Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

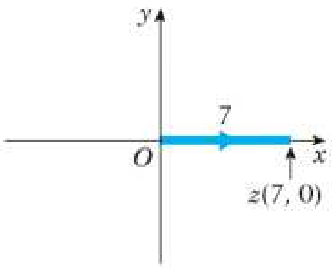
Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a 7 **b** $-5i$ **c** $\sqrt{3} + i$ **d** $2 + 2i$ **e** $1 - i$

f -8 **g** $3 - 4i$ **h** $-8 + 6i$ **i** $2 - \sqrt{3}i$

Solution:

a 7

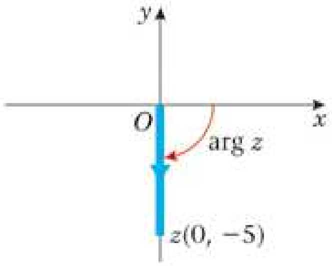


$r = 7$

$\theta = \arg z = 0$

$\therefore\ 7 = 7\ (\cos\theta + \mathrm{i}\sin\theta)$

b $-5\mathrm{i}$

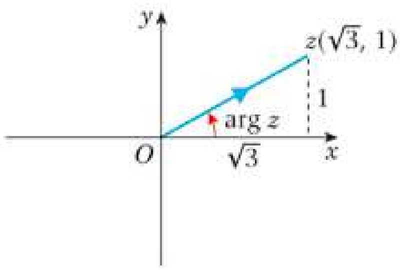


$r = 5$

$\theta = \arg z = -\frac{\pi}{2}$

$\therefore\ -5\mathrm{i} = 5\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

c $\sqrt{3} + \mathrm{i}$

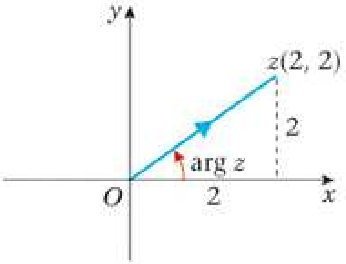


$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore\ \sqrt{3} + \mathrm{i} = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

d $2 + 2i$

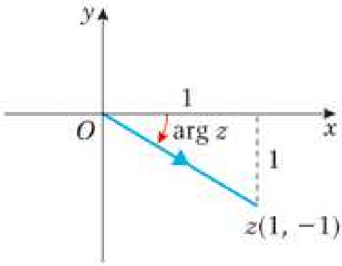


$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\therefore\ 2 + 2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

e $1 - i$

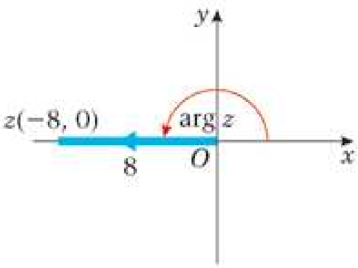


$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = -\frac{\pi}{4}$

$\therefore\ 1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

f -8

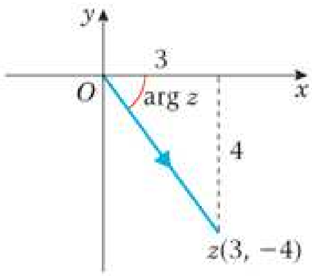


$r = 8$

$\theta = \arg z = \pi$

$\therefore\ -8 = 8(\cos\pi + i\sin\pi)$

g $3 - 4\text{i}$

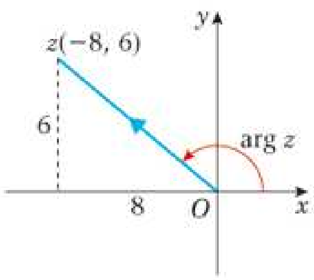


$r = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$\theta = \arg z = -\tan^{-1}\left(\frac{4}{3}\right) = -0.93^c$ (2 d.p.)

