

into the equation

$$x\frac{dv}{dx} = (2 + v)^2. \qquad ②$$

b Solve the differential equation ② to find v in terms of x.

c Hence show that

$$y = -2x - \frac{x}{\ln x + c}, \text{ where } c \text{ is an}$$

arbitrary constant, is a general solution of differential equation ①.

Solution:

a $y = vx$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Differentiating vx as a product, $\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v$, as $\frac{d}{dx}(x) = 1$.

Substituting $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into equation ① in the question

$$x\frac{dv}{dx} + v = \frac{(4x + vx)(x + vx)}{x^2}$$

$$= \frac{x^2(4 + v)(1 + v)}{x^2} = (4 + v)(1 + v) = 4 + 5v + v^2$$

$x\frac{dv}{dx} = 4 + 4v + v^2 = (2 + v)^2$, as required.

This is a separable equation and the first step in its solution is to separate the variables, by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side of the equation.

b $\int \frac{1}{(2 + v)^2}dv = \int \frac{1}{x}dx$

$$-\frac{1}{2 + v} = \ln x + c$$

$\int (2 + v)^{-2}dv = \frac{(2 + v)^{-1}}{-1} = -\frac{1}{2 + v}$

$$2 + v = -\frac{1}{\ln x + c}$$

$$v = -2 - \frac{1}{\ln x + c}$$

c $y = vx \Rightarrow v = \frac{y}{x}$

Substituting $v = \frac{y}{x}$ into the answer to part **b**

$$\frac{y}{x} = -2 - \frac{1}{\ln x + c}$$

$y = -2x - \frac{x}{\ln x + c}$, as required.

Multiply throughout by x to obtain the printed answer.

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Exercise A, Question 10

Question:

a Using the substitution $t = x^2$, or otherwise, find

$$\int x^3 \, e^{-x^2} \, dx.$$

b Find the general solution of the differential equation

$$x\frac{dy}{dx} + 3y = xe^{-x^2}.$$

Solution:

a $t = x^2 \Rightarrow \frac{dt}{dx} = 2x \Rightarrow x\frac{dx}{dt} = \frac{1}{2}$

$$\int x^3 e^{-x^2}\,dx = \int x^2 e^{-x^2}\left(x\frac{dx}{dt}\right)dt$$

$$= \int t\,e^{-t}\left(\tfrac{1}{2}\right)dt = \tfrac{1}{2}\int t\,e^{-t}\,dt$$

$$= -\frac{t\,e^{-t}}{2} + \int \frac{e^{-t}}{2}\,dt$$

$$= -\frac{t\,e^{-t}}{2} - \frac{e^{-t}}{2} + C$$

Returning to the original variable

$$\int x^3 e^{-x^2}\,dx = -\frac{x^2 e^{-x^2}}{2} - \frac{e^{-x^2}}{2} + C$$

The first part of this question is integration by substitution and could have been set on a C4 paper. Its purpose here is to help you with part **b**. Realising this helps you to check your work. When you come to the integration in part **b**, it should turn out to be the integration you have already carried out in part **a**. If it was not, you would need to check your work carefully.

b $x\frac{dy}{dx} + 3y = x\,e^{-x^2}$

$$\frac{dy}{dx} + \frac{3}{x}y = e^{-x^2} \quad ①$$

Divide throughout by x.

The integrating factor is

$$e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiply ① throughout by x^3

$$x^3\frac{dy}{dx} + 3x^2 y = x^3 e^{-x^2}$$

$$\frac{d}{dx}(yx^3) = x^3 e^{-x^2}$$

$$yx^3 = \int x^3 e^{-x^2}\,dx$$

$$= -\frac{x^2 e^{-x^2}}{2} - \frac{e^{-x^2}}{2} + C$$

$$y = -\frac{e^{-x^2}}{2x} - \frac{e^{-x^2}}{2x^3} + \frac{C}{x^3}$$

This is an exact equation, where one side is the exact derivative of a product and the other side is the expression you have already integrated in part **a**.

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Exercise A, Question 11

Question:

a Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = \cos^3 x.$$

b Show that, for $0 \leqslant x \leqslant 2\pi$, there are two points on the x-axis through which all the solution curves for this differential equation pass.

c Sketch the graph, $0 \leqslant x \leqslant 2\pi$, of the particular solution for which $y = 0$ at $x = 0$.

Solution:

a Dividing throughout by $\cos x$

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \cos^2 x \quad ①$$

$$\int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln \cos x = \ln \frac{1}{\cos x} = \ln \sec x$$

In C4 you learnt that $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$. As $-\sin x$ is the derivative of $\cos x$, $-\int \frac{-\sin x}{\cos x} dx = -\ln \cos x$.

As $\ln 1 = 0$, $-\ln \cos x = \ln 1 - \ln \cos x = \ln \frac{1}{\cos x}$, using the log law $\ln a - \ln b = \ln \frac{a}{b}$.

Hence the integrating factor is $e^{\ln \sec x} = \sec x$

Multiply ① by $\sec x$

$$\sec x \frac{dy}{dx} + \sec x \frac{\sin x}{\cos x} y = \cos^2 x \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x) y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x$$

$$y \sec x = \int \cos x \, dx = \sin x + C$$

Multiplying throughout by $\cos x$

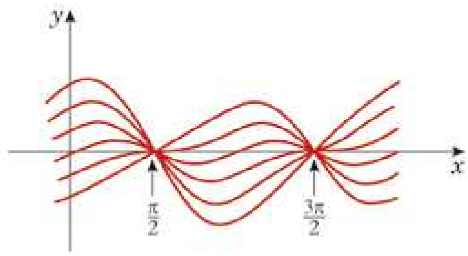
$$y = \sin x \cos x + C \cos x$$

b Where $\cos x = 0$ and $0 \leqslant x \leqslant 2\pi$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

The points $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ lie on all of the solution curves of the differential equation.

In general, for a given value of x, different values of c give different values of y. However, if $\cos x = 0$, the c will have no effect and y will be zero for any value of c.

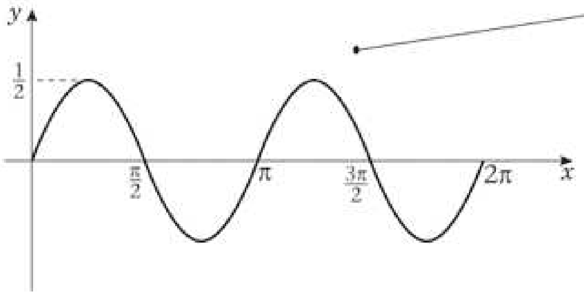


c $y = \sin x \cos x + C \cos x$

At $x = 0$, $y = 0$

$$0 = 0 + C \Rightarrow C = 0$$

$$y = \sin x \cos x = \tfrac{1}{2} \sin 2x$$



Using the identity $\sin 2x = 2 \sin x \cos x$. $\sin 2x$ is a function with period π. So the curve makes two complete oscillations in the interval $0 \leqslant x \leqslant 2\pi$

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Exercise A, Question 12

Question:

a Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = x.$$

Given that $y = 1$ at $x = 0$,

b find the exact values of the coordinates of the minimum point of the particular solution curve,

c draw a sketch of the particular solution curve.

Solution:

a The integrating factor is

$$e^{\int 2\,dx} = e^{2x}$$

Multiplying the differential equation throughout by e^{2x}

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = x\,e^{2x}$$

$$\frac{d}{dx}(y\,e^{2x}) = x\,e^{2x}$$

$$y\,e^{2x} = \int x\,e^{2x}\,dx$$

$$= \frac{x\,e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx = \frac{x\,e^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + C\,e^{-2x}$$

Integrate by parts.

b $y = 1$ at $x = 0$

$$1 = 0 - \tfrac{1}{4} + C \Rightarrow C = \tfrac{5}{4}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{5\,e^{-2x}}{4}$$

This is the particular solution of the differential equation for $y = 1$ at $x = 0$. You are asked to sketch this in part **c**.

For a minimum $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{1}{2} - \frac{5\,e^{-2x}}{2} = 0 \Rightarrow 5\,e^{-2x} = 1 \Rightarrow e^{2x} = 5$$

$$\ln e^{2x} = \ln 5 \Rightarrow 2x = \ln 5$$

$$x = \tfrac{1}{2}\ln 5$$

At the minimum, the differential equation reduces to

$$2y = x$$

The differential equation is $\frac{dy}{dx} + 2y = x$. At the minimum, $\frac{dy}{dx} = 0$ and so $2y = x$. If you did not see this you could, of course, substitute $x = \frac{1}{2}\ln 5$ into the particular solution and find y. This would take longer but would gain full marks.

Hence

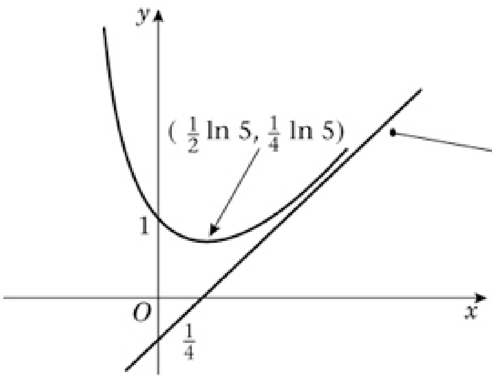
$$y = \tfrac{1}{2}x = \tfrac{1}{4}\ln 5$$

$$\frac{d^2y}{dx^2} = 5\,e^{-2x} > 0 \text{ for any real } x$$

This confirms the point is a minimum.

The coordinates of the minimum are $\left(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5\right)$.

c



As x increases, $e^{-2x} \to 0$ and so $\frac{x}{2} - \frac{1}{4} + \frac{5\,e^{-2x}}{4} \to \frac{x}{2} - \frac{1}{4}$. This means that $y = \frac{x}{2} - \frac{1}{4}$ is an asymptote of the curve. This has been drawn on the graph. It is not essential to do this, but if you recognise that this line is an asymptote, it helps you to draw the correct shape of the curve.

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Exercise A, Question 13

Question:

During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{dS}{dt} + \frac{2S}{120 - t} = \frac{1}{4}, \quad 0 \leq t < 120.$$

Given that $S = 6$ when $t = 0$,

a find S in terms of t,

b calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process.

Solution:

a $\int \frac{2}{120-t}\,dt = -2\ln(120-t) = \ln(120-t)^{-2} = \ln\frac{1}{(120-t)^2}$

Using the log law $n\log a = \log a^n$, with $n = -2$.

Hence the integrating factor is

$$e^{\int \frac{2}{120-t}dt} = e^{\ln\frac{1}{(120-t)^2}} = \frac{1}{(120-t)^2}$$

Multiply the equation throughout by $\frac{1}{(120-t)^2}$

$$\frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2}{(120-t)^3}S = \frac{1}{4(120-t)^2}$$

$$\frac{d}{dt}\left(\frac{S}{(120-t)^2}\right) = \tfrac{1}{4}(120-t)^{-2}$$

$\frac{d}{dt}\left(S(120-t)^{-2}\right)$
$= \frac{dS}{dt}(120-t)^{-2} - S\times(-2)(120-t)^{-3}$
$= \frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2}{(120-t)^3}S$
This product enables you to write the differential equation as a complete equation.

Integrating both sides with respect to t

$$\frac{S}{(120-t)^2} = \frac{1}{4}\int(120-t)^{-2}dt = -\frac{1}{4}\,\frac{(120-t)^{-1}}{-1} + C$$

$$\frac{S}{(120-t)^2} = \frac{1}{4(120-t)} + C$$

Multiply this equation by $(120-t)^2$.

$$S = \frac{120-t}{4} + C(120-t)^2$$

Remember to multiply the C by $(120-t)^2$. It is a common error to obtain C instead of $C(120-t)^2$ at this stage.

$S = 6$ when $t = 0$

$$6 = 30 + C\times 120^2 \Rightarrow C = -\frac{24}{120^2} = -\frac{1}{600}$$

$$S = \frac{120-t}{4} - \frac{(120-t)^2}{600}$$

b For a maximum value

$$\frac{dS}{dt} = -\frac{1}{4} + \frac{2(120-t)}{600} = 0$$

$$240 - 2t = 150 \Rightarrow t = \frac{240-150}{2} = 45$$

$$\frac{d^2S}{dt^2} = -\frac{1}{300} < 0 \Rightarrow \text{maximum}$$

Maximum value is given by

$$S = \frac{120-45}{4} - \frac{(120-45)^2}{600} = \frac{75}{4} - \frac{75}{8} = \frac{75}{8} = 9\frac{3}{8}$$

The maximum mass of salt predicted is $9\frac{3}{8}$ kg.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 14

Question:

A fertilized egg initially contains an embryo of mass m_0 together with a mass $100m_0$ of nutrient, all of which is available as food for the embryo. At time t the embryo has mass m and the mass of nutrient which has been consumed is $5(m - m_0)$.

a Show that, when three-quarters of the nutrient has been consumed, $m = 16m_0$.

The rate of increase of the mass of the embryo is a constant μ multiplied by the product of the mass of the embryo and the mass of the remaining nutrient.

b Show that $\frac{dm}{dt} = 5\mu m(21m_0 - m)$.

The egg hatches at time T, when three-quarters of the nutrient has been consumed.

c Show that $105\mu m_0 T = \ln 64$.

Solution:

a Three quarters of the nutrient is $\frac{3}{4} \times 100m_0 = 75m_0$

At time t, the nutrient consumed is $5(m - m_0)$

Hence

$$5(m - m_0) = 75m_0$$

$$5m - 5m_0 = 75m_0 \Rightarrow 5m = 80m_0$$

$$m = \frac{80m_0}{5} = 16m_0, \text{ as required}$$

b Rate of increase of mass = μ × mass × nutrient remaining

The nutrient remaining is the nutrient consumed, $5(m - m_0)$, subtracted from the original nutrient $100m_0$.

$$\frac{dm}{dt} = \mu \times m \times [100m_0 - 5(m - m_0)]$$

$$\frac{dm}{dt} = \mu m\,(100m_0 - 5m + 5m_0)$$

$$= \mu m\,(105m_0 - 5m)$$

$$= 5\,\mu m\,(21m_0 - m), \text{ as required}$$

c $\frac{dm}{dt} = 5\,\mu m\,(21m_0 - m)$

This is a separable equation.

$$\int 5\mu\,dt = \int \frac{1}{m(21m_0 - m)}\,dm$$

Separating the variables.

Let $\frac{1}{m(21m_0 - m)} = \frac{A}{m} + \frac{B}{21m_0 - m}$

To integrate the right hand side of this equation, you must break the expression up into partial fractions using one of the methods you learnt in C4.

Multiplying throughout by $m(21m_0 - m)$

$$1 = A(21m_0 - m) + Bm$$

Let $m = 0$

$$1 = A \times 21m_0 \Rightarrow A = \frac{1}{21m_0}$$

Let $m = 21m_0$

$$1 = B \times 21m_0 \Rightarrow B = \frac{1}{21m_0}$$

Hence

$$5\mu t = \frac{1}{21m_0}\int\left(\frac{1}{m} + \frac{1}{21m_0 - m}\right)dm$$

$$105\mu m_0 t = \int\left(\frac{1}{m} + \frac{1}{21m_0 - m}\right)dm = \ln m - \ln(21m_0 - m) + C$$

When $t = 0$, $m = m_0$

Initially, the mass of the embryo is m_0. This enables you to find the particular solution of the differential equation. The initial conditions are often known in scientific applications of mathematics.

$$0 = \ln m_0 - \ln 20m_0 + C$$

$$C = \ln 20m_0 - \ln m_0 = \ln\frac{20m_0}{m_0} = \ln 20$$

$$105\mu m_0 t = \ln m - \ln(21m_0 - m) + \ln 20 = \ln\left(\frac{20m}{21m_0 - m}\right)$$

Combining the logarithms at this stage simplifies the next stage of the calculation. The form of the simplification is $\ln a - \ln b + \ln c = \ln\frac{ac}{b}$

From part **a**, when $t = T$, $m = 16m_0$

$$105\mu m_0 T = \ln\left(\frac{20 \times 16m_0}{21m_0 - 16m_0}\right) = \ln\left(\frac{320m_0}{5m_0}\right)$$

$$= \ln 64, \text{ as required}$$

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Exercise A, Question 15

Question:

a Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{3x - 4y}{4x + 3y} \quad ①$$

into the differential equation

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4} \quad ②$$

b By solving differential equation ②, find the general solution of differential equation ②.

c Given that $y = 7$ at $x = 1$, show that the particular solution of differential equation ① can be written as

$$(3y - x)(y + 3x) = 200.$$

Solution:

a $y = vx$

$$\frac{dy}{dx} = x\frac{dx}{dx} + v$$

Differentiating vx as a product, $\frac{d}{dx}(vx) = \frac{dv}{dx}x + v\frac{d}{dx}(x) = x\frac{dv}{dx} + v$, as $\frac{d}{dx}(x) = 1$.

Substitute $y = vx$ and $\frac{dy}{dx} = x\frac{dv}{dx} + v$ into equation ① in the question

$$x\frac{dv}{dx} + v = \frac{3x - 4vx}{4x + 3vx} = \frac{x(3 - 4v)}{x(4 + 3v)}$$

$$x\frac{dv}{dx} = \frac{3 - 4v}{4 + 3v} - v = \frac{3 - 4v - 4v - 3v^2}{4 + 3v} = \frac{3 - 8v - 3v^2}{4 + 3v}$$

$$x\frac{dv}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4}, \text{ as required.}$$

This is a separable equation and in part **b** you solve it by collecting together the terms in v and dv on one side of the equation and the terms in x and dx on the other side.

b $\int \frac{3v + 4}{3v^2 + 8v - 3}dv = \frac{1}{2}\int \frac{6v + 8}{3v^2 + 8v - 3}dv = -\int \frac{1}{x}dx$

$$\tfrac{1}{2}\ln(3v^2 + 8v - 3) = -\ln x + A$$

$$\ln(3v^2 + 8v - 3) = -2\ln x + B$$

$$= \ln\frac{1}{x^2} + \ln C = \ln\frac{C}{x^2}$$

$\int \frac{f'(x)}{f(x)}dx = \ln f(x)$ is a standard formula you should know. As $6v + 8$ is the derivative of $3v^2 + 8v - 3$, $\int \frac{6v + 8}{3v^2 + 8v - 3}dv = \ln(3v^2 + 8v - 3)$.

Hence

$$3v^2 + 8v - 3 = \frac{C}{x^2}$$

An arbitrary constant B can be written as the logarithm of another arbitrary constant $\ln C$.

c $y = xv \Rightarrow v = \frac{y}{x}$

Substituting into the answer to part **b**

$$\frac{3y^2}{x^2} + \frac{8y}{x} - 3 = \frac{C}{x^2}$$

$$3y^2 + 8yx - 3x^2 = C$$

Multiply each term in the equation by x^2.

$y = 7$ at $x = 1$

$$3 \times 49 + 56 - 3 = C \Rightarrow C = 200$$

Factorising the left hand side of the equation

$$(3y - x)(y + 3x) = 200, \text{ as required.}$$

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Exercise A, Question 16

Question:

a Use the substitution $u = y^{-2}$ to transform the differential equation

$$\frac{dy}{dx} + 2xy = xe^{-x^2} y^3 \qquad ①$$

into the differential equation

$$\frac{du}{dx} - 4xu = -2xe^{-x^2}. \qquad ②$$

b Find the general solution of differential equation ②.

c Hence obtain the solution of differential equation ① for which $y = 1$ at $x = 0$.