$\therefore\ 3 - 4\text{i} = 5(\cos(-0.93^c) + \text{i}\sin(-0.93^c))$

h $-8 + 6\text{i}$

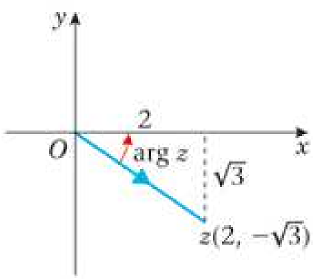


$r = \sqrt{(-8)^2 + 6^2} = \sqrt{100} = 10$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{6}{8}\right) = 2.50^c$ (2 d.p.)

$\therefore\ -8 + 6\text{i} = 10(\cos(2.50^c) + \text{i}\sin(2.50^c))$

i $2 - \sqrt{3}\text{i}$



$r = \sqrt{2^2 + (-\sqrt{3})^2} = \sqrt{7}$

$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) = -0.71^c$ (2 d.p.)

$\therefore\ 2 - \sqrt{3}\text{i} = \sqrt{7}\,(\cos(-0.71^c) + \text{i}\sin(-0.71^c))$

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Exercise A, Question 2

Question:

Express the following in the form $x + \text{i}y$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $5\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$

b $\frac{1}{2}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$

c $6\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + \text{i}\sin\left(-\frac{2\pi}{3}\right)\right)$

e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$

f $-4\left(\cos\frac{7\pi}{6} + \text{i}\sin\frac{7\pi}{6}\right)$

Solution:

a $5\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 5(0 + i)$

$= 5i$

b $\frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= \frac{\sqrt{3}}{4} + \frac{1}{4}i$

c $6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 6\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -3\sqrt{3} + 3i$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= 2 - 2i$

f $-4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= -4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= 2\sqrt{3} + 2i$

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Exercise A, Question 3

Question:

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$. Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a -3

b $6i$

c $-2\sqrt{3} - 2i$

d $-8 + i$

e $2 - 5i$

f $-2\sqrt{3} + 2\sqrt{3}i$

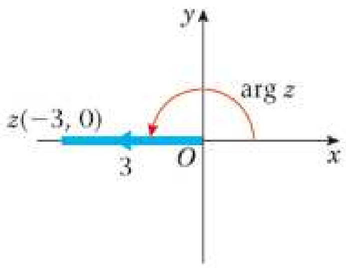
g $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

h $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

i $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

Solution:

a -3

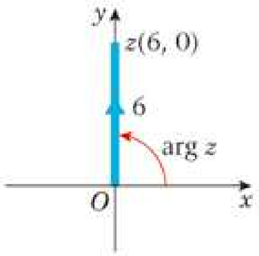


$r = 3$

$\theta = \arg z = \pi$

$\therefore\ -3 = 3e^{\pi i}$

b 6i

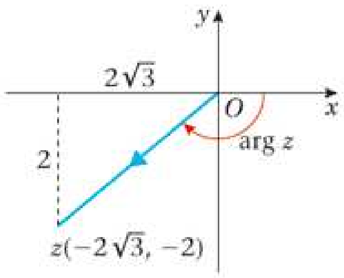


$r = 6$

$\theta = \arg z = \frac{\pi}{2}$

$\therefore\ 6i = 6e^{\frac{\pi i}{2}}$

c $-2\sqrt{3} - 2i$

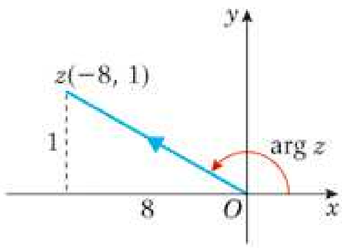


$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$

$\theta = \arg z = -\pi + \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

$\therefore\ -2\sqrt{3} - 2i = 4e^{\frac{-5\pi i}{6}}$

d $-8 + \mathrm{i}$

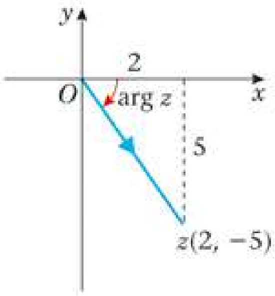


$r = \sqrt{(-8)^2 + 1^2} = \sqrt{65}$

$\theta = \pi - \tan^{-1}\left(\frac{1}{8}\right) = 3.02^c$ (2 d.p.)

$\therefore -8 + \mathrm{i} = \sqrt{65}\,\mathrm{e}^{3.02\mathrm{i}}$

e $2 - 5\mathrm{i}$

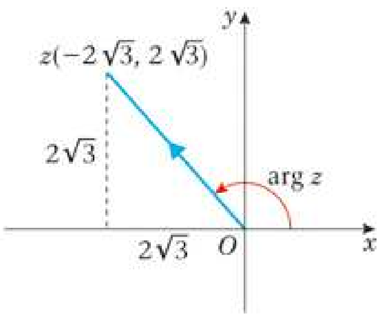


$r = \sqrt{2^2 + (-5)^2} = \sqrt{29}$

$\theta = -\tan^{-1}\left(\frac{5}{2}\right) = -1.19^c$ (2 d.p.)

$\therefore 2 - 5\mathrm{i} = \sqrt{29}\,\mathrm{e}^{-1.19\mathrm{i}}$

f $-2\sqrt{3} + 2\sqrt{3}\mathrm{i}$



$r = \sqrt{(-2\sqrt{3})^2 + (2\sqrt{3})^2} = \sqrt{12 + 12} = \sqrt{24}$

$= \sqrt{4}\sqrt{6} = 2\sqrt{6}$

$\theta = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2\sqrt{3}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore\ -2\sqrt{3} + 2\sqrt{3}\,\text{i} = 2\sqrt{6}\,\text{e}^{\frac{3\pi\text{i}}{4}}$

g $\sqrt{8}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\text{e}^{\frac{\pi\text{i}}{4}}$

$r = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$
$\theta = \frac{\pi}{4}$

h $8\left(\cos\frac{\pi}{6} - \text{i}\sin\frac{\pi}{6}\right)$

$= 8\left(\cos\left(-\frac{\pi}{6}\right) + \text{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$= 8\text{e}^{-\frac{\pi\text{i}}{6}}$

$r = 8,\ \theta = -\frac{\pi}{6}$

i $2\left(\cos\frac{\pi}{5} - \text{i}\sin\frac{\pi}{5}\right)$

$= 2\left(\cos\left(-\frac{\pi}{5}\right) + \text{i}\sin\left(-\frac{\pi}{5}\right)\right)$

$= 2\text{e}^{-\frac{\pi\text{i}}{5}}$

$r = 2,\ \theta = -\frac{\pi}{5}$

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Exercise A, Question 4

Question:

Express the following in the form $x + \mathrm{i}y$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

a $e^{\frac{\pi}{3}\mathrm{i}}$ **b** $4e^{\pi\mathrm{i}}$ **c** $3\sqrt{2}\,e^{\frac{\pi\mathrm{i}}{4}}$

d $8e^{\frac{\pi\mathrm{i}}{6}}$ **e** $3e^{-\frac{\pi\mathrm{i}}{2}}$ **f** $e^{\frac{5\pi\mathrm{i}}{6}}$

g $e^{-\pi\mathrm{i}}$ **h** $3\sqrt{2}e^{\frac{-3\pi}{4}\mathrm{i}}$ **i** $8e^{-\frac{4\pi\mathrm{i}}{3}}$

Solution:

a $e^{\frac{\pi\mathrm{i}}{3}} = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$

b $4e^{\pi\mathrm{i}} = 4(\cos\pi + \mathrm{i}\sin\pi)$

$= 4(-1 + \mathrm{i}(0))$

$= -4$

c $3\sqrt{2}\,e^{\frac{\pi\mathrm{i}}{4}} = 3\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

$= 3\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)$

$= 3 + 3\mathrm{i}$

d $8e^{\frac{\pi\mathrm{i}}{6}} = 8\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

$= 8\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$

$= 4\sqrt{3} + 4\mathrm{i}$

e $3e^{-\frac{\pi\mathrm{i}}{2}} = 8\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$= 3(0 - \mathrm{i})$

$= -3\mathrm{i}$

f $e^{\frac{5\pi\mathrm{i}}{6}} = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$

g $e^{-\pi i} = \cos(-\pi) + i\sin(-\pi)$

$= -1 + i(0)$

$= -1$

h $3\sqrt{2}e^{-\frac{3\pi}{4}i} = 3\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