Solution:

a $u = y^{-2}$

$$\frac{du}{dx} = -2 \times y^{-3} \times \frac{dy}{dx}$$

Differentiate both sides implicitly with respect to x.

Hence

$$\frac{dy}{dx} = -\frac{y^3}{2}\frac{du}{dx}$$

You transform this equation, making $\frac{dy}{dx}$ the subject of the formula as you need to substitute for $\frac{dy}{dx}$ in ①.

Substituting in equation ① in the question

$$-\frac{y^3}{2}\frac{du}{dx} + 2xy = x\,e^{-x^2}y^3$$

Divide by y^3

$$-\frac{1}{2}\frac{du}{dx} + \frac{2x}{y^2} = x\,e^{-x^2}$$

As $u = \frac{1}{y^2}$

$$-\frac{1}{2}\frac{du}{dx} + 2xu = x\,e^{-x^2}$$

Multiply by (-2)

$$\frac{du}{dx} - 4xu = -2x\,e^{-x^2}$$, as required

b The integrating factor of ② is

$$e^{\int -4x\,dx} = e^{-2x^2}$$

Multiplying ② throughout by e^{-2x^2}

$$e^{-2x^2}\frac{du}{dx} - 4xu\,e^{-2x^2} = -2x\,e^{-x^2} \times e^{-2x^2} = -2x\,e^{-3x^2}$$

$$\frac{d}{dx}(u\,e^{-2x^2}) = -2x\,e^{-3x^2}$$

$$u\,e^{-2x^2} = -2\int x\,e^{-3x^2}\,dx = \tfrac{1}{3}e^{-3x^2} + C$$

This integration can be carried out by inspection. As $\frac{d}{dx}(e^{-3x^2}) = -6x\,e^{-3x^2}$, then $\int x\,e^{-3x^2}\,dx = -\frac{1}{6}e^{-3x^2}$.

Multiplying throughout by e^{2x^2}

$$u = \tfrac{1}{3}e^{-x^2} + C\,e^{2x^2}$$

c As $u = \frac{1}{y^2}$

$$\frac{1}{y^2} = \tfrac{1}{3}e^{-x^2} + C\,e^{2x^2}$$

$y = 1$ at $x = 0$

$$1 = \tfrac{1}{3} + C \Rightarrow C = \tfrac{2}{3}$$

$$\frac{1}{y^2} = \frac{1}{3}e^{-x^2} + \frac{2}{3}e^{2x^2}$$

As no form of the answer has been specified in the question, this is an acceptable answer for the particular solution of ①.

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Exercise A, Question 17

Question:

Given that θ satisfies the differential equation

$$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 5\theta = 0$$

and that, when $t = 0$, $\theta = 3$ and $\frac{d\theta}{dt} = -6$, express θ in terms of t.

Solution:

The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m + 2)^2 = -1$$
$$m = -2 \pm i$$

The general solution is

$$\theta = e^{-2t}(A\cos t + B\sin t)$$

If the solutions to the auxiliary equation are $\alpha \pm i\beta$, you may quote the result that the general solution of the differential equation is $e^{\alpha t}(A\cos\beta t + B\sin\beta t)$.

$t = 0$, $\theta = 3$

$$3 = A$$

Using $\sin 0 = 0$ and $\cos 0 = 1$.

$$\frac{d\theta}{dt} = -2e^{-2t}(A\cos t + B\sin t) + e^{-2t}(-A\sin t + B\cos t)$$

$t = 0$, $\frac{d\theta}{dt} = -6$

$$-6 = -2A + B$$
$$B = 2A - 6 = 0$$

As $A = 3$

The particular solution is

$$\theta = 3e^{-2t}\cos t$$

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Exercise A, Question 18

Question:

Given that $3x \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = k \cos 2x,$$

where k is a constant,

a calculate the value of k,

b find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which at $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$.

Solution:

a $y = 3x \sin 2x \Rightarrow \frac{dy}{dx} = 3 \sin 2x + 6x \cos 2x$

$\frac{d^2y}{dx^2} = 6 \cos 2x + 6 \cos 2x - 12x \sin 2x$

$= 12 \cos 2x - 12x \sin 2x$

Use the product rule for differentiating.

Substituting into the differential equation

$12 \cos 2x - \cancel{12x \sin 2x} + \cancel{12x \sin 2x} = k \cos 2x$

Hence

$k = 12$

b The auxiliary equation is

$m^2 + 4 = 0$

$m = \pm 2\text{i}$

The complementary function is given by

$y = A \cos 2x + B \sin 2x$

If the solutions to the auxiliary equation are $m = \pm \alpha\text{i}$, you may quote the result that the complementary function is $A \cos \alpha x + B \sin \alpha x$.

From **a**, the general solution is

$y = A \cos 2x + B \sin 2x + 3x \sin 2x$

Part **a** of the question gives you that $3x \sin 2x$ is a particular integral of the differential equation and general solution = complementary function + particular integral.

$x = 0, y = 2$

$2 = A$

$x = \frac{\pi}{4}, y = \frac{\pi}{2}$

$\frac{\pi}{2} = A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2} + 3 \times \frac{\pi}{4} \sin \frac{\pi}{2}$

$\frac{\pi}{2} = B + \frac{3\pi}{4} \Rightarrow B = -\frac{\pi}{4}$

Use $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$.

The particular solution is

$y = 2 \cos 2x - \frac{\pi}{4} \sin 2x + 3x \sin 2x$

Solutionbank FP2

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Exercise A, Question 19

Question:

Given that $a + bx$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 16 + 4x,$$

a find the values of the constants a and b.

b Find the particular solution of this differential equation for which $y = 8$ and $\frac{dy}{dx} = 9$ at $x = 0$.

Solution:

a $y = a + bx \Rightarrow \frac{dy}{dx} = b$ and $\frac{d^2y}{dx^2} = 0$

Substituting into the differential equation

$0 - 4b + 4a + 4bx = 16 + 4x$

Equating the coefficients of x

$4b = 4 \Rightarrow b = 1$

Equating the constant coefficients

$-4b + 4a = 16$

$-4 + 4a = 16 \Rightarrow a = 5$ — Use $b = 1$.

$a = 5,\ b = 1$

b The auxiliary equation is

$m^2 - 4m + 4 = 0$

$(m - 2)^2 = 0$

$m = 2$, repeated

The complementary function is given by

$y = e^{2x}(A + Bx)$ — If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha x}(A + Bx)$. You can quote this result.

The general solution is

$y = e^{2x}(A + Bx) + 5 + x$ — general solution = complementary function + particular integral.

$y = 8,\ x = 0$

$8 = A + 5 \Rightarrow A = 3$

$\frac{dy}{dx} = 2e^{2x}(A + Bx) + Be^{2x} + 1$

$\frac{dy}{dx} = 9,\ x = 0$

$9 = 2A + B + 1 \Rightarrow B = 8 - 2A = 2$ — Use $A = 3$

The particular solution is

$y = e^{2x}(3 + 2x) + 5 + x$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 20

Question:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 65 \sin 2x, \quad x > 0.$$

a Find the general solution of the differential equation.

b Show that for large values of x this general solution may be approximated by a sine function and find this sine function.

Solution:

a The auxiliary equation is

$$m^2 + 4m + 5 = 0$$
$$m^2 + 4m + 4 = -1$$
$$(m + 2)^2 = -1$$
$$m = -2 \pm \mathrm{i}$$

The complementary function is given by

$$y = \mathrm{e}^{-2x}(A\cos x + B\sin x)$$

For a particular integral, let $y = p\cos 2x + q\sin 2x$

If the right hand side of the second order differential equation is a sine or cosine function, then you should try a particular integral of the form $p\cos\omega x + q\sin\omega x$, with an appropriate ω. Here $\omega = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -2p\sin 2x + 2q\cos 2x$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4p\cos 2x - 4q\sin 2x$$

Substituting into the differential equation

$$-4p\cos 2x - 4q\sin 2x - 8p\sin 2x + 8q\cos 2x + 5p\cos 2x + 5q\sin 2x = 65\sin 2x$$
$$(-4p + 8q + 5p)\cos 2x + (-4q - 8p + 5q)\sin 2x = 65\sin 2x$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$

The coefficients of $\cos 2x$ and $\sin 2x$ can be equated separately. The coefficient of $\cos 2x$ on the right hand side of this equation is zero.

$\cos 2x$: $\quad -4p + 8q + 5p = 0 \Rightarrow p + 8q = 0 \quad$ ①

$\sin 2x$: $\quad -4q - 8p + 5q = 65 \Rightarrow -8p + q = 65 \quad$ ②

$$8p + 64q = 0 \quad ③$$
$$65q = 65 \Rightarrow q = 1$$

Multiply ① by 8 and add the result to ②.

Substitute $q = 1$ into ①

$$p + 8 = 0 \Rightarrow p = -8$$

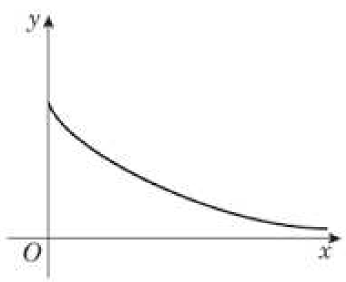
A particular integral is $-8\cos 2x + \sin 2x$

The general solution is

$$y = \mathrm{e}^{-2x}(A\cos x + B\sin x) + \sin 2x - 8\cos 2x$$

b As $x \to \infty$, $\mathrm{e}^{-2x} \to 0$ and, hence,

$$y \to \sin 2x - 8\cos 2x$$



The graph of e^{-2x} against x has this shape. As x becomes larger e^{-2x} is close to zero, so $\mathrm{e}^{-2x}(A\cos x + B\sin x)$ is also small.

Let

$$\sin 2x - 8\cos 2x = R\sin(2x - \alpha)$$
$$= R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$

$1 = R\cos\alpha \ldots \quad$ ④

$8 = R\sin\alpha \ldots \quad$ ⑤

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 8^2 = 65$$
$$R^2 = 65 \Rightarrow R = \sqrt{65}$$

Add ④ squared to ⑤ squared and use the identity $\cos^2\alpha + \sin^2\alpha = 1$.

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{8}{1} \Rightarrow \tan\alpha = 8$$

Divide ⑤ by ④.

Hence, for large x, y can be approximated by the sine function $\sqrt{65}\sin(2x - \alpha)$, where $\tan\alpha = 8$ ($\alpha \approx 82.9°$)

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Exercise A, Question 21

Question:

a Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2e^{-t}.$$

b Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ at $t = 0$.

Solution:

a The auxiliary equation is

$$m^2 + 2m + 2 = 0$$
$$m^2 + 2m + 1 = -1$$
$$(m + 1)^2 = -1$$
$$m = -1 \pm i$$

The complementary function is

$$y = e^{-t}(A\cos t + B\sin t)$$

Try a particular integral $y = k\,e^{-t}$

If the right hand side of the differential equation is $\lambda\,e^{at+b}$, where λ is any constant, then a possible form of the particular integral is $k\,e^{at+b}$.

$$\frac{dy}{dt} = -k\,e^{-t}, \quad \frac{d^2y}{dt^2} = k\,e^{-t}$$

Substituting into the differential equation

$$k\,e^{-t} - 2k\,e^{-t} + 2k\,e^{-t} = 2\,e^{-t}$$

Divide throughout by e^{-t}.

$$k - 2k + 2k = 2 \Rightarrow k = 2$$

A particular integral is $2\,e^{-t}$

The general solution is

$$y = e^{-t}(A\cos t + B\sin t) + 2\,e^{-t}$$

b $y = 1,\ t = 0$

$$1 = A + 2 \Rightarrow A = -1$$

Substitute the boundary condition $y = 1$, $t = 0$ into the general solution gives you an equation for one arbitrary constant.

$$\frac{dy}{dt} = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t) - 2\,e^{-t}$$

Use the product rule for differentiating.

$$\frac{dy}{dt} = 1,\ t = 0$$

$$1 = -A + B - 2 \Rightarrow B = 3 + A = 2$$

As $A = -1$.

The particular solution is

$$y = e^{-t}(2\sin t - \cos t) + 2\,e^{-t}$$

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Exercise A, Question 22

Question:

a Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0$$

b Given that $x = 1$ and $\frac{dx}{dt} = 1$ at $t = 0$, find the particular solution of the differential equation, giving your answer in the form $x = f(t)$.

c Sketch the curve with equation $x = f(t)$, $0 \leqslant t \leqslant \pi$, showing the coordinates, as multiples of π, of the points where the curve cuts the t-axis.

Solution:

a The auxiliary equation is

$$m^2 + 2m + 5 = 0$$
$$m^2 + 2m + 1 = -4$$
$$(m+1)^2 = -4$$
$$m = -1 \pm 2\text{i}$$

You may use any appropriate method to solve the quadratic. Completing the square works efficiently when the coefficient of m is even.

The general solution is

$$x = \text{e}^{-t}(A\cos 2t + B\sin 2t)$$

b $x = 1,\ t = 0$

$$1 = A$$

Use the product rule for differentiation.

$$\frac{\text{d}x}{\text{d}t} = -\text{e}^{-t}(A\cos 2t + B\sin 2t) + 2\,\text{e}^{-t}(-A\sin 2t + B\cos 2t)$$

$\frac{\text{d}x}{\text{d}t} = 1,\ t = 0$

$$1 = -A + 2B \Rightarrow 2B = A + 1 = 2 \Rightarrow B = 1$$

The particular solution is

$$x = \text{e}^{-t}(\cos 2t + \sin 2t)$$

Both A and B are 1.

c The curve crosses the t-axis where

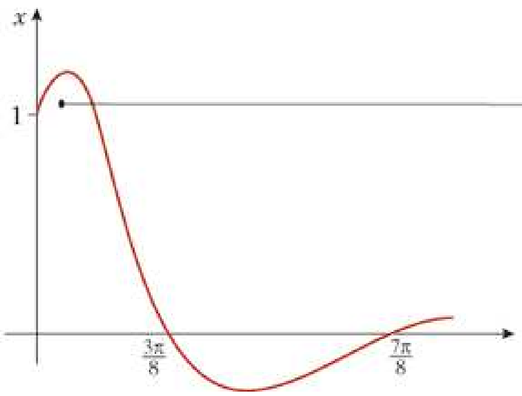
$$\text{e}^{-t}(\cos 2t + \sin 2t) = 0$$

e^{-t} can never be zero.

$$\cos 2t + \sin 2t = 0$$
$$\sin 2t = -\cos 2t$$
$$\tan 2t = -1$$

Divide both sides by $\cos 2t$ and use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$2t = \frac{3\pi}{4}, \frac{7\pi}{4}$$
$$t = \frac{3\pi}{8}, \frac{7\pi}{8}$$



The boundary conditions give you that at $t = 0$, $x = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the two points where $t = \frac{3\pi}{8}$ and $\frac{7\pi}{8}$.

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Exercise A, Question 23

Question:

a Find the general solution of the differential equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 3y = 3t^2 + 11t$$

b Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ when $t = 0$.

c For this particular solution, calculate the value of y when $t = 1$.

Solution:

a The auxiliary equation is

$$2m^2 + 7m + 3 = 0$$
$$(2m + 1)(m + 3) = 0$$
$$m = -\tfrac{1}{2}, -3$$

The complementary function is given by

$y = A\,e^{-\frac{1}{2}t} + B\,e^{-3t}$

If the auxiliary equation has two real solutions α and β, the complementary function is $y = A\,e^{\alpha t} + B\,e^{\beta t}$. You can quote this result.

For a particular integral, try $y = at^2 + bt + c$

$\frac{dy}{dt} = 2at + b, \frac{d^2y}{dt^2} = 2a$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic, so you try the general quadratic $at^2 + bt + c$.

Substitute into the differential equation

$4a + 14at + 7b + 3at^2 + 3bt + 3c = 3t^2 + 11t$

$3at^2 + (14a + 3b)t + 4a + 7b + 3c = 3t^2 + 11t$

Equating the coefficients of t^2

$3a = 3 \Rightarrow a = 1$

Equating the coefficients of t

$14a + 3b = 11 \Rightarrow 3b = 11 - 14a = -3 \Rightarrow b = -1$

Use $a = 1$.

Equating the constant coefficients

$4a + 7b + 3c = 0 \Rightarrow 3c = -4a - 7b = 3 \Rightarrow c = 1$

Use $a = 1$ and $b = -1$.

A particular integral is $t^2 - t + 1$.

The general solution is $y = A\,e^{-\frac{1}{2}t} + B\,e^{-3t} + t^2 - t + 1$.

b $y = 1, t = 0$

$1 = A + B + 1 \Rightarrow A + B = 0$ ①

$\frac{dy}{dt} = -\frac{1}{2}A\,e^{-\frac{1}{2}t} - 3B\,e^{-3t} + 2t - 1$

Differentiate the general solution in part **a** with respect to t.

$\frac{dy}{dt} = 1, t = 0$

$1 = -\frac{1}{2}A - 3B - 1 \Rightarrow \frac{1}{2}A + 3B = -2$ ②

$A + 6B = -4$ ③

$5B = -4 \Rightarrow B = -\frac{4}{5}$

Multiply ② by 2 and then subtract ① from ③.

Substituting $B = -\frac{4}{5}$ into ①

$A - \frac{4}{5} = 0 \Rightarrow A = \frac{4}{5}$

The particular solution is $y = \frac{4}{5}\left(e^{-\frac{1}{2}t} - e^{-3t}\right) + t^2 - t + 1$.

c When $t = 1$, $y = \frac{4}{5}\left(e^{-\frac{1}{2}} - e^{-3}\right) + 1 = 1.45$ (3 s.f.)

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 24

Question:

a Find the value of λ for which $\lambda x \cos 3x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 9y = -12 \sin 3x$$

b Hence find the general solution of this differential equation.

The particular solution of the differential equation for which $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$, is $y = g(x)$.

c Find $g(x)$.

d Sketch the graph of $y = g(x)$, $0 \leqslant x \leqslant \pi$.