$= 3\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= -3 - 3i$

i $8e^{-\frac{4\pi i}{3}} = 8\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right)$

$= 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -4 + 4\sqrt{3}\,i$

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Exercise A, Question 5

Question:

Express the following in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$.

a $e^{\frac{16\pi}{13}i}$ **b** $4e^{\frac{17\pi}{5}i}$ **c** $5e^{-\frac{9\pi}{8}i}$

Solution:

a $e^{\frac{16\pi i}{13}} = \cos\left(\frac{16\pi}{13}\right) + i\sin\left(\frac{16\pi}{13}\right)$

$= \cos\left(-\frac{10\pi}{13}\right) + i\sin\left(-\frac{10\pi}{13}\right)$

-2π from the argument.

b $4e^{\frac{17\pi i}{5}} = 4\left(\cos\left(\frac{17\pi}{5}\right) + i\sin\left(\frac{17\pi}{5}\right)\right)$

$= 4\left(\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right)$

$= 4\left(\cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)\right)$

c $5e^{\frac{-9\pi i}{8}} = 5\left(\cos\left(-\frac{9\pi}{8}\right) + i\sin\left(-\frac{9\pi}{8}\right)\right)$

$= 5\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)$

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Exercise A, Question 6

Question:

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Solution:

$e^{i\theta} = \cos\theta + i\sin\theta$ ①

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ ②

① − ② $\Rightarrow e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin\theta$

$\therefore \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ (as required)

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Exercise B, Question 1

Question:

Express the following in the form $x + \mathrm{i}y$.

a $(\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos 3\theta + \mathrm{i}\sin 3\theta)$

b $\left(\cos\frac{3\pi}{11} + \mathrm{i}\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + \mathrm{i}\sin\frac{8\pi}{11}\right)$

c $3\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos\left(\frac{-\pi}{12}\right) + \mathrm{i}\sin\left(\frac{-\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

e $4\left(\cos\left(\frac{-5\pi}{9}\right) + \mathrm{i}\sin\left(\frac{-5\pi}{9}\right)\right) \times \frac{1}{2}\left(\cos\left(\frac{-5\pi}{18}\right) + \mathrm{i}\sin\left(\frac{-5\pi}{18}\right)\right)$

f $6\left(\cos\frac{\pi}{10} + \mathrm{i}\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)$

g $(\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos\theta - \mathrm{i}\sin\theta)$

h $3\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$

Solution:

a $(\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos 3\theta + \mathrm{i}\sin 3\theta)$

$= \cos(2\theta + 3\theta) + \mathrm{i}\sin(2\theta + 3\theta)$

$= \cos 5\theta + \mathrm{i}\sin 5\theta$

b $\left(\cos\frac{3\pi}{11} + \mathrm{i}\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + \mathrm{i}\sin\frac{8\pi}{11}\right)$

$= \cos\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right) + \mathrm{i}\sin\left(\frac{3\pi}{11} + \frac{8\pi}{11}\right)$

$= \cos\pi + \mathrm{i}\sin\pi$

$= -1 + \mathrm{i}(0)$

$= -1$

c $3\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

$= 3(2)\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$

$= 6\left(\frac{1}{2}\ 1\ \frac{\sqrt{3}}{2}\mathrm{i}\right)$

$= 3 + 3\sqrt{3}\,\mathrm{i}$

d $\sqrt{6}\left(\cos\left(-\frac{\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

$= (\sqrt{6})(\sqrt{3})\left(\cos\left(-\frac{\pi}{12} + \frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{12} + \frac{\pi}{3}\right)\right)$

$= \sqrt{18}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

$= 3\left(\sqrt{2}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right)$

$= 3 + 3\mathrm{i}$

$\sqrt{18} = \sqrt{9}\sqrt{2}$
$= 3\sqrt{2}$

e $4\left(\cos\left(-\frac{5\pi}{9}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{9}\right)\right)\times\frac{1}{2}\left(\cos\left(-\frac{5\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{18}\right)\right)$

$=4\left(\frac{1}{2}\right)\left(\cos\left(-\frac{5\pi}{9}+-\frac{5\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{5\pi}{9}+-\frac{5\pi}{18}\right)\right)$

$=2\left(\cos\left(-\frac{15\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{15\pi}{18}\right)\right)$

$=2\left(\cos\left(-\frac{5\pi}{6}\right)\ 1\ \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)$

$=-\sqrt{3}-\mathrm{i}$

f $6\left(\cos\frac{\pi}{10}+\mathrm{i}\sin\frac{\pi}{10}\right)\times 5\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\times\frac{1}{3}\left(\cos\frac{2\pi}{5}+\mathrm{i}\sin\frac{2\pi}{5}\right)$

$=6(5)\left(\frac{1}{3}\right)\left(\cos\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{10}+\frac{\pi}{3}+\frac{2\pi}{5}\right)\right)$

$=10\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$

$=10\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)$

$=-5\sqrt{3}+5\mathrm{i}$

g $(\cos 4\theta+\mathrm{i}\sin 4\theta)(\cos\theta-\mathrm{i}\sin\theta)$

$=(\cos 4\theta+\mathrm{i}\sin 4\theta)(\cos(-\theta)+\mathrm{i}\sin(-\theta))$

$=\cos(4\theta+-\theta)+\mathrm{i}\sin(4\theta+-\theta)$

$=\cos 3\theta+\mathrm{i}\sin 3\theta$

h $3\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)\times\sqrt{2}\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)$

$=3\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)\times\sqrt{2}\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$

$=3(\sqrt{2})\left(\cos\left(\frac{\pi}{12}-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{12}-\frac{\pi}{3}\right)\right)$

$=3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

$=3\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\mathrm{i}\right)$

$=3-3\mathrm{i}$

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Exercise B, Question 2

Question:

Express the following in the form $x + iy$.

a $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$

b $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$

c $\dfrac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

d $\dfrac{\cos 2\theta - i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$

Solution:

a $\dfrac{\cos 5\theta + i \sin 5\theta}{\cos 2\theta + i \sin 2\theta}$

$= \cos(5\theta - 2\theta) + i\sin(5\theta - 2\theta)$

$= \cos 3\theta + i\sin 3\theta$

b $\dfrac{\sqrt{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}$

$= \dfrac{\sqrt{2}}{\left(\frac{1}{2}\right)}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right)$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2 + 2i$

c $\dfrac{3\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)}{4\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)}$

$= \frac{3}{4}\left(\cos\left(\frac{\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{5\pi}{6}\right)\right)$

$= \frac{3}{4}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$= \frac{3}{4}(0 - i)$

$= -\frac{3}{4}i$

d $\dfrac{\cos 2\theta - i \sin 2\theta}{\cos 3\theta + i \sin 3\theta}$

$= \dfrac{\cos(-2\theta) + i\sin(-2\theta)}{\cos 3\theta + i\sin 3\theta}$

$= \cos(-2\theta - 3\theta) + i\sin(-2\theta - 3\theta)$

$= \cos(-5\theta) + i\sin(-5\theta)$ *or* $\cos 5\theta - i\sin 5\theta$

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Exercise B, Question 3

Question:

z and w are two complex numbers where

$$z = -9 + 3\sqrt{3}\text{i},\ |w| = \sqrt{3} \text{ and } \arg w = \frac{7\pi}{12}.$$

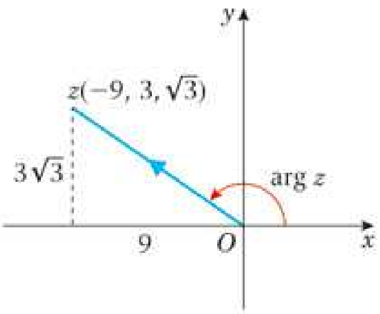
Express the following in the form $r(\cos\theta + \text{i}\sin\theta)$,

a z, **b** w, **c** zw, **d** $\frac{z}{w}$,

where $-\pi < \theta \leq \pi$.