Solution:

a Let $y = \lambda x \cos 3x$

$$\frac{dy}{dx} = \lambda \cos 3x - 3\lambda x \sin 3x$$

$$\frac{d^2y}{dx^2} = -3\lambda \sin 3x - 3\lambda \sin 3x - 9\lambda x \cos 3x$$

$$= -6\lambda \sin 3x - 9\lambda x \cos 3x$$

Use the product rule for differentiation

$$\frac{d}{dx}(x \sin 3x) = \frac{d}{dx}(x) \sin 3x + x \frac{d}{dx}(\sin 3x)$$

$$= \sin 3x + 3x \cos 3x$$

Substituting into the differential equation

$$-6\lambda \sin 3x - 9\lambda x \cos 3x + 9\lambda x \cos 3x = -12 \sin 3x$$

Hence

$$\lambda = 2$$

b The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m^2 = -9$$

$$m = \pm 3i$$

The complementary function is given by

$$y = A \cos 3x + B \sin 3x$$

The general solution is

$$y = A \cos 3x + B \sin 3x + 2x \cos 3x$$

Part **a** shows that $2x \cos 3x$ is a particular integral of the differential equation and general solution = complementary function + particular integral

c $y = 1, x = 0$

$$1 = A$$

$$\frac{dy}{dx} = -3A \sin 3x + 3B \cos 3x + 2 \cos 3x - 6x \sin 3x$$

Differentiate the general solution in part **b** with respect to x.

$$\frac{dy}{dx} = 2, x = 0$$

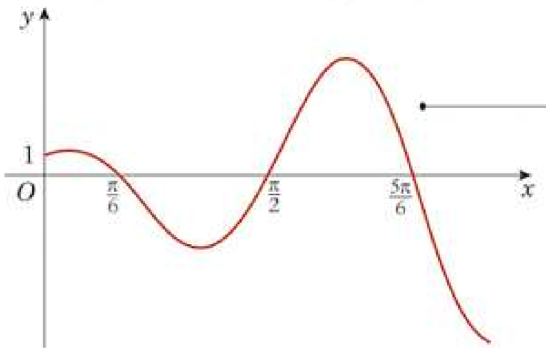
$$2 = 3B + 2 \Rightarrow B = 0$$

The particular solution is

$$y = \cos 3x + 2x \cos 3x = (1 + 2x) \cos 3x$$

d For $x > 0$, the curve crosses the x–axis at $\cos 3x = 0$

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



The boundary conditions give you that at $x = 0$, $y = 1$ and the curve has a positive gradient. The curve must then turn down and cross the axis at the three points where $x = \frac{\pi}{6}, \frac{\pi}{2}$ and $\frac{5\pi}{6}$.

The $(1 + 2x)$ factor in the general solution means that the size of the oscillations increases as x increases.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 25

Question:

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t},\ t \geqslant 0$$

a Show that $Kt^2 e^{3t}$ is a particular integral of the differential equation, where K is a constant to be found.

b Find the general solution of the differential equation.

Given that a particular solution satisfies

$y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$,

c find this solution.

Another particular solution which satisfies

$y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$, has equation $y = (1 - 3t + 2t^2)e^{3t}$

d For this particular solution, draw a sketch graph of y against t, showing where the graph crosses the t-axis. Determine also the coordinates of the minimum point on the sketch graph.

Solution:

a If $y = Kt^2 e^{3t}$

$$\frac{dy}{dt} = 2Kt\,e^{3t} + 3Kt^2 e^{3t}$$

$$\frac{d^2y}{dt^2} = 2K\,e^{3t} + 6Kt\,e^{3t} + 6Kt\,e^{3t} + 9Kt^2 e^{3t}$$

$$= 2K\,e^{3t} + 12Kt\,e^{3t} + 9Kt^2 e^{3t}$$

Substituting into the differential equation

$$2K\,e^{3t} + \cancel{12Kt\,e^{3t}} + \cancel{9Kt^2 e^{3t}} - \cancel{12Kt\,e^{3t}} - \cancel{18Kt^2 e^{3t}} + \cancel{9Kt^2 e^{3t}} = 4\,e^{3t}$$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

Hence

$2K = 4 \Rightarrow K = 2$

$2t^2 e^{3t}$ is a particular integral of the differential equation.

b The auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$(m - 3)^2 = 0$$

$$m = 3, \text{ repeated}$$

The complementary function is given by

$y = e^{3t}(A + Bt)$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$y = e^{3t}(A + Bt) + 2t^2 e^{3t} = (A + Bt + 2t^2)e^{3t}$

c $y = 3,\ t = 0$

$3 = A$

$$\frac{dy}{dt} = (B + 4t)e^{3t} + 3(A + Bt + 2t^2)e^{3t}$$

$\frac{dy}{dt} = 1,\ t = 0$

$1 = B + 3A \Rightarrow B = 1 - 3A \Rightarrow B = -8$

As $A = 3$.

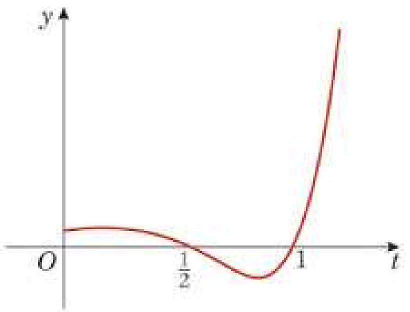
The particular solution is

$y = (3 - 8t + 2t^2)e^{3t}$

d This particular solution crosses the t-axis where

$1 - 3t + 2t^2 = (1 - 2t)(1 - t) = 0$

$t = \frac{1}{2}, 1$



For a minimum $\frac{dy}{dt} = 0$

$(-3 + 4t)\,e^{3t} + (1 - 3t + 2t^2)3\,e^{3t} = 0$

e^{3t} cannot be zero, so you can divide throughout by e^{3t}.

$-3 + 4t + 3 - 9t + 6t^2 = 0$

$6t^2 - 5t = t(6t - 5) = 0 \Rightarrow t = 0, \frac{5}{6}$

From the digram $t = \frac{5}{6}$ gives the minimum

It is clear from the diagram that there is a minimum point between $t = \frac{1}{2}$ and $t = 1$. You do not have to consider the second derivative to show that it is a minimum.

At $t = \frac{5}{6}$

$y = \left(1 - 3 \times \frac{5}{6} + 2 \times \left(\frac{5}{6}\right)^2\right)e^{3 \times \frac{5}{6}} = -\frac{1}{9}\,e^{\frac{5}{2}}$

The coordinates of the minumum point are

$\left(\frac{5}{6}, -\frac{1}{9}\,e^{\frac{5}{2}}\right)$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 26

Question:

a Find the general solution of the differential equation

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9$$

b Find the particular solution of this differential equation for which $x = 3$ and $\frac{dx}{dt} = -1$ when $t = 0$.

The particular solution in part **b** is used to model the motion of the particle P on the x-axis. At time t seconds ($t \geqslant 0$), P is x metres from the origin O.

c Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum.

Solution:

a The auxiliary equation is

$$2m^2 + 5m + 2 = 0$$
$$(2m + 1)(m + 2) = 0$$
$$m = -\tfrac{1}{2}, -2$$

The complementary function is given by

$$x = A\,e^{-\frac{1}{2}t} + B\,e^{-2t}$$

If the auxiliary equation has two real solutions α and β, the complementary function is $x = A\,e^{\alpha t} + B\,e^{\beta t}$. You can quote this result.

For a particular integral, try $x = pt + q$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is linear, so you try the general linear function $pt + q$.

$$\frac{dx}{dt} = p, \frac{d^2x}{dt^2} = 0$$

Substituting into the differential equation

$$0 + 5p + 2pt + 2q = 2t + 9$$

Equating the coefficients of t

$$2p = 2 \Rightarrow p = 1$$

Equating the constant coefficients

$$5p + 2q = 9 \Rightarrow q = \frac{9 - 5p}{2} \Rightarrow q = 2$$

A particular integral is $t + 2$

The general solution is

$$x = A\,e^{-\frac{1}{2}t} + B\,e^{-2t} + t + 2$$

b $x = 3, t = 0$

$3 = A + B + 2 \Rightarrow A + B = 1$ ①

$\frac{dx}{dt} = -\frac{1}{2}A\,e^{-\frac{1}{2}t} - 2B\,e^{-2t} + 1$

Differentiating the general solution in part **a**.

$\frac{dx}{dt} = -1, t = 0$

$-1 = -\frac{1}{2}A - 2B + 1 \Rightarrow \frac{1}{2}A + 2B = 2$ ②

$A + 4B = 4$ ③

$3B = 3 \Rightarrow B = 1$

Multiplying ② by 2 and subtracting ① from ③.

Substituting $B = 1$ into ①

$A + 1 = 1 \Rightarrow A = 0$

The particular solution is

$x = e^{-2t} + t + 2$

c For a minimum

$\frac{dx}{dt} = -2\,e^{-2t} + 1 = 0$

$e^{-2t} = \frac{1}{2}$

You take logarithms of both sides of this equation and use $e^{\ln f(x)} = f(x)$.

$-2t = \ln\frac{1}{2} = -\ln 2$

$\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$, as $\ln 1 = 0$.

$t = \frac{1}{2}\ln 2$

$\frac{d^2x}{dt^2} = 4\,e^{-2t} > 0$, for any real t

Hence the stationary value is a minimum value

When $t = \frac{1}{2}\ln 2$

$x = e^{-\ln 2} + \frac{1}{2}\ln 2 + 2 = \frac{1}{2} + \frac{1}{2}\ln 2 + 2 = \frac{5}{2} + \frac{1}{2}\ln 2$

$e^{-\ln 2} = e^{\ln 1 - \ln 2} = e^{\ln\frac{1}{2}} = \frac{1}{2}$

The minimum distance is $\frac{1}{2}(5 + \ln 2)$ m, as required.

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 27

Question:

Given that $x = At^2 e^{-t}$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t},$$

a find the value of A.

b Hence find the solution of the differential equation for which $x = 1$ and $\frac{dx}{dt} = 0$ at $t = 0$.

c Use your solution to prove that for $t \geq 0$, $x \leq 1$.

Solution:

a If $x = At^2 e^{-t}$

$$\frac{dx}{dt} = 2At\,e^{-t} - At^2\,e^{-t}$$

$$\frac{d^2x}{dt^2} = 2A\,e^{-t} - 2At\,e^{-t} - 2At\,e^{-t} + At^2\,e^{-t}$$

$$= 2A\,e^{-t} - 4At\,e^{-t} + At^2\,e^{-t}$$

Substituting into the differential equation

$$2A\,e^{-t} - \cancel{4At\,e^{-t}} + \cancel{At^2\,e^{-t}} + \cancel{4At\,e^{-t}} - \cancel{2At^2\,e^{-t}} + \cancel{At^2\,e^{-t}} = e^{-t}$$

e^{-t} cannot be zero, so you can divide throughout by e^{-t}.

Hence

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

b The auxiliary equation is

$$m^2 + 2m + 1 = (m+1)^2 = 0$$

$$m = -1, \text{ repeated}$$

The complementary function is given by

$$x = e^{-t}(A + Bt)$$

If the auxiliary equation has a repeated root α, then the complementary function is $e^{\alpha t}(A + Bt)$. You can quote this result.

The general solution is

$$x = e^{-t}(A + Bt) + \tfrac{1}{2}t^2\,e^{-t} = (A + Bt + \tfrac{1}{2}t^2)e^{-t}$$

From part **a**, $\frac{1}{2}t^2\,e^{-t}$ is a particular integral of the differential equation.

$x = 1,\ t = 0$

$$1 = A$$

$$\frac{dx}{dt} = (B + t)\,e^{-t} - (A + Bt + \tfrac{1}{2}t^2)e^{-t}$$

$\frac{dx}{dt} = 0,\ t = 0$

$$0 = B - A \Rightarrow B = A = 1$$

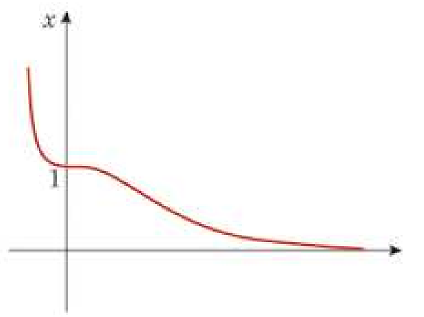
The particular solution is

$$x = \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$$

c $\frac{dx}{dt} = (1 + t)\,e^{-t} - \left(1 + t + \tfrac{1}{2}t^2\right)e^{-t}$

$$= -\tfrac{1}{2}t^2\,e^{-t} \leqslant 0, \text{ for all real } t.$$

When $t = 0$, $x = 1$ and x has a negative gradient for all positive t, x is a decreasing function of t. Hence, for $t \geqslant 0$, $x \leqslant 1$, as required.



The graph of x against t, shows the curve crossing the x-axis at $x = 1$ and then decreasing. For all positive t, x is less than 1.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 28

Question:

Given that $y = kx$ is a particular solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 3x,$$

a find the value of the constant k.

b Find the most general solution of this differential equation for which $y = 0$ at $x = 0$.

c Prove that all curves given by this solution pass through the point $(\pi, 3\pi)$ and that they all have equal gradients when $x = \frac{\pi}{2}$.

d Find the particular solution of the differential equation for which $y = 0$ at $x = 0$ and at $x = \frac{\pi}{2}$.

e Show that a minimum value of the solution in part **d** is

$$3 \arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{(\pi^2 - 4)}$$

Solution:

a $y = kx \Rightarrow \frac{dy}{dx} = k \Rightarrow \frac{d^2y}{dx^2} = 0$

Substituting into $\frac{d^2y}{dx^2} + y = 3x$

$$0 + kx = 3x$$
$$k = 3$$

b The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complementary function is given by

$$y = A\sin x + B\cos x$$

and the general solution is

$$y = A\sin x + B\cos x + 3x$$

$y = 0, x = 0$

$$0 = B + 0 \Rightarrow B = 0$$

The most general solution is

$$y = A\sin x + 3x$$

In part **b**, only one condition is given, so only one of the arbitrary constants can be found. The solution is a family of functions, some of which are illustrated in the diagram below.

c At $x = \pi$

$$y = A\sin \pi + 3\pi = 3\pi$$

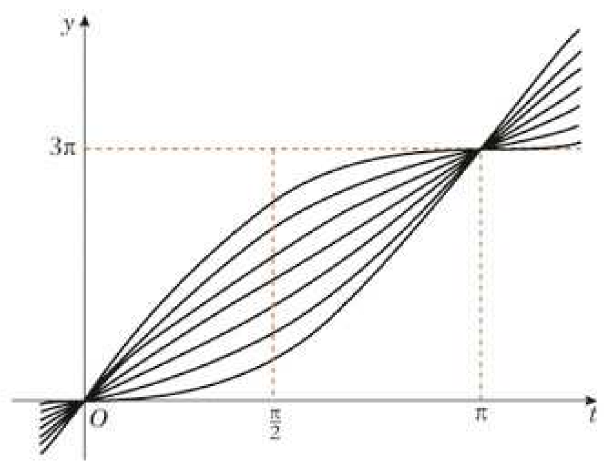
This is independent of the value of A. Hence, all curves given by the solution in part **a** pass through $(\pi, 3\pi)$.

$$\frac{dy}{dx} = A\cos x + 3$$

At $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = A\cos\frac{\pi}{2} + 3 = 3$$

This is independent of the value of A. Hence, all curves given by the solution in part **a** have an equal gradient of 3 at $x = \frac{\pi}{2}$.



As is illustrated by this diagram, the family of curves $y = A\sin x + 3x$ all go through (0, 0) and $(\pi, 3\pi)$. The tangent to the curves at $x = \frac{\pi}{2}$ are all parallel to each other.

d $y = 0, x = \frac{\pi}{2}$

Substituting into $y = A\sin x + 3x$

$$0 = A\sin\frac{\pi}{2} + \frac{3\pi}{2} = A + \frac{3\pi}{2} \Rightarrow A = -\frac{3\pi}{2}$$

The particular solution is

$$y = 3x - \frac{3\pi}{2}\sin x$$

e For a minimum

$$\frac{dy}{dx} = 3 - \frac{3\pi}{2}\cos x = 0$$

$$\cos x = \frac{2}{\pi} \Rightarrow x = \arccos\left(\frac{2}{\pi}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3\pi}{2}\sin x$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} > 0 \Rightarrow \text{minimum}$$

$\cos x = \frac{2}{\pi}$ has an infinite number of solutions. This shows that the solution in the first quadrant gives a minimum as $\sin x$ is positive in that quadrant.

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{4}{\pi^2} = \frac{\pi^2 - 4}{\pi^2}$$

In the interval $0 \leqslant x \leqslant \frac{\pi}{2}$

$$\sin x = +\left(\frac{\pi^2 - 4}{\pi^2}\right)^{\frac{1}{2}} = \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$y = 3\arccos\left(\frac{2}{\pi}\right) - \frac{3\pi}{2} \times \frac{\sqrt{\pi^2 - 4}}{\pi}$$

$$= 3\arccos\left(\frac{2}{\pi}\right) - \frac{3}{2}\sqrt{\pi^2 - 4}\text{, as required.}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 29

Question:

a Show that the transformation $y = xv$ transforms the equation

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + 9x^2)y = x^5, \qquad ①$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad ②$$

b Solve the differential equation ② to find v as a function of x.

c Hence state the general solution of the differential equation ①.

Solution:

a

$$y = xv$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Use the product rule for differentiation
$\frac{d}{dx}(xv) = \frac{d}{dx}(x) \times v + x \times \frac{dv}{dx} = 1 \times v + x\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$$

Substituting for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into ①

$$x^2\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + (2 + 9x^2)vx = x^5$$

$$x^3\frac{d^2v}{dx^2} + \cancel{2x^2\frac{dv}{dx}} - \cancel{2xv} - \cancel{2x^2\frac{dv}{dx}} + \cancel{2xv} + 9x^3v = x^5$$

$$x^3\frac{d^2v}{dx^2} + 9x^3v = x^5$$

Divide by x^3.

$$\frac{d^2v}{dx^2} + 9v = x^2, \text{ as required}$$

b The auxiliary equation is

$$m^2 + 9 = 0 \Rightarrow m^2 = -9$$
$$m = \pm 3\text{i}$$

The complementary function is given by

$$v = A\cos 3x + B\sin 3x$$

For a particular integral, try $v = px^2 + qx + r$

$$\frac{\text{d}v}{\text{d}x} = 2px + q,\ \frac{\text{d}^2v}{\text{d}x^2} = 2p$$

Substituting into ②

$$2p + 9px^2 + 9qx + 9r = x^2$$

Equating coefficients of x^2

$$9p = 1 \Rightarrow p = \tfrac{1}{9}$$

Equating coefficients of x

$$9q = 0 \Rightarrow q = 0$$

Equating constant coefficients

$$2p + 9r = 0 \Rightarrow 9r = -2p = -\tfrac{2}{9} \Rightarrow r = -\tfrac{2}{81}$$

A particular integral is $\frac{1}{9}x^2 - \frac{2}{81}$

A general solution of ② is

$$v = A\cos 3x + B\sin 3x + \tfrac{1}{9}x^2 - \tfrac{2}{81}$$

c $\dfrac{y}{x} = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$

$y = Ax\cos 3x + Bx\sin 3x + \frac{1}{9}x^3 - \frac{2}{81}x$

If the right hand side of the differential equation is a polynomial of degree n, then you can try a particular integral of the same degree. Here the right hand side is a quadratic x^2, so you try a general quadratic $px^2 + qx + r$.

As $p = \frac{1}{9}$.

$y = vx \Rightarrow v = \dfrac{y}{x}$.

The question does not ask for a particular form of the answer in part **c**, so this would be an acceptable answer.

Solutionbank FP2
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Exercise A, Question 30

Question:

Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$, and that y is a function of x,

a find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t.

Assuming that $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$,

b show that the substitution $x = t^{\frac{1}{2}}$, transforms the differential equation

$$\frac{d^2y}{dx^2} + \left(6x - \frac{1}{x}\right)\frac{dy}{dx} - 16x^2y = 4x^2e^{2x^2} \quad ①$$

into the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$$

c Hence find the general solution of ① giving y in terms of x.

Solution:

a $x = t^{\frac{1}{2}} \Rightarrow \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2t^{\frac{1}{2}}}$

$\frac{dt}{dx} = \frac{1}{\frac{1}{2t^{\frac{1}{2}}}} = 2t^{\frac{1}{2}}$

Use $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times 2t^{\frac{1}{2}} = 2t^{\frac{1}{2}}\frac{dy}{dt}$

You obtain an expression for $\frac{dy}{dx}$ using the chain rule.

b Substituting $x = t^{\frac{1}{2}}$, the result of part **a** and the given $\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}$ into ①

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}}\frac{dy}{dt} - 16ty = 4te^{2t}$

$\left(6t^{\frac{1}{2}} - \frac{1}{t^{\frac{1}{2}}}\right)2t^{\frac{1}{2}} = 6t^{\frac{1}{2}} \times 2t^{\frac{1}{2}} - \frac{2t^{\frac{1}{2}}}{t^{\frac{1}{2}}} = 12t - 2$

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 12t\frac{dy}{dt} - 2\frac{dy}{dt} - 16ty = 4te^{2t}$

$4t\frac{d^2y}{dt^2} + 12t\frac{dy}{dt} - 16ty = 4te^{2t}$

Divide throughout by $4t$.

$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$, as required

c The auxiliary equation is

$$m^2 + 3m - 4 = (m - 1)(m + 4) = 0$$

$$m = 1, -4$$

The complementary function is

$$y = Ae^t + Be^{-4t}$$

For a particular integral try, $y = ke^{2t}$

If the right hand side of the equation is $e^{\alpha t}$, you can try $ke^{\alpha t}$ as a particular integral. This will work unless α is a solution of the auxiliary equation.

$$\frac{dy}{dt} = 2ke^{2t}, \frac{d^2y}{dt^2} = 4ke^{2t}$$

Substituting into $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 4y = e^{2t}$

$$\cancel{4ke^{2t}} + 6ke^{2t} - \cancel{4ke^{2t}} = e^{2t}$$

As e^{2t} cannot be zero, you can divide throughout by e^{2t}.

$$6k = 1 \Rightarrow k = \frac{1}{6}$$

A particular integral is $\frac{1}{6}e^{2t}$

The general solution of the differential equation in y and t is

$$y = Ae^t + Be^{-4t} + \frac{1}{6}e^{2t}$$

$x = t^{\frac{1}{2}} \Rightarrow t = x^2$

The general solution of ① is

$$y = Ae^{x^2} + Be^{-4x^2} + \frac{1}{6}e^{2x^2}$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 31

Question:

A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at time t hours satisfies the differential equation

$$2x\frac{d^2x}{dt^2} - 6\left(\frac{dx}{dt}\right)^2 = x^2 - 3x^4, \quad x > 0.$$

a Show that the substitution $y = \frac{1}{x^2}$ transforms this differential equation into

$$\frac{d^2y}{dt^2} + y = 3. \qquad ①$$

b Find the general solution of differential equation ①.

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$,

c find an expression for x in terms of t,

d write down the maximum value of x as t varies.

Solution:

a $y = x^{-2}$

Differentiating implicitly with respect to t

$$\frac{dy}{dt} = -2x^{-3}\frac{dx}{dt}$$

Use $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Differentiating again implicitly with respect to t.

$$\frac{d^2y}{dt^2} = 6x^{-4}\left(\frac{dx}{dt}\right)^2 - 2x^{-3}\frac{d^2x}{dt^2} \quad ②$$

This expression is closely related to the left hand side of the original differential equation in the question. This suggests to you that if you divide the original equation by $-x^4$, then the left hand side can just be replaced by $\frac{d^2x}{dt^2}$

Dividing the differential equation given in the question by $-x^4$, it becomes

$$-2x^{-3}\frac{d^2x}{dt^2} + 6x^{-4}\left(\frac{dx}{dt}\right)^2 = -x^{-2} + 3$$

Using equation ② and $y = x^{-2}$

$$\frac{d^2y}{dt^2} = -y + 3$$

$$\frac{d^2y}{dt^2} + y = 3, \text{ as required}$$

b The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

The complementary function is given by

$$y = A\cos t + B\sin t$$

By inspection, a particular integral of ① is 3

As $\frac{d^2}{dt^2}(3) = 0$, $y = 3$ satisfies $\frac{d^2y}{dt^2} + y = 3$, by inspection and you need not write down any working.

The general solution of ② is

$$y = A\cos t + B\sin t + 3$$

c The general solution of the differential equation in x and t is

$$\frac{1}{x^2} = A\cos t + B\sin t + 3 \quad ③$$

When $t = 0$, $x = \frac{1}{2}$

$$4 = A + 3 \Rightarrow A = 1$$

Differentiating ③ implicitly with respect to t

$$-\frac{2}{x^3}\frac{dx}{dt} = -A\sin t + B\cos t$$

Use the chain rule $\frac{d}{dt}(x^{-2}) = \frac{d}{dx}(x^{-2}) \times \frac{dx}{dt} = -2x^{-3}\frac{dx}{dt}$.

When $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$

$$0 = B$$

The particular solution is

$$\frac{1}{x^2} = \cos t + 3$$

As $x > 0$, $t > 0$

$$x = \frac{1}{\sqrt{(\cos t + 3)}}$$

As x and t are both positive, the negative square root need not be considered.

d The maximum value of x is

$$x = \frac{1}{\sqrt{(-1 + 3)}} = \frac{1}{\sqrt{2}}$$

The maximum value of this fraction is when the denominator has its least value. The smallest possible value of $\cos t$ is -1. So you can write down the maximum value without using calculus.

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Exercise A, Question 32

Question:

Given that $x = \ln t$, $t > 0$, and that y is a function of x,

a find $\frac{dy}{dx}$ in terms of $\frac{dy}{dt}$ and t,

b show that $\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$.

c Show that the substitution $x = \ln t$ transforms the differential equation

$$\frac{d^2y}{dx^2} - (1 - 6e^x)\frac{dy}{dx} + 10ye^{2x} = 5e^{2x}\sin 2e^x \quad ①$$

into the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5\sin 2t \quad ②$$

d Hence find the general solution of ①, giving your answer in the form $y = f(x)$.

Solution:

a $x = \ln t \Rightarrow \frac{dx}{dt} = \frac{1}{t} \Rightarrow \frac{dt}{dx} = t$

$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \times t$

$\frac{dy}{dx} = t\frac{dy}{dt}$

b $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx} \times \frac{d}{dt}\left(\frac{dy}{dx}\right)$

It is a common error to proceed from $\frac{dy}{dx} = t\frac{dy}{dt}$ to $\frac{d^2y}{dx^2} = \frac{dy}{dt} + t\frac{d^2y}{dt^2}$. This is incorrect because the left hand side has been differentiated with respect to x and the right hand side with respect to t. The version of the chain rule given here must be used.

$= t\frac{d}{dt}\left(t\frac{dy}{dt}\right)$

$= t\left(\frac{dy}{dt} + t\frac{d^2y}{dt^2}\right)$

$= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$, as required

c Substituting $x = \ln t$, $\frac{dy}{dx} = t\frac{dy}{dt}$ and

$\frac{d^2y}{dx^2} = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}$ into ①

$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - (1 - 6t)t\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

$e^{2\ln t} = e^{\ln t^2} = t^2$, using the log rule $n \ln a = \ln a^n$ and $e^{\ln f(t)} = f(t)$.

$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} - t\frac{dy}{dt} + 6t^2\frac{dy}{dt} + 10yt^2 = 5t^2 \sin 2t$

After cancelling, divide throughout by t^2.

$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 5 \sin 2t$, as required

d The auxiliary equation of ① is

$m^2 + 6m + 10 = 0$

$m^2 + 6m + 9 = -1$

$(m + 3)^2 = -1$

$m + 3 = \pm i$

$m = -3 \pm i$

The complementary function is given by

$y = e^{-3t}(A\cos t + B\sin t)$

For a particular integral try $y = p\sin 2t + q\cos 2t$

$\frac{dy}{dt} = 2p\cos 2t - 2q\sin 2t$

$\frac{d^2y}{dt^2} = -4p\sin 2t - 4q\cos 2t$

If the right hand side of the second order differential equation is a $k\sin nt$ or $k\cos nt$ function, then you should try a particular integral of the form $p\cos nt + q\sin nt$.

Substituting into ②

$-4p\sin 2t - 4q\cos 2t + 12p\cos 2t - 12q\sin 2t + 10p\sin 2t + 10q\cos 2t = 5\sin 2t$

$(-4p - 12q + 10p)\sin 2t + (-4q + 12p + 10q)\cos 2t = 5\sin 2t$

$(6p - 12q)\sin 2t + (12p + 6q)\cos 2t = 5\sin 2t$

Equating the coefficients of $\sin 2t$

$6p - 12q = 5$ ③

$12p + 6q = 0$ ④

You can solve the simultaneous equations by any appropriate method.

From ④ $p = -\frac{6}{12}q = -\frac{1}{2}q$

Substitute into ③

$-3q - 12q = -15q = 5 \Rightarrow q = -\frac{1}{3}$

Hence $p = -\frac{1}{2}q = -\frac{1}{2} \times -\frac{1}{3} = \frac{1}{6}$

The general solution of ② is

$y = e^{-3t}(A\cos t + B\sin t) + \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$

$x = \ln t \Rightarrow t = e^x$

The general solution of ① is

$y = e^{-3e^x}(A\cos(e^x) + B\sin(e^x)) + \frac{1}{6}\sin(2e^x) - \frac{1}{3}\cos(2e^x)$

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Exercise A, Question 33

Question:

Given that x is so small that terms in x^3 and higher powers of x may be neglected, show that

$$11\sin x - 6\cos x + 5 = A + Bx + Cx^2,$$

stating the values of the constants A, B and C.

Solution:

a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$= 1 - \frac{x^2}{2}$, neglecting terms in x^3 and higher powers

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$= x$, neglecting terms in x^3 and higher powers

The series of $\cos x$ and $\sin x$ are both given in the formulae book and may be quoted without proof, unless the question specifically asks for a proof.

$$11\sin x - 6\cos x + 5 = 11x - 6\left(1 - \frac{x^2}{2}\right) + 5$$
$$= 11x - 6 + 3x^2 + 5$$
$$= -1 + 11x + 3x^2$$

You substitute the abbreviated series into the expression and collect together terms.

$A = -1$, $B = 11$, $C = 3$

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Exercise A, Question 34

Question:

Show that for $x > 1$,

$$\ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x = \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots .$$

Solution:

a LHS $= \ln(x^2 - x + 1) + \ln(x + 1) - 3\ln x$

$= \ln[(x^2 - x + 1)(x + 1)] - \ln x^3$

$= \ln\left(\frac{x^3 + 1}{x^3}\right) = \ln\left(1 + \frac{1}{x^3}\right)$

You collect together the three terms of the left hand side (LHS) of the expression into a single logarithm using all three log rules; $\log x + \log y = \log xy$

$\log x - \log y = \log\left(\frac{x}{y}\right)$,

and $n\log x = \log x^n$.

$(x^2 - x + 1)(x + 1) = x^3 + x^2 - x^2 - x + x + 1$
$= x^3 + 1$

Substituting $\frac{1}{x^3}$ for x and n for r in the series

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$$

This series is given in the formulae booklet. It is valid for $-1 < x \leqslant 1$ and, if $x > 1$, then $0 < \frac{1}{x^3} < 1$ so the series is valid for this question.

LHS $= \frac{1}{x^3} - \frac{1}{2x^6} + \ldots + \frac{(-1)^{n-1}}{nx^{3n}} + \ldots$, as required

$(-1)^{n+1} = (-1)^{n-1}$. If n is odd, both sides are 1. If n is even, both sides are -1.

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Exercise A, Question 35

Question:

Given that x is so small that terms in x^4 and higher powers of x may be neglected, find the values of the constants A, B, C and D for which

$$e^{-2x}\cos 5x = A + Bx + Cx^2 + Dx^3.$$

Solution:

a $e^{-2x} = 1 + (-2x) + \dfrac{(-2x)^2}{2!} + \dfrac{(-2x)^3}{3!} + \ldots$

$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots$

Substituting $-2x$ for x in the formula $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and ignoring terms in x^4 and higher powers.

$\cos 5x = 1 - \dfrac{(5x)^2}{2!} + \ldots$

$= 1 - \frac{25}{2}x^2 + \ldots$

Substituting $5x$ for x in the formula $\cos x = 1 - \frac{x^2}{2!} + \ldots$ and ignoring terms in x^4 and higher powers.

$e^{-2x}\cos 5x = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\right)\left(1 - \frac{25}{2}x^2 + \ldots\right)$

$= 1 - \frac{25}{2}x^2 - 2x + 25x^3 + 2x^2 - \frac{4}{3}x^3 + \ldots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $2x^2$ by $-\frac{25}{2}x^2$ gives $-25x^4$ and you just ignore his term.

$= 1 - 2x + \left(-\frac{25}{2} + 2\right)x^2 + \left(25 - \frac{4}{3}\right)x^3 + \ldots$

$= 1 - 2x - \frac{21}{2}x^2 + \frac{71}{3}x^3 + \ldots$

$A = 1, B = -2, C = -\frac{21}{2}, D = \frac{71}{3}$

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Exercise A, Question 36

Question:

a Find the first four terms of the expansion, in ascending powers of x, of

$(2x+3)^{-1}$, $|x| < \frac{2}{3}$.

b Hence, or otherwise, find the first four non-zero terms of the expansion, in ascending powers of x, of

$\frac{\sin 2x}{3+2x}$, $|x| < \frac{2}{3}$.

Solution:

a $(2x+3)^{-1} = 3^{-1}\left(1+\frac{2x}{3}\right)^{-1}$

$= \frac{1}{3}\left(1 - \frac{2x}{3} + \frac{(-1)(-2)}{2.1}\left(\frac{2x}{3}\right)^2 + \frac{(-1)(-2)(-3)}{3.2.1}\left(\frac{2x}{3}\right)^3 + \ldots\right)$

$= \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \ldots\right)$

$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots$

Part **a** is a binomial series with a rational index. This is in the C3 specification. The FP2 specification prerequisites states 'A knowledge of the specifications for C1, C2, C3, C4 and FP1, their prerequisites, preambles and associated formulae is assumed and may be tested.' In part **b**, this series is combined with a series in the FP2 specification.

b $\frac{\sin 2x}{3+2x} = \sin 2x(3+2x)^{-1}$

$= \left(2x - \frac{(2x)^3}{3!} + \ldots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots\right)$

$= \left(2x - \frac{4}{3}x^3 + \ldots\right)\left(\frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots\right)$

$= \frac{2}{3}x - \frac{4}{9}x^2 + \frac{8}{27}x^3 - \frac{16}{81}x^4 - \frac{4}{9}x^3 + \frac{8}{27}x^4 + \ldots$

$= \frac{2}{3}x - \frac{4}{9}x^2 + \left(\frac{8}{27} - \frac{4}{9}\right)x^3 + \left(\frac{8}{27} - \frac{16}{81}\right)x^4 + \ldots$

$= \frac{2}{3}x - \frac{4}{9}x^2 - \frac{4}{27}x^3 + \frac{8}{81}x^4 + \ldots$

When multiplying out the brackets, you discard terms in x^4 and higher powers. For example, multiplying $-\frac{4}{3}x^3$ by $\frac{4}{27}x^2$ gives $-\frac{16}{81}x^5$ and you ignore this term.

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Exercise A, Question 37

Question:

a By using the power series expansion for $\cos x$ and the power series expansion for $\ln(1 + x)$, find the series expansion for $\ln(\cos x)$ in ascending powers of x up to and including the term in x^4.

b Hence, or otherwise, obtain the first two non-zero terms in the series expansion for $\ln(\sec x)$ in ascending powers of x.

Solution:

a $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$

$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)$ ①,

neglecting terms above x^4

The expression $-\frac{x^2}{2!} + \frac{x^4}{4!}$ is used to replace the x in the standard series for $\ln(1 + x)$.

$\ln(1 + x) = x - \frac{x^2}{2} + \ldots$

Using the expansion ①

$\ln(\cos x) = \ln\left(1 + \left(-\frac{x^2}{2} + \frac{x^4}{24}\right)\right)$

$= \left(-\frac{x^2}{2} + \frac{x^4}{24}\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 + \ldots$

$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + \ldots$

$= -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$

$-\frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24}\right)^2 = -\frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{1152}$

but, as the expansion is only required up to the term in x^4, you only need the first of the three terms.

b $\ln(\sec x) = \ln\left(\frac{1}{\cos x}\right) = \ln 1 - \ln\cos x$

$= -\ln\cos x$

Using the log rule $\log\left(\frac{a}{b}\right) = \log a - \log b$ and the fact that $\ln 1 = 0$.

Using the result to part **a**

$\ln(\sec x) = -\left(-\frac{x^2}{2} - \frac{x^4}{12} - \ldots\right) = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$

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Exercise A, Question 38

Question:

a Find the Taylor expansion of $\cos 2x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^5$.

b Use your answer to part **a** to obtain an estimate of $\cos 2$, giving your answer to 6 decimal places.

Solution:

a Let $f(x) = \cos 2x$ $\quad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -2\sin 2x \quad f'\left(\frac{\pi}{4}\right) = -2\sin\frac{\pi}{2} = -2$

$f''(x) = -4\cos 2x \quad f''\left(\frac{\pi}{4}\right) = -4\cos\frac{\pi}{2} = 0$

$f'''(x) = 8\sin 2x \quad f'''\left(\frac{\pi}{4}\right) = 8\sin\frac{\pi}{2} = 8$

$f^{(iv)}(x) = 16\cos 2x \quad f^{(iv)}\left(\frac{\pi}{4}\right) = 16\cos\frac{\pi}{2} = 0$

$f^{(v)}(x) = -32\sin 2x \quad f^{(v)}\left(\frac{\pi}{4}\right) = -32\sin\frac{\pi}{2} = -32$

Taylor's and Maclaurin's series need repeated differentiation and substitution. You need to display these in a systematic form, both to help you substitute correctly and to show your working clearly so that the examiner can award you marks.

$f^{(iv)}(x)$ and $f^{(v)}(x)$ are symbols which can be used for the fourth and fifth derivatives of $f(x)$ respectively.

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(iv)}(a) + \frac{(x-a)^5}{5!}f^{(v)}(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \cos 2x$ and $a = \frac{\pi}{4}$

$$\cos 2x = \left(x - \frac{\pi}{4}\right) \times (-2) + \frac{\left(x-\frac{\pi}{4}\right)^3}{6} \times 8 + \frac{\left(x-\frac{\pi}{4}\right)^5}{120} \times (-32) + \ldots$$

$$= -2\left(x - \frac{\pi}{4}\right) + \frac{4}{3}\left(x - \frac{\pi}{4}\right)^3 - \frac{4}{15}\left(x - \frac{\pi}{4}\right)^5 + \ldots$$

All of the even derivatives are zero at $x = \frac{\pi}{4}$.

b Let $x = 1$, then $x - \frac{\pi}{4} = 0.2146\ldots$

Substituting into the result of part **a**

$$\cos 2 = -2(0.2146\ldots) + \frac{4}{3}(0.2146\ldots)^3 - \frac{4}{15}(0.2146\ldots)^5 + \ldots$$

$$\approx -0.416147 \text{ (6 d.p.)}$$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

This is a very accurate estimate and is correct to 6 decimal places.

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Exercise A, Question 39

Question:

a Find the Taylor expansion of $\ln(\sin x)$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^3$.

b Use your answer to part **a** to obtain an estimate of $\ln(\sin 0.5)$, giving your answer to 6 decimal places.

Solution:

a Let $f(x) = \ln(\sin x)$ $\qquad f\left(\frac{\pi}{6}\right) = \ln\frac{1}{2} = -\ln 2$

$f'(x) = \frac{\cos x}{\sin x} = \cot x \qquad f'\left(\frac{\pi}{6}\right) = \cot\frac{\pi}{6} = \sqrt{3}$

$f''(x) = -\operatorname{cosec}^2 x \qquad f''\left(\frac{\pi}{6}\right) = -4$

$\operatorname{cosec}\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$

$f'''(x) = 2\operatorname{cosec}^2 x \cot x \qquad f'''\left(\frac{\pi}{6}\right) = 2 \times 2^2 \times \sqrt{3} = 8\sqrt{3}$

Using the chain rule,

$\frac{d}{dx}(-\operatorname{cosec}^2 x) = -2\operatorname{cosec} x \frac{d}{dx}(\operatorname{cosec} x)$

$= -2\operatorname{cosec} x \times -\operatorname{cosec} x \cot x$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$$

This is the appropriate form of Taylor's series for this question. It is given in the formula booklet.

Substituting $f(x) = \ln(\sin x)$ and $a = \frac{\pi}{6}$

$$\ln(\sin x) = -\ln 2 + \left(x - \frac{\pi}{6}\right) \times \sqrt{3} + \frac{1}{2}\left(x - \frac{\pi}{6}\right)^2 \times (-4) + \frac{1}{6}\left(x - \frac{\pi}{6}\right)^3 \times 8\sqrt{3} + \ldots$$

$$= -\ln 2 + \sqrt{3}\left(x - \frac{\pi}{6}\right) - 2\left(x - \frac{\pi}{6}\right)^2 + \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^3 + \ldots$$

b Let $x = 0.5$, then $x - \frac{\pi}{6} = -0.023\,598\,7\ldots$

Work out $x - \frac{\pi}{4}$ on your calculator and then use the ANS button to complete the calculation.

Substituting into the result of part **a**

$$\ln(\sin 0.5) = -\ln 2 + \sqrt{3}(-0.023\,598\ldots) - 2(-0.023\,598\ldots)^2 + \frac{4\sqrt{3}}{3}(-0.023\,598\ldots)^3 + \ldots$$

$$\approx -0.735166 \text{ (6 d.p.)}$$

Solutionbank FP2
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Exercise A, Question 40

Question:

Given that $y = \tan x$,

a find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

b Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

c Hence show that

$$\tan \frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}.$$

Solution:

a $y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\frac{d^2y}{dx^2} = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \times \sec x \tan x$

$= 2\sec^2 x \tan x$

Using the chain rule for differentiation.

$\frac{d^3y}{dx^3} = \tan x \frac{d}{dx}(2\sec^2 x) + 2\sec^2 x \frac{d}{dx}(\tan x)$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

Using the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = 2\sec^2 x$ and $v = \tan x$.

b Let $y = f(x) = \tan x$

$f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$

Using the results in part **a**

$f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$f''(x) = 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 2 \times (\sqrt{2})^2 \times 1 = 4$

$f'''\left(\frac{\pi}{4}\right) = 4\sec^2\frac{\pi}{4}\tan^2\frac{\pi}{4} + 2\sec^4\frac{\pi}{4}$

$= 4(\sqrt{2})^2 \times 1^2 + 2(\sqrt{2})^4 = 8 + 8 = 16$

$\sec\frac{\pi}{4} = \sqrt{2}$ and $\tan\frac{\pi}{4} = 1$

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \ldots$

This is the first four terms of Taylor's series.

Substituting $f(x) = \tan x$ and $x = \frac{\pi}{4}$

$\tan x = 1 + \left(x - \frac{\pi}{4}\right) \times 2 + \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 \times 4 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3 \times 16 + \ldots$

$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots$

You are expanding $\tan x$ about the point $x = \frac{\pi}{4}$, using Taylor's series.

c Let $x = \frac{3\pi}{10}$, then $x - \frac{\pi}{4} = \frac{3\pi}{10} - \frac{\pi}{4} = \frac{\pi}{20}$

Substituting into the result in part **b**

$\tan\frac{3\pi}{10} = 1 + 2\left(\frac{\pi}{20}\right) + 2\left(\frac{\pi}{20}\right)^2 + \frac{8}{3}\left(\frac{\pi}{20}\right)^3 + \ldots$

$\approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$, as required

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 41

Question:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$$

At $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$.

a Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$.

b Express y as a series in ascending powers of x, up to and including the term in x^3.

Solution:

a $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$ ①

Differentiate ① throughout with respect to x

$$-2x\frac{d^2y}{dx^2} + (1-x^2)\frac{d^3y}{dx^3} - \frac{dy}{dx} - x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \quad ②$$

Substituting $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$ into ②

$$0 + \frac{d^3y}{dx^3} + 1 - 0 - 2 = 0$$

At $x = 0$, $\frac{d^3y}{dx^3} = 1$

Using the product rule for differentiation

$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with

$u = 1 - x^2$ and $v = \frac{d^2y}{dx^2}$,

$$\frac{d}{dx}\left((1-x^2)\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^2y}{dx^2}\frac{d}{dx}(1-x^2) + (1-x^2)\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d^2y}{dx^2} \times -2x + (1-x^2)\frac{d^3y}{dx^3}$$

b Let $y = f(x)$

From the data in the question

$f(0) = 2$, $f'(0) = -1$

At $x = 0$, ① above becomes

$f''(0) + 2 \times 2 = 0 \Rightarrow f''(0) = -4$

And the result to part **a** becomes

$f'''(0) = 1$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

$$y = 2 + x \times (-1) + \frac{x^2}{2} \times (-4) + \frac{x^3}{6} \times 1 + \ldots$$

$$= 2 - x - 2x^2 + \frac{1}{6}x^3 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 42

Question:

$(1 + 2x)\frac{dy}{dx} = x + 4y^2.$

a Show that

$$(1 + 2x)\frac{d^2y}{dx^2} = 1 + 2(4y - 1)\frac{dy}{dx} \qquad ①$$

b Differentiate equation ① with respect to x to obtain an equation involving

$\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}, x$ and y.

Given that $y = \frac{1}{2}$ at $x = 0$,

c find a series solution for y, in ascending powers of x, up to and including the term in x^3.

Solution:

a $(1+2x)\frac{dy}{dx} = x + 4y^2 \quad \star$

> You need to differentiate $4y^2$ implicitly with respect to x.
> $\frac{d}{dx}(4y^2) = \frac{dy}{dx} \times \frac{d}{dy}(4y^2) = 8y\frac{dy}{dx}$.

Differentiate $\star$ throughout with respect to x

$$2\frac{dy}{dx} + (1+2x)\frac{d^2y}{dx^2} = 1 + 8y\frac{dy}{dx}$$

$$(1+2x)\frac{d^2y}{dx^2} = 1 + 8y\frac{dy}{dx} - 2\frac{dy}{dx}$$

$$= 1 + 2(4y-1)\frac{dy}{dx} \quad ① \text{ as required.}$$

b Differentiate ① throughout with respect to x

$$2\frac{d^2y}{dx^2} + (1+2x)\frac{d^3y}{dx^3} = 8\left(\frac{dy}{dx}\right)^2 + 2(4y-1)\frac{d^2y}{dx^2} \ldots \quad ②$$

> When using the product rule for differentiation
> $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with
> $u = 2(4y-1)$ and $v = \frac{dy}{dx}$,
> $2(4y-1)$ must be differentiated implicitly with respect to x. So
> $\frac{d}{dx}\left(2(4y-1)\frac{dy}{dx}\right)$
> $= 8\frac{dy}{dx} \times \frac{dy}{dx} + 2(4y-1)\frac{d}{dx}\left(\frac{dy}{dx}\right)$
> $= 8\left(\frac{dy}{dx}\right)^2 + 2(4y-1)\frac{d^2y}{dx^2}$.

c Let $y = f(x)$
From the data in the question

$$f(0) = \frac{1}{2}$$

At $x = 0$, $y = \frac{1}{2}$, $\star$ becomes

$$f'(0) = 4 \times \left(\frac{1}{2}\right)^2 = 1$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{dy}{dx} = 1$, ① becomes

$$f''(0) = 1 + 2\left(4 \times \frac{1}{2} - 1\right) \times 1 = 3$$

At $x = 0$, $y = \frac{1}{2}$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 3$, ② becomes

$$2 \times 3 + f'''(0) = 8 \times 1^2 + 2\left(4 \times \frac{1}{2} - 1\right) \times 3$$

$$6 + f'''(0) = 8 + 6 \Rightarrow f'''(0) = 8$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

> The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

$$y = \frac{1}{2} + x \times 1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 8 + \ldots$$

$$= \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 43

Question:

$\frac{dy}{dx} = y^2 + xy + x$, $y = 1$ at $x = 0$

a Use the Taylor series method to find y as a series in ascending powers of x, up to and including the term in x^3.

b Use your series to find y at $x = 0.1$, giving your answer to 2 decimal places.

Solution:

a Let $y = \mathrm{f}(x)$

From the data in the question

$\mathrm{f}(0) = 1$

$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + xy + x$ ①

At $x = 0$, $y = 1$, ① becomes

$\mathrm{f}'(0) = 1^2 + 0 + 0 = 1$

Differentiate ① throughout by x

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 1$ ②

At $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, ② becomes

$\mathrm{f}''(0) = 2 \times 1 \times 1 + 1 + 0 + 1 = 4$

Differentiate ② throughout by x

$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ③

At $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4$, ③ becomes

$\mathrm{f}'''(0) = 2 \times 1^2 + 2 \times 1 \times 4 + 1 + 1 + 0 = 12$

$\mathrm{f}(x) = \mathrm{f}(0) + x\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$

$y = 1 + x \times 1 + \frac{x^2}{2} \times 4 + \frac{x^3}{6} \times 12 + \ldots$

$= 1 + x + 2x^2 + 2x^3 + \ldots$

> y^2 has to be differentiated implicitly by x. So
> $\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}}{\mathrm{d}y}(y^2) = \frac{\mathrm{d}y}{\mathrm{d}x} \times 2y$

> Using the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = x$ and $v = y$,
> $\frac{\mathrm{d}}{\mathrm{d}x}(xy) = y\frac{\mathrm{d}x}{\mathrm{d}x} + x\frac{\mathrm{d}y}{\mathrm{d}x} = y \times 1 + x\frac{\mathrm{d}y}{\mathrm{d}x}$.

> Using the product rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$ with $u = 2y$ and $v = \frac{\mathrm{d}y}{\mathrm{d}x}$,
> $\frac{\mathrm{d}}{\mathrm{d}x}\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(2y) + 2y\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$
> $= \frac{\mathrm{d}y}{\mathrm{d}x} \times 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

b At 0.1,

$y = 1 + 0.1 + 2(0.1)^2 + 2(0.1)^3 + \ldots$

$\approx 1 + 0.1 + 0.02 + 0.002 = 1.122$

$y \approx 1.12$ (2 d.p.)

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 44

Question:

$$y\frac{dy}{dx}=\frac{x+3}{y+1}$$

Given that $y = 1.5$ at $x = 0$,

a Use the Taylor series method to find the series solution for y, in ascending powers of x, up to and including the term in x^3.

b Use your result to **a** to estimate, to 3 decimal places, the value of y at $x = 0.1$.

Solution:

a Rearranging the differential equation in the question

$$(y^2 + y)\frac{dy}{dx} = x + 3 \quad ①$$

> The right hand side of the equation in the question would be hard to repeatedly differentiate as a quotient, so multiply both sides by $y + 1$.

Let $y = f(x)$

From the data in the question

$f(0) = 1.5$

At $x = 0$, $y = 1.5$, ① becomes

$$(1.5^2 + 1.5)\,f'(0) = 0 + 3 \Rightarrow f'(0) = \frac{3}{3.75} = 0.8$$

Differentiate ① throughout by x

$$(2y + 1)\left(\frac{dy}{dx}\right)^2 + (y^2 + y)\frac{d^2y}{dx^2} = 1 \quad ②$$

At $x = 0$, $y = 1.5$, $\frac{dy}{dx} = 0.8$, ② becomes

$$4 \times 0.8^2 + (1.5^2 + 1.5)\,f''(0) = 1$$

$$f''(0) = \frac{1 - 4 \times 0.8^2}{3.75} = -0.416$$

Differentiate ② throughout by x

> Differentiating $\left(\frac{dy}{dx}\right)^2$ by x, using the chain rule
>
> $\frac{d}{dx}\left(\left(\frac{dy}{dx}\right)^2\right) = 2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}$.

$$2\left(\frac{dy}{dx}\right)^3 + (2y + 1)\,2 \times \frac{dy}{dx} \times \frac{d^2y}{dx^2} + (2y + 1)\frac{dy}{dx} \times \frac{d^2y}{dx^2} + (y^2 + y)\frac{d^3y}{dx^3} = 0.$$

$$2\left(\frac{dy}{dx}\right)^3 + 3(2y + 1)\frac{dy}{dx}\frac{d^2y}{dx^2} + (y^2 + y)\frac{d^3y}{dx^3} = 0 \quad ③$$

At $x = 0$, $y = 1.5$, $\frac{dy}{dx} = 0.8$, $\frac{d^2y}{dx^2} = -0.416$, ② becomes

$$2 \times 0.8^3 + 3 \times 4 \times 0.8 \times -0.416 + (1.5^2 + 1.5)\,f'''(0) = 0$$

$$1.204 - 3.9936 + 3.75\,f'''(0) = 0$$

$$f'''(0) = \frac{3.9936 - 1.024}{3.75} = 0.79189\dot{3}$$

> This is a recurring decimal. There is an exact fraction $\frac{7424}{9375}$.

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$y = 1.5 + x \times 0.8 + \frac{x^2}{2} \times -0.416 + \frac{x^3}{6} \times 0.79189\dot{3} + \dots$$

$$= 1.5 + 0.8x - 0.208x^2 + 0.13198\dot{2}x^3 + \dots$$

b At 0.1,

$$y = 1.5 + 0.08 - 0.00208 + 0.000\,131\,98 \dots$$

$$\approx 1.578 \text{ (3 d.p.)}$$

> The fourth term is small and this justifies you using the truncated series to make the approximation.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 45

Question:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$

a Find an expression for $\frac{d^3y}{dx^3}$.

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

b find the series solution for y, in ascending powers of x, up to and including the term in x^3.

c Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

Solution:

a $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$ ①

Differentiate ① throughout with respect to x

$$\frac{dy}{dx} \times \frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$y\frac{d^3y}{dx^3} = -3\frac{dy}{dx}\frac{d^2y}{dx^2} - \frac{dy}{dx} = -\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right)$$

$$\frac{d^3y}{dx^3} = -\frac{1}{y}\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right) \quad ②$$

Using the product rule for differentiation

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

with $u = y$ and $v = \frac{d^2y}{dx^2}$,

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right) = \frac{d^2y}{dx^2} \times \frac{dy}{dx} + y \times \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{dy}{dx} \times \frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3}$$

The wording of the question requires you to make $\frac{d^3y}{dx^3}$ the subject of the formula. There are many possible alternative forms for the answer.

b Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 1$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, ① becomes

$1 \times f''(0) + 1^2 + 1 = 0 \Rightarrow f''(0) = -2$

At $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = -2$, ② becomes

$f'''(0) = -\frac{1}{1} \times 1(3 \times -2 + 1) = -1(-6 + 1) = 5$

$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

$y = 1 + x \times 1 + \frac{x^2}{2} \times -2 + \frac{x^3}{6} \times 5 + \ldots$

$= 1 + x - x^2 + \frac{5}{6}x^3 + \ldots$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^3.

c The series expansion up to and including the term in x^3 can be used to estimate y if x is small. So it would be sensible to use it at $x = 0.2$ but not at $x = 50$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 46

Question:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y^2 = 6, \text{ with } y = 1 \text{ and } \frac{dy}{dx} = 0 \text{ at } x = 0.$$

a Use the Taylor series method to obtain y as a series of ascending powers of x, up to and including the term in x^4.

b Hence find the approximate value for y when $x = 0.2$.

Solution:

a $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y^2 = 6$ ①

Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 0$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0$, ① becomes

$f''(0) - 4 \times 0 + 3 \times 1^2 = 6 \Rightarrow f''(0) = 3$

Differentiate ① throughout with respect to x

$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 6y\frac{dy}{dx} = 0$ ②

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 3$, ② becomes

$f'''(0) - 4 \times 3 + 6 \times 1 \times 0 = 0 \Rightarrow f'''(0) = 12$

Differentiate ② throughout with respect to x

$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2} = 0$ ③

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 0,\ \frac{d^2y}{dx^2} = 3,\ \frac{d^3y}{dx^3} = 12$,

③ becomes

$f^{(iv)}(0) - 4 \times 12 + 6 \times 0^2 + 6 \times 1 \times 3 = 0$

$f^{(iv)}(0) = 48 - 18 = 30$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \ldots$

$y = 1 + x \times 0 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times 12 + \frac{x^4}{24} \times 30 + \ldots$

$= 1 + \frac{3}{2}x^2 + 2x^3 + \frac{5}{4}x^4 + \ldots$

$3y^2$ has to be differentiated implicitly with respect to x. So
$\frac{d}{dx}(3y^2) = \frac{dy}{dx} \times \frac{d}{dy}(3y^2) = \frac{dy}{dx} \times 6y$

Using the product rule for differentiation
$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with
$u = 6y$ and $v = \frac{dy}{dx}$,

$\frac{d}{dx}\left(6y\frac{dy}{dx}\right)$
$= \frac{dy}{dx}\frac{d}{dy}(6y) + 6y\frac{d}{dx}\left(\frac{dy}{dx}\right)$
$= 6\left(\frac{dy}{dx}\right)^2 + 6y\frac{d^2y}{dx^2}$.

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

b At $x = 0.2$

$y = 1 + 0.06 + 0.016 + 0.002 + \ldots \approx 1.078$

$y \approx 1.08$ (2 d.p.)

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 47

Question:

Given that

$$f(x) = \ln(1 + \cos 2x), \quad 0 \leqslant x < \frac{\pi}{2},$$

Show that

a $f'(x) = -2 \tan x$

b $f''''(x) = -[f'''(x)\, f'(x) + (f''(x))^2]$.

c Use Maclaurin's theorem to find the expansion of $f(x)$, in ascending powers of x, up to and including the term in x^4.

Solution:

a Let $u = 1 + \cos 2x$, then $\mathrm{f}(x) = \ln u$

$$\frac{du}{dx} = -2\sin 2x$$

$$\mathrm{f}'(x) = \mathrm{f}'(u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{1+\cos 2x} \times -2\sin 2x$$

$$= \frac{-4\sin x\cos x}{2\cos^2 x}$$

$$= \frac{-2\sin x}{\cos x} = -2\tan x, \text{ as required}$$

Using the identities $\sin 2x = 2\sin x\cos x$ and $\cos 2x = 2\cos^2 x - 1$.

b $\mathrm{f}''(x) = -2\sec^2 x$

$$\mathrm{f}'''(x) = -4\sec^2 x\tan x$$

$$\mathrm{f}''''(x) = -8\sec x \,.\, \sec x\tan x \,.\, \tan x - 4\sec^2 x \,.\, \sec^2 x$$

$$= -8\sec^2 x\tan^2 x - 4\sec^4 x$$

$$= -\left[-4\sec^2 x\tan x \times -2\tan x + (-2\sec^2 x)^2\right]$$

$$= -\left[\mathrm{f}'''(x)\,\mathrm{f}'(x) + (\mathrm{f}''(x))^2\right], \text{ as required}$$

$\mathrm{f}''''(x)$ is a symbol used for the fourth derivative of $\mathrm{f}(x)$ with respect to x. The symbol $\mathrm{f}^{(iv)}(x)$ is also used for the fourth derivative.

You use the product rule for differentiation $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$ with $u = -4\sec^2 x$ and $v = \tan x$.
You also use the chain rule $\frac{d}{dx}(\sec^2 x) = 2\sec x\frac{d}{dx}(\sec x)$
$= 2\sec x \times \sec x\tan x$.

c $\mathrm{f}(0) = \ln(1 + \cos 0) = \ln 2$

$$\mathrm{f}'(0) = -2\tan 0 = 0$$

$$\mathrm{f}''(0) = -2\sec^2 0 = -2$$

$$\mathrm{f}'''(0) = -4\sec^2 0\tan 0 = 0$$

$$\mathrm{f}''''(0) = -\left[\mathrm{f}'''(0)\,\mathrm{f}'(0) + (\mathrm{f}''(0))^2\right]$$

$$= -\left[0 \times 0 + (-2)^2\right] = -4$$

Using the result for part **b**.

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \frac{x^4}{4!}\mathrm{f}^{(iv)}(0) + \ldots$$

$$= \ln 2 + x \times 0 + \frac{x^2}{2} \times -2 + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times -4 + \ldots$$

$$= \ln 2 - x^2 - \frac{1}{6}x^4 + \ldots$$

The formula for Maclaurin's series is given in the formulae booklet. For this question, you need the terms up to and including the term in x^4.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 48

Question:

a Use the Taylor series method to obtain a solution in a series of ascending powers of x, up to and including the term in x^4, of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{x^2},$$

given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$.

b Working to a least 4 decimal places, use the series obtained in part **a** to obtain the value of y at **i** $x = 0.1$, **ii** $x = 0.2$.

c By differentiating the series obtained for y, obtain estimates for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0.1$.

Solution:

a $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{x^2}$ ①

Let $y = f(x)$

From the data in the question

$f(0) = 1,\ f'(0) = 1$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1$, ① becomes

$f''(0) - 3 \times 1 + 2 \times 1 = e^0 = 1$

$f''(0) = 1 + 3 - 2 = 2$

Differentiate ① throughout with respect to x

$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 2x\,e^{x^2}$ ②

$\frac{d}{dx}\left(e^{x^2}\right) = \frac{d}{dx}(x^2) \times e^{x^2} = 2x\,e^{x^2}$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = 2$, ② becomes

$f'''(0) - 3 \times 2 + 2 \times 1 = 0$

$f'''(0) = 6 - 2 = 4$

Differentiate ② throughout with respect to x

$\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 2\,e^{x^2} + 4x^2\,e^{x^2}$ ③

$\frac{d}{dx}\left(2x\,e^{x^2}\right) = e^{x^2}\frac{d}{dx}(2x) + 2x\frac{d}{dx}\left(e^{x^2}\right)$

At $x = 0,\ y = 1,\ \frac{dy}{dx} = 1,\ \frac{d^2y}{dx^2} = 2,\ \frac{d^3y}{dx^3} = 4,$

③ becomes

$f^{(iv)}(0) - 3 \times 4 + 2 \times 2 = 2 + 0 \Rightarrow f^{(iv)}(0) = 10$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$

$y = 1 + x \times 1 + \frac{x^2}{2} \times 2 + \frac{x^3}{6} \times 4 + \frac{x^4}{24} \times 10 + \dots$

$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \dots$

b i At $x = 0.1$

$$y = 1 + 0.1 + 0.01 + 0.000\,666\ldots + 0.000\,041\ldots$$
$$\approx 1.110\,708 = 1.1107 \text{ (4 d.p.)}$$

ii At $x = 0.2$

$$y = 1 + 0.2 + 0.04 + 0.005\,333\ldots + 0.000\,666\ldots$$
$$\approx 1.2460 \text{ (4 d.p.)}$$

As x gets larger, the approximation gets less accurate, so the answer to **ii** will be less accurate than the answer to **i**. In this case the value at 0.1 is accurate to 4 decimal paces. The approximation at 0.2 is a very good one but the accurate answer, 1.246 064…, is 1.2641 to 4 decimal places.

c $y = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \ldots$

Differentiating term by term

$$\frac{dy}{dx} = 1 + 2x + 2x^2 + \frac{5}{3}x^3 + \ldots$$

At $x = 0.1$

$$\frac{dy}{dx} = 1 + 0.2 + 0.02 + 0.001\,666\ldots$$
$$\approx 1.222 \text{ (3 d.p.)}$$

$$\frac{d^2y}{dx^2} = 2 + 4x + 5x^2 + \ldots$$

At $x = 0.1$

$$\frac{d^2y}{dx^2} = 2 + 0.4 + 0.05 + \ldots$$
$$\approx 2.45 \text{ (2 d.p.)}$$

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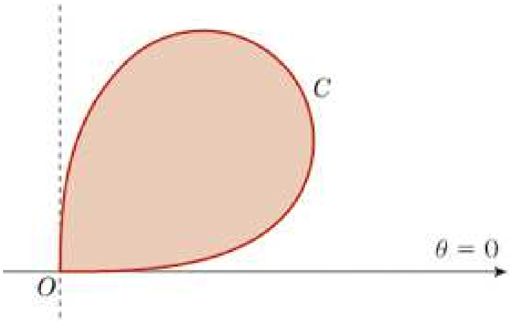
Exercise A, Question 49

Question:



The figure shows a sketch of the curve C with polar equation

$$r^2 = a^2 \sin 2\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

where a is a constant.

Find the area of the shaded region enclosed by C.

Solution:

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta$$

You need to know the formula for the area of polar curves $A = \frac{1}{2}\int r^2 \, d\theta$. In this question, the diagram shows that the limits are 0 and $\frac{\pi}{2}$.

$$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int a^2 \sin 2\theta \, d\theta$$

$$= \frac{a^2}{2}\left[-\frac{\cos 2\theta}{2}\right]$$

$$A = \frac{a^2}{4}\left[-\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{4}\left[1 - (-1)\right]$$

$\cos\left(2 \times \frac{\pi}{2}\right) = \cos \pi = -1$ and $\cos 0 = 1$.

$$= \frac{1}{2}a^2$$

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Exercise A, Question 50

Question:

Relative to the origin O as pole and initial line $\theta = 0$, find an equation in polar coordinate form for

a a circle, centre O and radius 2,

b a line perpendicular to the initial line and passing through the point with polar coordinates (3, 0).

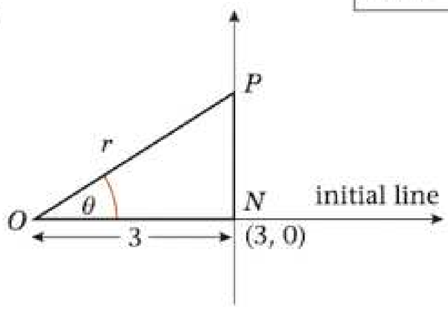
c a straight line through the points with polar coordinates (4, 0) and $\left(4, \frac{\pi}{3}\right)$.

Solution:

a $r = 2$

You can just write the answer to part **a** down. The equation $r = k$ is the equation of a circle centre O and radius k, for any positive k.

b



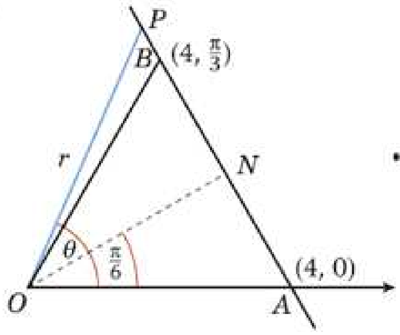
For any point P on the line

$$\frac{3}{r} = \cos\theta$$

$$r = \frac{3}{\cos\theta} = 3\sec\theta$$

If the point (3, 0) is labelled N, trigonometry on the right-angled triangle ONP gives the polar equation of the line.

c



In this diagram, the point (4, 0) is labelled A, the point $\left(4, \frac{\pi}{3}\right)$ is labelled B and the foot of the perpendicular from O to AB is labelled N. The triangle OAB is equilateral and $\angle AON = \frac{1}{2} \times 60° = 30° = \frac{\pi}{6}$ radians.

In the triangle ONA

$$\frac{ON}{OA} = \frac{ON}{4} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$ON = 2\sqrt{3}$$

In the triangle ONP,

$$\frac{ON}{OP} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$\frac{2\sqrt{3}}{r} = \cos\left(\theta - \frac{\pi}{6}\right)$$

$$r = 2\sqrt{3}\sec\left(\theta - \frac{\pi}{6}\right)$$

This relation is true for any point P on the line and, as $OP = r$ this gives you the polar equation of the line.

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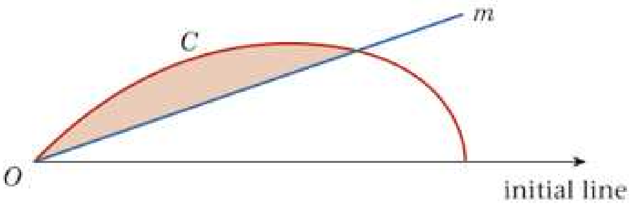
Exercise A, Question 51

Question:



The figure shows a curve C with polar equation $r = 4a\cos 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$, and a line m with polar equation $\theta = \frac{\pi}{8}$. The shaded region, shown in the figure, is bounded by C and m. Use calculus to show that the area of the shaded region is $\frac{1}{2}a^2(\pi - 2)$.

Solution:

$$A = \frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} r^2 \, d\theta$$

The lower limit, $\frac{\pi}{8}$, is given by the polar equation of m. The upper limit, $\frac{\pi}{4}$, can be identified from the domain of definition, $0 \leqslant \theta \leqslant \frac{\pi}{4}$ given in the question and the diagram.

$$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int 16a^2\cos^2 2\theta \, d\theta$$

$$= 8a^2 \int \cos^2 2\theta \, d\theta$$

$$= 8a^2 \int \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$$= 4a^2 \left[\theta + \frac{\sin 4\theta}{4}\right]$$

$$A = 4a^2 \left[\theta + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= 4a^2 \left[\left(\frac{\pi}{4} - \frac{\pi}{8}\right) + \left(0 - \frac{1}{4}\right)\right]$$

$\sin\left(4 \times \frac{\pi}{4}\right) = \sin \pi = 0$ and $\sin\left(4 \times \frac{\pi}{8}\right) = \sin\frac{\pi}{2} = 1$.

$$= 4a^2 \left[\frac{\pi}{8} - \frac{1}{4}\right]$$

$$= \frac{1}{2}a^2(\pi - 2)$$

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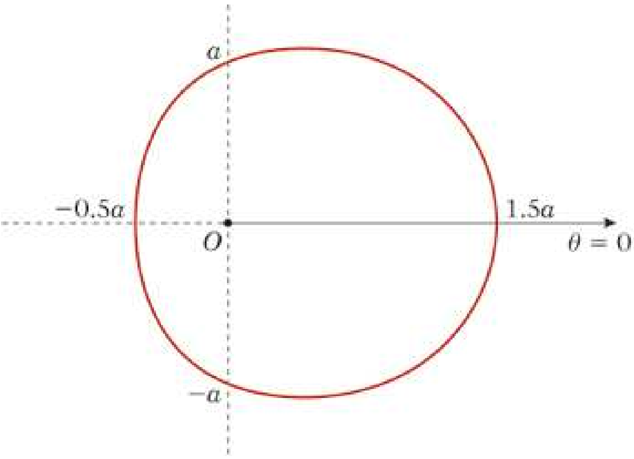
Exercise A, Question 52

Question:



The curve shown in the figure has polar equation

$r = a\left(1 + \frac{1}{2}\cos\theta\right), \quad a > 0, \quad 0 < \theta \leqslant 2\pi.$

Determine the area enclosed by the curve, giving your answer in terms of a and π.

Solution:

$$A = 2 \times \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$$

The method used here is to find twice the area above the initial line.

$$= \int_0^{\pi} a^2\left(1 + \frac{1}{2}\cos\theta\right)^2 d\theta$$

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{4}\cos^2\theta\right) d\theta$$

Use $\cos 2\theta = 2\cos^2\theta - 1$.

$$= a^2\int_0^{\pi}\left(1 + \cos\theta + \frac{1}{8}\cos 2\theta + \frac{1}{8}\right) d\theta$$

$$= a^2\int_0^{\pi}\left(\frac{9}{8} + \cos\theta + \frac{1}{8}\cos 2\theta\right) d\theta$$

$$= a^2\left[\frac{9}{8}\theta + \sin\theta + \frac{\sin 2\theta}{16}\right]_0^{\pi}$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{9}{8}\theta$, which has a non-zero value at π.

$$= a^2 \times \frac{9}{8}\pi = \frac{9}{8}\pi a^2$$

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Exercise A, Question 53

Question:

a Sketch the curve with polar equation

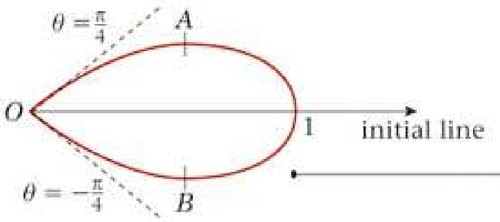
$$r = \cos 2\theta,\ -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$$

At the distinct points A and B on this curve, the tangents to the curve are parallel to the initial line, $\theta = 0$.

b Determine the polar coordinates of A and B, giving your answers to 3 significant figures.

Solution:

a



At $\theta = -\frac{\pi}{4}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $\cos 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

b $y = r\sin\theta = \cos 2\theta \sin\theta$

$$\frac{dy}{d\theta} = -2\sin 2\theta \sin\theta + \cos 2\theta \cos\theta = 0$$

$$-4\sin\theta\cos\theta\sin\theta + (1 - 2\sin^2\theta)\cos\theta = 0$$

$$\cos\theta(-4\sin^2\theta + 1 - 2\sin^2\theta) = 0$$

At A and B, $\cos\theta \neq 0$

$$6\sin^2\theta = 1$$

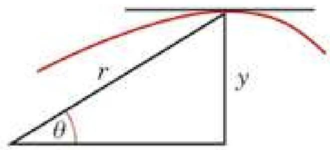
$$\sin\theta = \pm\frac{1}{\sqrt{6}}$$

$$\theta = \pm 0.420\,534\ldots$$

$$r = \cos 2\theta = 1 - 2\sin^2\theta = 1 - \tfrac{2}{6} = \tfrac{2}{3}$$

To 3 significant figures, the polar coordinates of A and B are

(0.667, 0.421) and (0.667, −0.421).



Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. The diagram above shows that $y = r\sin\theta$. You find the polar coordinates θ of the points by finding the values of θ for which $r\sin\theta$ has a maximum or minimum value.

r has an exact value but the question specifically asks for 3 significant figures. Unless the questions specifies otherwise, in polar coordinates, you should always give the value of the angle in radians.

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Exercise A, Question 54

Question:

a Sketch the curve with polar equation

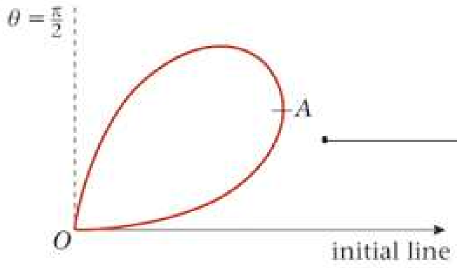
$r = \sin 2\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{2}$

At the point A, where A is distinct from O, on this curve, the tangent to the curve is parallel to $\theta = \frac{\pi}{2}$.

b Determine the polar coordinates of the point A, giving your answer to 3 significant figures.

Solution:

a



At $\theta = 0$, $r = 0$. As θ increases, r increases until $\theta = \frac{\pi}{4}$. For $\theta = \frac{\pi}{4}$, $\sin 2\theta$ has its greatest value of 1. After that, as θ increases, r decreases to $\sin\left(2\times\frac{\pi}{2}\right) = \sin\pi = 0$ at $\theta = \frac{\pi}{2}$.

b $x = r\cos\theta = \sin 2\theta\cos\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 2(2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 4\cos^3\theta - 2\cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 6\cos^3\theta - 4\cos\theta = 0$

$2\cos\theta(3\cos^2\theta - 2) = 0$

At A, $\cos\theta \neq 0$

$\cos^2\theta = \frac{2}{3}$

$\cos\theta = \left(\frac{2}{3}\right)^{\frac{1}{2}}$, for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

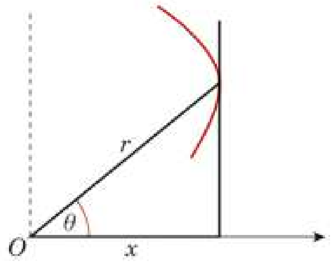
$\theta = 0.615\,479\ldots$

By calculator

$r = \sin 2\theta = 0.942\,809\ \ldots$

To 3 significant figures, the coordinates of A are

$(0.943, 0.615)$



Where the tangent at a point is parallel to $\theta = \frac{\pi}{2}$ (which is the same as being perpendicular to the initial line), the distance x from the half line $\theta = \frac{\pi}{2}$ has a stationary value. The diagram above shows that $x = r\cos\theta$. You find the polar coordinates θ of such points by finding the values of θ for which $r\cos\theta$ has a maximum or minimum value.

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Exercise A, Question 55

Question:

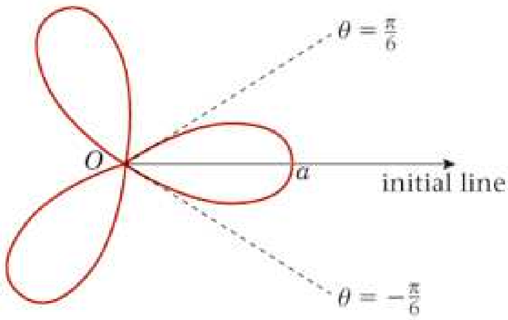
a Sketch the curve with polar equation

$r = a\cos 3\theta, \quad 0 \leqslant \theta < 2\pi$

b Find the area enclosed by one loop of this curve.

Solution:

a



At $\theta = -\frac{\pi}{6}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $a\cos 6\theta$ has its greatest value of a. Then, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{6}$. Between $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{2}$, $\cos 6\theta$ is negative and, as $r \geqslant 0$, the curve does not exist. The pattern repeats itself in the other intervals where the curve exists.

b $A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} r^2\,d\theta$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 3\theta$.

$$\frac{1}{2}\int a^2\cos^2 3\theta\,d\theta = \frac{a^2}{2}\int\left(\frac{1}{2}\cos 6\theta + \frac{1}{2}\right)d\theta$$

$$= \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]$$

$\sin\left(6 \times \frac{\pi}{6}\right) = \sin\pi = 0$

$$A = \frac{a^2}{4}\left[\frac{\sin 6\theta}{6} + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{a^2}{4}\left[\frac{1}{6}(0 - 0) + \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right]$$

$$= \frac{a^2}{4} \times \frac{\pi}{3} = \frac{\pi}{12}a^2$$

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Exercise A, Question 56

Question:

The curve C has polar equation

$r = 6\cos\theta, \qquad -\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2},$

and the line D has polar equation

$r = 3\sec\left(\frac{\pi}{3} - \theta\right), \qquad -\frac{\pi}{6} \leqslant \theta < \frac{5\pi}{6}$

a Find a Cartesian equation of C and a Cartesian equation of D.

b Sketch on the same diagram the graphs of C and D, indicating where each cuts the initial line.

The graphs of C and D intersect at the points P and Q.

c Find the polar coordinates of P and Q.

Solution:

a $r = 6\cos\theta$

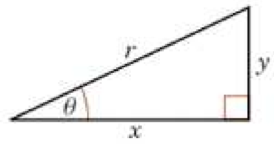
Multiplying the equation by r

$r^2 = 6r\cos\theta$

$x^2 + y^2 = 6x$

$x^2 - 6x + 9 + y^2 = 9$

$(x-3)^2 + y^2 = 9$



This diagram illustrates the relations between polar and Cartesian coordinates. The relations you need to solve the question are

$r^2 = x^2 + y^2$,

$x = r\cos\theta$ and $y = r\sin\theta$.

This is an acceptable answer but putting the equation into a form which shows that the curve is a circle, centre (3, 0) and radius 3, helps you to draw the sketch in part **b**.

$r = 3\sec\left(\frac{\pi}{3} - \theta\right)$

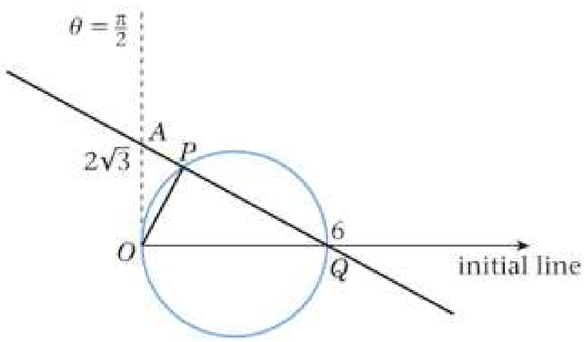
$$3 = r\cos\left(\frac{\pi}{3} - \theta\right) = r\cos\frac{\pi}{3}\cos\theta + r\sin\frac{\pi}{3}\sin\theta$$

$$= \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$$

$$= \frac{1}{2}x + \frac{\sqrt{3}}{2}y$$

$x + \sqrt{3}y = 6$

b



The initial line is the positive x-axis and the half-line $u = \frac{\pi}{2}$ is the positive y-axis. At $x = 0$, $x + \sqrt{3}y = 6$ gives $y = \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

c By inspection, the polar coordinates of Q are (6, 0)

The question does not say which point is P and which is Q. You can choose which is which.

$\angle OPQ = 90°$

The angle in a semi-circle is a right angle.

In the triangle OAQ

$$\tan AQO = \frac{OA}{OQ} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} \Rightarrow \angle AQO = 30°$$

In the triangle OPQ

$OP = OQ\sin PQO = 6\sin 30° = 3$

$$\angle POQ = 180° - 90° - 30° = 60° = \frac{\pi}{3}$$

Hence the polar coordinates of P are

$$(OP, \angle POQ) = \left(3, \frac{\pi}{3}\right)$$

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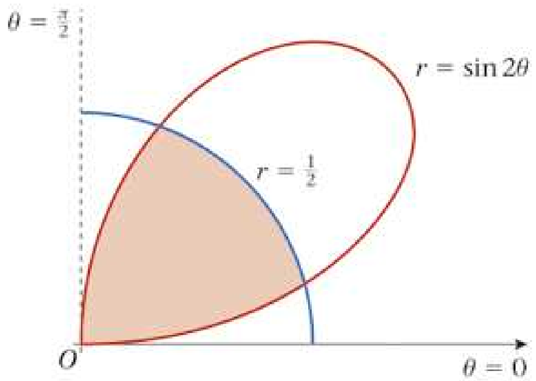
Exercise A, Question 57

Question:



The figure show the half lines $\theta = 0$, $\theta = \frac{\pi}{2}$ and the curves with polar equations

$r = \frac{1}{2}$, $\quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$,

$r = \sin 2\theta$, $\quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

a Find the exact values of θ at the two points where the curves cross.

b Find by integration the area of the shaded region, shown in the figure, which is bounded by both curves.

Solution:

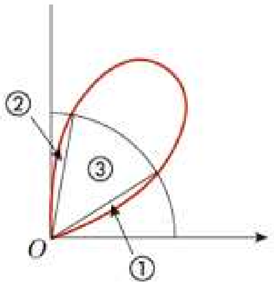
a The curves intersect at

$$\frac{1}{2} = \sin 2\theta$$

$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

b



The shaded area can be broken up into three parts. You can find the small areas labelled ① and ②, which are equal in area, by integration. The larger area is a sector of a circle and you find this using $A = \frac{1}{2}r^2\theta$, where θ is in radians.

The area of the sector ③ is given by

$$A_3 = \frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times \frac{\pi}{3} = \frac{\pi}{24}$$

The radius of the sector is $\frac{1}{2}$ and the angle is $\frac{5\pi}{12} - \frac{\pi}{12} = \frac{\pi}{3}$.

The area of ① is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{12}} r^2 \, d\theta$$

Using $\cos 2A = 1 - 2\sin^2 A$ with $A = 2\theta$.

$$\frac{1}{2}\int \sin^2 2\theta \, d\theta = \frac{1}{2}\int\left(\frac{1}{2} - \frac{1}{2}\cos 4\theta\right)d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$A_1 = \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{12}}$$

$\sin\left(4 \times \frac{\pi}{12}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$= \frac{1}{4}\left[\frac{\pi}{12} - 0 - \frac{1}{4}\left(\frac{\sqrt{3}}{2} - 0\right)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right]$$

The area of the shaded region is given by

$$2 \times A_1 + A_3 = \frac{1}{2}\left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \frac{\pi}{24} = \frac{\pi}{12} - \frac{\sqrt{3}}{16}$$

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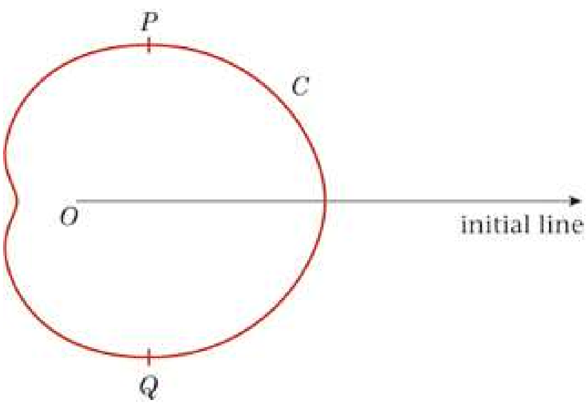
Exercise A, Question 58

Question:



The curve C, shown in the figure, has polar equation

$r = a(3 + \sqrt{5}\cos\theta),\ -\pi \leqslant \theta < \pi$

a Find the polar coordinates of the points P and Q where the tangents to C are parallel to the initial line.

The curve C represents the perimeter of the surface of a swimming pool. The direct distance from P to Q is 20 m.

b Calculate the value of a.

c Find the area of the surface of the pool.

Solution:

a Let $y = r\sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of the point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

$y = a(3 + \sqrt{5}\cos\theta)\sin\theta$

$= 3a\sin\theta + \sqrt{5}a\cos\theta\sin\theta = 3a\sin\theta + \frac{\sqrt{5}a}{2}\sin 2\theta$

$\frac{dy}{d\theta} = 3a\cos\theta + \sqrt{5}a\cos 2\theta = 0$

$3\cos\theta + \sqrt{5}(2\cos^2\theta - 1) = 0$

$2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$

$\cos\theta = \frac{-3 \pm \sqrt{(9+40)}}{4\sqrt{5}}$

As $|\cos\theta| \leqslant 1$, you reject the value $-\frac{10}{4\sqrt{5}} \approx -1.118$.

$= \frac{-3+7}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$

By calculator

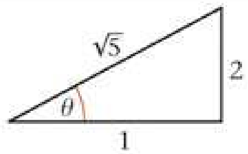
$\theta = \pm 1.107$ (3 d.p.)

At $\cos\theta = \frac{1}{\sqrt{5}}$

$r = a(3 + \sqrt{5}\cos\theta) = a\left(3 + \sqrt{5} \times \frac{1}{\sqrt{5}}\right) = 4a$

The polar coordinates are

$P:(4a, 1.107)$, $Q:(4a, -1.107)$



As $1^2 + 2^2 = (\sqrt{5})^2$, the diagram illustrates that if $\cos\theta = \frac{1}{\sqrt{5}}$ then $\sin\theta = \frac{2}{\sqrt{5}}$.

b $PQ = 2y = 2r\sin\theta$

$= 2 \times 4a \times \frac{2}{\sqrt{5}} = \frac{16}{\sqrt{5}}a = 20\,\text{m}$, given

$a = \frac{20\sqrt{5}}{16}\,\text{m} = \frac{5\sqrt{5}}{4}\,\text{m}$

c $A = 2 \times \frac{1}{2}\int_0^{\pi} r^2\,d\theta$

The method used here is to find twice the area above the initial line.

$\int a^2(3 + \sqrt{5}\cos\theta)^2\,d\theta = \int a^2(9 + 6\sqrt{5}\cos\theta + 5\cos^2\theta)\,d\theta$

$= a^2\int\left(9 + 6\sqrt{5}\cos\theta + \frac{5}{2}\cos 2\theta + \frac{5}{2}\right)d\theta$

Using $\cos 2\theta = 2\cos^2\theta - 1$.

$= a^2\int\left(\frac{23}{2} + 6\sqrt{5}\cos\theta + \frac{5}{2}\cos 2\theta\right)d\theta$

$= a^2\left[\frac{23}{2}\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta\right]$

$A = a^2\left[\frac{23}{2}\theta + 6\sqrt{5}\sin\theta + \frac{5}{4}\sin 2\theta\right]_0^{\pi} = \frac{23\pi}{2}a^2$

You use the value of a you found in part **b**.

$= \frac{23\pi}{2}\left(\frac{5\sqrt{5}}{4}\right)^2\text{m}^2 = \frac{2875\pi}{32}\text{m}^2 \approx 282\,\text{m}^2$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 59

Question:

a Sketch the curve with polar equation

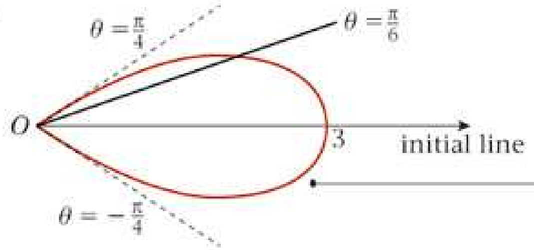
$$r = 3\cos 2\theta,\ -\frac{\pi}{4} \leqslant \theta < \frac{\pi}{4}$$

b Find the area of the smaller finite region enclosed between the curve and the half-line $\theta = \frac{\pi}{6}$.

c Find the exact distance between the two tangents which are parallel to the initial line.

Solution:

a



At $\theta = -\frac{\pi}{4}$, $r = 0$. As θ increases, r increases until $\theta = 0$. For $\theta = 0$, $3\cos 2\theta$ has its greatest value of 3. After that, as θ increases, r decreases to 0 at $\theta = \frac{\pi}{4}$.

b $A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2 \, d\theta$

$\frac{1}{2}\int r^2 \, d\theta = \frac{1}{2}\int 9\cos^2 2\theta \, d\theta$

$= \frac{9}{2}\int\left(\frac{\cos 4\theta}{2} + \frac{1}{2}\right)d\theta = \frac{9}{4}\int(\cos 4\theta + 1)\,d\theta$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = 2\theta$.

$= \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]$

$A = \frac{9}{4}\left[\frac{\sin 4\theta}{4} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{9}{4}\left[\frac{1}{4}\left(0 - \frac{\sqrt{3}}{2}\right) + \left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right]$

$\sin\left(4 \times \frac{\pi}{6}\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$= -\frac{9\sqrt{3}}{32} + \frac{3\pi}{16} = \frac{3}{32}(2\pi - 3\sqrt{3})$

c Let $y = r\sin\theta = 3\cos 2\theta \sin\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinate θ of such a point by finding the value of θ for which $y = r\sin\theta$ has a stationary value.

$\frac{dy}{d\theta} = -6\sin 2\theta \sin\theta + 3\cos 2\theta\cos\theta = 0$

$2\sin 2\theta \sin\theta = \cos 2\theta \cos\theta$

$\frac{\sin 2\theta \sin\theta}{\cos 2\theta\cos\theta} = \tan 2\theta \tan\theta = \frac{1}{2}$

$\frac{2\tan^2\theta}{1 - \tan^2\theta} = \frac{1}{2}$

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$.

$4\tan^2\theta = 1 - \tan^2\theta$

$5\tan^2\theta = 1$

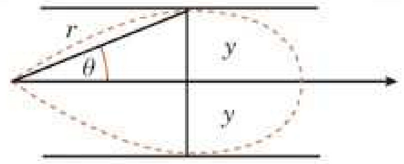
$\tan\theta = \frac{1}{\sqrt{5}}$

One value of $\tan\theta$ is sufficient to complete the question. r is not needed.

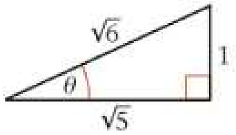
The distance between the two tangents is given by
$2y = 2r\sin\theta = 6\cos 2\theta \sin\theta = 6(2\cos^2\theta - 1)\sin\theta$

$= 6 \times \left(2 \times \frac{5}{6} - 1\right) \times \frac{1}{\sqrt{6}} = 6 \times \frac{2}{3} \times \frac{1}{\sqrt{6}}$

$= \frac{2\sqrt{6}}{3}$



This sketch shows you that the distance between the two tangents parallel to the initial line is given by $2y = 2r\sin\theta$.

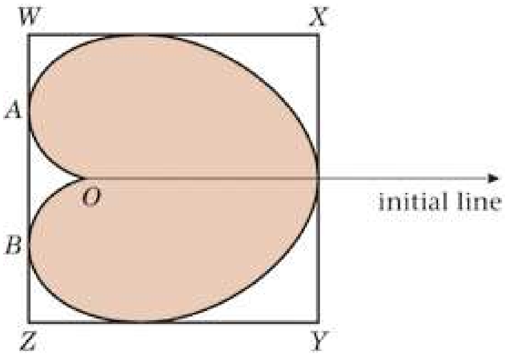


As $(\sqrt{5})^2 + 1^2 = (\sqrt{6})^2$, if $\tan\theta = \frac{1}{\sqrt{5}}$, then $\sin\theta = \frac{1}{\sqrt{6}}$ and $\cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 60

Question:



The figure shows a sketch of the cardioid C with equation $r = a(1 + \cos\theta)$, $-\pi < \theta \leqslant \pi$.
Also shown are the tangents to C that are parallel and perpendicular to the initial line.
These tangents form a rectangle $WXYZ$.

a Find the area of the finite region, shaded in the figure, bounded by the curve C.

b Find the polar coordinates of the points A and B where WZ touches the curve C.

c Hence find the length of WX.

Given that the length of WZ is $\frac{3\sqrt{3}a}{2}$,

d find the area of the rectangle $WXYZ$.

A heart-shape is modelled by the cardioid C, where $a = 10$ cm. The heart shape is cut from the rectangular card $WXYZ$, shown the figure.

e Find a numerical value for the area of card wasted in making this heart shape.

Solution:

a $A = 2 \times \frac{1}{2}\int_0^{\pi} r^2\,d\theta$

The total area is twice the area above the initial line.

$$\int r^2\,d\theta = \int a^2(1+\cos\theta)^2\,d\theta = \int a^2(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A = a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi} = \frac{3}{2}\pi a^2$$

As $\sin\pi = \sin 2\pi = 0$ and $\sin 0 = 0$, all of the terms are zero at both the lower and the upper limit except for $\frac{3}{2}\theta$, which has a non-zero value at π.

b Let $x = r\cos\theta$

$$= a(1+\cos\theta)\cos\theta = a\cos\theta + a\cos^2\theta$$

When the tangent at a point is perpendicular to the initial line, you find the polar coordinates θ of the points by finding any values of θ for which $r\cos\theta$ has a stationary value.

$$\frac{dx}{d\theta} = -a\sin\theta - 2a\sin\theta\cos\theta = 0$$

$$\sin\theta(2\cos\theta+1) = 0$$

$$\cos\theta = -\frac{1}{2}$$

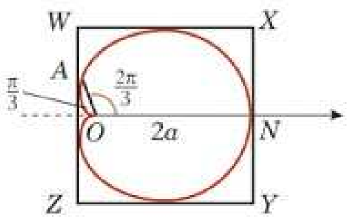
$\sin\theta = 0$ corresponds to the point where XY cuts the curve C and can be rejected as a solution to part **b**.

$$\theta = \pm\frac{2\pi}{3}$$

At A and B

$$r = a(1+\cos\theta) = a\left(1-\frac{1}{2}\right) = \frac{1}{2}a$$

$$A{:}\left(\frac{1}{2}a, \frac{2\pi}{3}\right),\ B{:}\left(\frac{1}{2}a, -\frac{2\pi}{3}\right)$$



This sketch illustrates the geometry which is used to solve parts **c** and **d**.

c $WX = AO\cos\frac{\pi}{3} + ON$

$$= \frac{1}{2}a \times \frac{1}{2} + 2a = \frac{9}{4}a$$

d Area of rectangle $WXYZ$ is given by

$$WX \times WZ = \frac{9}{4}a \times \frac{3\sqrt{3}}{2}a = \frac{27\sqrt{3}}{8}a^2$$

e The area wasted is given by

The area wasted is the answer to part **a** subtracted from the answer to part **d**.

$$\frac{27\sqrt{3}}{8}a^2 - \frac{3}{2}\pi a^2 = \left(\frac{27\sqrt{3}}{8} - \frac{3\pi}{2}\right)a^2 = \left(\frac{27\sqrt{3}}{8} - \frac{3\pi}{2}\right)10^2\,\text{cm}^2$$

$$= 113\,\text{cm}^2\ (3\text{ s.f.})$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 61

Question:

a Sketch, on the same diagram, the curves defined by the polar equations $r = a$ and $r = a(1 + \cos\theta)$, where a is a positive constant and $-\pi < \theta \leqslant \pi$.

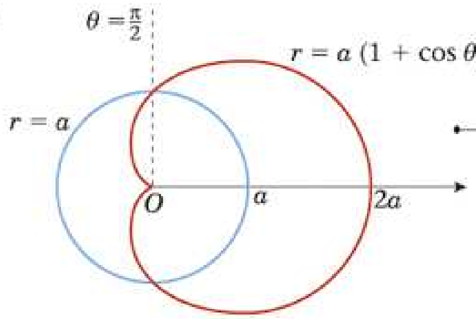
b By considering the stationary values of $r\sin\theta$, or otherwise, find equations of the tangents to the curve $r = a(1 + \cos\theta)$ which are parallel to the initial line.

c Show that the area of the region for which

$$a < r < a(1 + \cos\theta) \text{ is } \frac{(\pi + 8)a^2}{4}.$$

Solution:

a



$r = a(1 + \cos\theta)$ is a cardioid and $r = a$ is a circle centre O, radius a.

b Let $y = r\sin\theta = a(1+\cos\theta)\sin\theta$

$= a\sin\theta + a\cos\theta\sin\theta = a\sin\theta + \frac{a}{2}\sin 2\theta$

Where the tangent at a point is parallel to the initial line, the distance y from the point to the initial line has a stationary value. You find the polar coordinates θ of such points by finding the values of θ for which $y = r\sin\theta$ has stationary values.

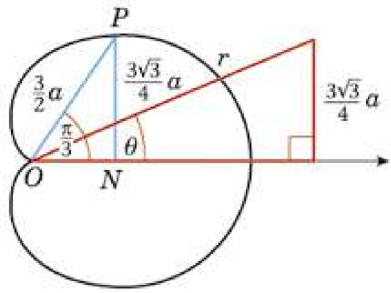
$\frac{dy}{d\theta} = a\cos\theta + a\cos 2\theta = 0$

$\cos 2\theta + \cos\theta = 2\cos^2\theta - 1 + \cos\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}, \cos\theta = -1$

$\theta = \pm\frac{\pi}{3}, \theta = \pi$



At $\theta = \frac{\pi}{3}$,

$r = a\left(1+\cos\frac{\pi}{3}\right) = a\left(1+\frac{1}{2}\right) = \frac{3}{2}a$

And $y = r\sin\frac{\pi}{3} = \frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a$

You find the distance (labelled PN in the diagram above) from the point where the tangent meets the curve to the initial line.

The polar equation of the tangent is given by

$r\sin\theta = \frac{3\sqrt{3}}{4}a$

The polar equation is found by trigonometry in the triangle marked in red on the diagram above.

$r = \frac{3\sqrt{3}a}{4}\operatorname{cosec}\theta$

Similarly at $\theta = -\frac{\pi}{3}$, the equation of the tangent is $r = -\frac{3\sqrt{3}a}{4}\operatorname{cosec}\theta$.

At $\theta = \pi$, the equation of the tangent is

$\theta = \pi$.

It is easy to overlook this case. The half-line $\theta = \pi$ does touch the cardioid at the pole.

: The circle and the cardioid meet when

$a = a(1 + \cos\theta) \Rightarrow \cos\theta = \theta$

$\theta = \pm\frac{\pi}{2}$

To find the area of the cardioid between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$

$A = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta$

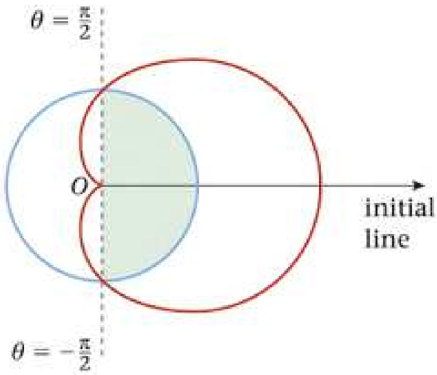
The total area is twice the area above the initial line.

$$\int r^2\,d\theta = \int a^2(1 + \cos\theta)^2\,d\theta = \int a^2(1 + 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= a^2\int\left(1 + 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]$$



$$A = a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= a^2\left(\frac{3\pi}{4} + 2\right)$$

The required area is A less half of the circle

$$\left(\frac{3\pi}{4} + 2\right)a^2 - \frac{1}{2}\pi a^2 = \frac{1}{4}\pi a^2 + 2a^2$$

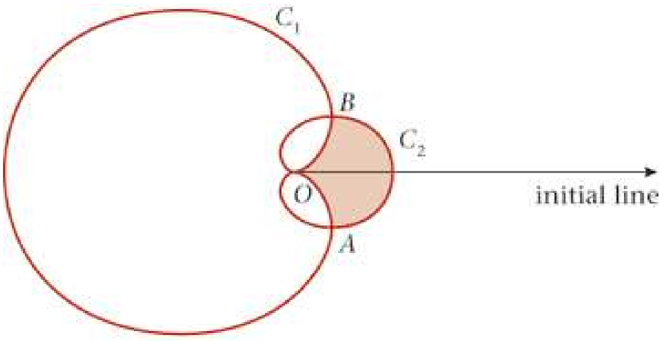
$$= \left(\frac{\pi + 8}{4}\right)a^2, \text{ as required}$$

The area you are asked to find is inside the cardioid and outside the circle. You find it by subtracting the shaded semi-circle from the area of the cardioid bounded by the half-lines $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 62

Question:



The figure is a sketch of two curves C_1 and C_2 with polar equations

$$C_1 : r = 3a(1 - \cos\theta), \qquad -\pi \leqslant \theta < \pi$$

and $$C_2 : r = a(1 + \cos\theta), \qquad -\pi \leqslant \theta < \pi$$

The curves meet at the pole O and at the points A and B.

a Find, in terms of a, the polar coordinates of the points A and B.

b Show that the length of the line AB is $\frac{3\sqrt{3}}{2}a$.

The region inside C_2 and outside C_1 is shaded in the figure.

c Find, in terms of a, the area of this region.

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

d calculate the area of this badge, giving your answer to 3 significant figures.

Solution:

a C_1 and C_2 intersect where

$$3a(1 - \cos\theta) = a(1 + \cos\theta)$$

$$3 - 3\cos\theta = 1 + \cos\theta$$

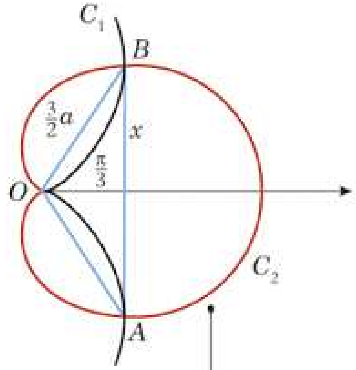
$$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

Where $\cos\theta = \frac{1}{2}$

$$r = a(1 + \cos\theta) = a\left(1 + \frac{1}{2}\right) = \frac{3}{2}a$$

$$A{:}\left(\frac{3}{2}a, -\frac{\pi}{3}\right),\ B{:}\left(\frac{3}{2}a, \frac{\pi}{3}\right)$$



b $AB = 2 \times \frac{3}{2}a\sin\frac{\pi}{3} = 3a \times \frac{\sqrt{3}}{2}$

$$= \frac{3\sqrt{3}}{2}a, \text{ as required}$$

Referring to the diagram,

$$\frac{x}{\frac{3}{2}a} = \sin\frac{\pi}{3} \Rightarrow x = \frac{3}{2}a\sin\frac{\pi}{3}$$

and $AB = 2x$.

c The area A_1 enclosed by OB and C_1 is given by

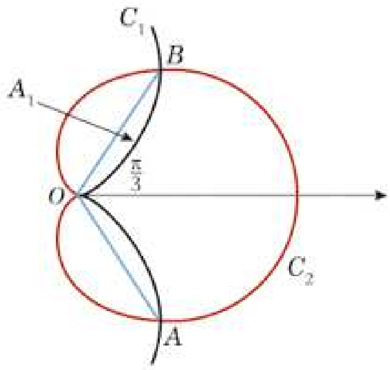
$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$$

$$\int r^2\,d\theta = \int 9a^2(1 - \cos\theta)^2\,d\theta = \int 9a^2(1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 9a^2\int\left(1 - 2\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{2}\right)d\theta$$

$$= 9a^2\int\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

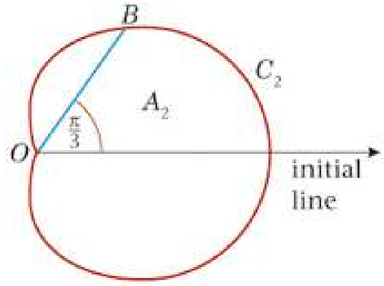
$$= 9a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]$$



$$A_1 = \frac{1}{2} \times 9a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{9}{2}a^2\left[\frac{\pi}{2} - \sqrt{3} + \frac{\sqrt{3}}{8}\right] = \frac{9a^2}{16}(4\pi - 7\sqrt{3})$$

The area A_2 enclosed by the initial line, C_2 and OB is given by



$$A_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$$

$$\int r^2\,d\theta = \int a^2(1+\cos\theta)^2\,d\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)d\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_2 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$

The required area R is given by

$$R = 2(A_2 - A_1)$$

$$= 2\left[\frac{a^2}{16}(4\pi+9\sqrt{3}) - \frac{9a^2}{16}(4\pi-7\sqrt{3})\right]$$

$$= \frac{2a^2}{16}[4\pi+9\sqrt{3}-(36\pi-63\sqrt{3})]$$

$$= \frac{a^2}{8}[72\sqrt{3}-32\pi] = (9\sqrt{3}-4\pi)a^2$$

d $\frac{3\sqrt{3}}{2}a = 4.5\text{ cm}$

$a = \frac{9}{3\sqrt{3}}\text{ cm} = \sqrt{3}\text{ cm}$

The area of the badge is

$(9\sqrt{3}-4\pi)a^2 = (9\sqrt{3}-4\pi)\times 3\text{ cm}^2$

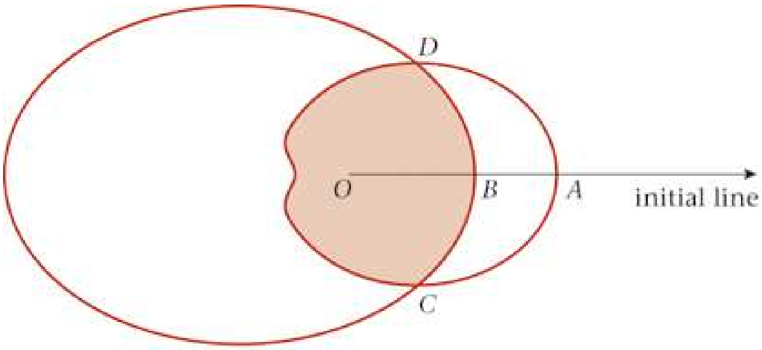
$= 9.07\text{ cm}^2$ (3 s.f.)

You use the result from part **b** to find a and substitute the value of a into the result of part **c**.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 63

Question:



A logo is designed which consists of two overlapping closed curves.

The polar equations of these curves are

$r = a(3 + 2\cos\theta)$ and

$r = a(5 - 2\cos\theta)$ $0 \leqslant \theta < 2\pi$

The figure is a sketch (not to scale) of these two curves.

a Write down the polar coordinates of the points A and B where the curves meet the initial line.

b Find the polar coordinates of the points C and D where the two curves meet.

c Show that the area of the overlapping region, which is shaded in the figure, is

$$\frac{a^2}{3}(49\pi - 48\sqrt{3})$$

Solution:

a $A{:}(5a, 0)$, $B{:}(3a, 0)$

For A, at $\theta = 0$, $r = a(3 + 2\cos 0) = a(3 + 2) = 5a$.
For B, at $\theta = 0$, $r = a(5 - 2\cos 0) = a(5 - 2) = 3a$.

b The curves intersect where

$$\not{a}(3 + 2\cos\theta) = \not{a}(5 - 2\cos\theta)$$

$$4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

In this question $0 \leqslant \theta < 2\pi$.

Where $\cos\theta = \frac{1}{2}$

$$r = a(3 + 2\cos\theta) = a\left(3 + 2 \times \frac{1}{2}\right) = 4a$$

$$C{:}\left(4a, \frac{5\pi}{3}\right), D{:}\left(4a, \frac{\pi}{3}\right)$$

c The area A_1 enclosed by $r = a(3 + 2\cos\theta)$ and the half-lines $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ is given by

$$A_1 = 2 \times \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\,\mathrm{d}\theta$$

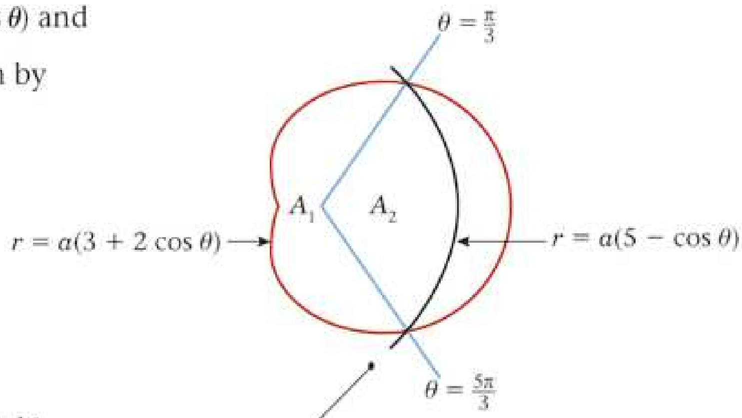
$$\int r^2\,\mathrm{d}\theta = \int a^2(3 + 2\cos\theta)^2\,\mathrm{d}\theta$$

$$= a^2\int(9 + 12\cos\theta + 4\cos^2\theta)\,\mathrm{d}\theta$$

$$= a^2\int(9 + 12\cos\theta + 2\cos 2\theta + 2)\,\mathrm{d}\theta$$

$$= a^2\int(11 + 12\cos\theta + 2\cos 2\theta)\,\mathrm{d}\theta$$

$$= a^2[11\theta + 12\sin\theta + \sin 2\theta]$$



The shaded area in the question is the sum of the two areas A_1 and A_2 shown in the diagram above. It is important that you carefully distinguish which curve is which.

$$A_1 = a^2\Big[11\theta + 12\sin\theta + \sin 2\theta\Big]_{\frac{\pi}{3}}^{\pi}$$

$$= a^2\left[11\left(\pi - \frac{\pi}{3}\right) + 12\left(0 - \frac{\sqrt{3}}{2}\right) + \left(0 - \frac{\sqrt{3}}{2}\right)\right]$$

$$= a^2\left[\frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right]$$

The area A_2 enclosed by $r = a(5 - 2\cos\theta)$ and the half-lines $\theta = \frac{5\pi}{3}$ and $\theta = \frac{\pi}{3}$ is given by

$$A_2 = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,d\theta$$

$$\int r^2\,d\theta = \int a^2(5 - 2\cos\theta)^2\,d\theta = a^2\int(25 - 20\cos\theta + 4\cos^2\theta)\,d\theta$$

$$= a^2\int(25 - 20\cos\theta + 2\cos 2\theta + 2)\,d\theta$$

$$= a^2\int(27 - 20\cos\theta + 2\cos 2\theta)\,d\theta$$

$$= a^2[27\theta - 20\sin\theta + \sin 2\theta]$$

The double angle formulae, here $\cos 2\theta = 2\cos^2\theta - 1$, are used in all questions involving the areas of cardioids.

$$A_2 = a^2\Big[27\theta - 20\sin\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= a^2\left[27 \times \frac{\pi}{3} - 20 \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= a^2\left[\frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right]$$

The area of the overlapping region is given by

$$A_1 + A_2 = a^2\left(\frac{22\pi}{3} - \frac{13\sqrt{3}}{2} + \frac{27\pi}{3} - \frac{19\sqrt{3}}{2}\right)$$

$$= a^2\left(\frac{49\pi}{3} - 16\sqrt{3}\right)$$

$$= \frac{a^2}{3}(49\pi - 48\sqrt{3}), \text{ as required}$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 64

Question:

The curve C has polar equation $r = 3a\cos\theta$, $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$. The curve D has polar equation $r = a(1 + \cos\theta)$, $-\pi \leqslant \theta < \pi$. Given that a is positive,

a sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line.

The graphs of C and D intersect at the pole O and at the points P and Q.

b Find the polar coordinates of P and Q.

c Use integration to find the exact value of the area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$.

The region R contains all points which lie outside D and inside C.

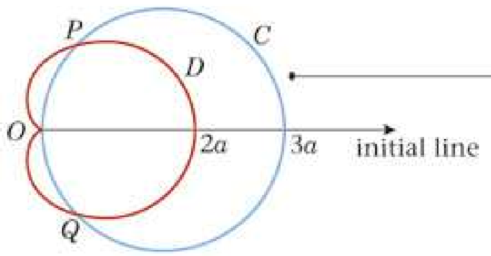
Given that the value of the smaller area enclosed by the curve C and the line $\theta = \frac{\pi}{3}$ is

$\frac{3a^2}{16}(2\pi - 3\sqrt{3})$,

d show that the area of R is πa^2.

Solution:

a



The curve C is a circle of diameter $3a$ and the curve D is a cardioid. The points of intersection of C and D have been marked on the diagram. The question does not specify which is P and which is Q. They could be interchanged. This would make no substantial difference to the solution of the question.

b The points of intersection of C and D are given by

$3\not{a}\cos\theta = \not{a}(1 + \cos\theta)$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

In this question $-\frac{\pi}{2} \leqslant \theta < \frac{\pi}{2}$.

Where $\cos\theta = \frac{1}{2}$

$r = 3a\cos\frac{\pi}{3} = 3a \times \frac{1}{2} = \frac{3}{2}a$

$P:\left(\frac{3}{2}a, \frac{\pi}{3}\right)$, $Q:\left(\frac{3}{2}a, -\frac{\pi}{3}\right)$

c The area between D, the initial line and OP is given by

$$A_1 = \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\,\mathrm{d}\theta$$

$$\int r^2\,\mathrm{d}\theta = \int a^2(1+\cos\theta)^2\,\mathrm{d}\theta = a^2\int(1+2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta$$

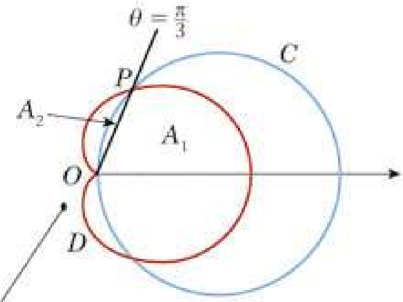
$$= a^2\int\left(1+2\cos\theta+\frac{1}{2}\cos 2\theta+\frac{1}{2}\right)\mathrm{d}\theta$$

$$= a^2\int\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)\mathrm{d}\theta$$

$$= a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]$$

$$A_1 = \frac{1}{2}\times a^2\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{2}+\sqrt{3}+\frac{\sqrt{3}}{8}\right] = \frac{a^2}{16}(4\pi+9\sqrt{3})$$



d Let the smaller area enclosed by C and the half-line $\theta = \frac{\pi}{3}$ be A_2.

By the symmetry of the figure, to find the area inside C but outside D, you subtract two areas A_1 and two areas A_2 from the area inside C. C is a circle of radius $\frac{3a}{2}$.

$$R = \pi\left(\frac{3a}{2}\right)^2 - 2A_1 - 2A_2$$

$$= \frac{9a^2\pi}{4} - \frac{2a^2}{16}(4\pi+9\sqrt{3}) - \frac{6a^2}{16}(2\pi-3\sqrt{3})$$

This is twice the area you are given in the question.

$$= \frac{9a^2\pi}{4} - \frac{\pi a^2}{2} - \frac{9\sqrt{3}a^2}{8} - \frac{3\pi a^2}{4} + \frac{9\sqrt{3}a^2}{8} = \pi a^2, \text{ as required}$$