Solution:

a $z = -9 + 3\sqrt{3}\text{i}$



$$r = \sqrt{(-9)^2 + (3\sqrt{3})^2} = \sqrt{81 + 27} = \sqrt{108}$$
$$= \sqrt{36}\sqrt{3} = 6\sqrt{3}$$

$$\theta = \arg z = \pi - \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right) = \frac{5\pi}{6}$$

$$z = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$$

b $r = |w| = \sqrt{3}$

$$\theta = \arg w = \frac{7\pi}{12}$$

$$w = \sqrt{3}\left(\cos\frac{7\pi}{12} + \text{i}\sin\frac{7\pi}{12}\right)$$

c $zw = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) \times \sqrt{3}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$= (6\sqrt{3})(\sqrt{3})\left(\cos\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right) + i\sin\left(\frac{5\pi}{6} + \frac{7\pi}{12}\right)\right)$

$= 18\left(\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right)$

$= 18\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)$

d $\frac{z}{w} = \frac{6\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{\sqrt{3}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)}$

$= \frac{6\sqrt{3}}{\sqrt{3}}\left(\cos\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right) + i\sin\left(\frac{5\pi}{6} - \frac{7\pi}{12}\right)\right)$

$= 6\left(\cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)\right)$

$= 6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

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Exercise C, Question 1

Question:

Use de Moivre's theorem to simplify each of the following:

a $(\cos\theta + i\sin\theta)^6$

b $(\cos 3\theta + i\sin 3\theta)^4$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

f $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

g $\frac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$

h $\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

i $\frac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

j $\frac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

k $\frac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$

l $\frac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3}$

Solution:

a $(\cos\theta + i\sin\theta)^6$
$= \cos 6\theta + i\sin 6\theta$

b $(\cos 3\theta + i\sin 3\theta)^4$
$= \cos(4(3\theta)) + i\sin(4(3\theta))$
$= \cos 12\theta + i\sin 12\theta$

c $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

$= \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

e $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$

$= \cos\frac{10\pi}{5} + i\sin\frac{10\pi}{5}$

$= \cos 2\pi + i\sin 2\pi$
$= \cos 0 + i\sin 0$
$= 1 + 1(0)$
$= 1$

f $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

$= \left(\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right)^{15}$

$= \cos\left(-\frac{15\pi}{10}\right) + i\sin\left(-\frac{15\pi}{10}\right)$

$= \cos\left(\frac{5\pi}{10}\right) + i\sin\left(\frac{5\pi}{10}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= 0 + i$
$= i$

g $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$

$= \dfrac{\cos 5\theta + i\sin 5\theta}{\cos 4\theta + i\sin 4\theta}$

$= \cos(5\theta - 4\theta) + i\sin(5\theta - 4\theta)$

$= \cos\theta + i\sin\theta$

h $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

$= \dfrac{\cos 14\theta + i\sin 14\theta}{\cos 12\theta + i\sin 12\theta}$

$= \cos(14\theta - 12\theta) + i\sin(14\theta - 12\theta)$

$= \cos 2\theta + i\sin 2\theta$

i $\dfrac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

$= (\cos 2\theta + i\sin 2\theta)^{-3}$

$= \cos(-6\theta) + i\sin(-6\theta)$

$= \cos 6\theta - i\sin 6\theta$

j $\dfrac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$

$= \dfrac{\cos 8\theta + i\sin 8\theta}{\cos 9\theta + i\sin 9\theta}$

$= \cos(8\theta - 9\theta) + i\sin(8\theta - 9\theta)$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

k $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta + i\sin 3\theta)^2}$

$= \dfrac{\cos 5\theta + i\sin 5\theta}{(\cos(-3\theta) + i\sin(-3\theta))^2}$

$= \dfrac{\cos 5\theta + i\sin 5\theta}{\cos(-6\theta) + i\sin(-6\theta)}$

$= \cos(5\theta - -6\theta) + i\sin(5\theta - -6\theta)$

$= \cos 11\theta - i\sin 11\theta$

l $\dfrac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3}$

$= \dfrac{\cos(-\theta) - i\sin(-\theta)}{(\cos(-2\theta) - i\sin(-2\theta))^3}$

$= \dfrac{\cos(-\theta) - i\sin(-\theta)}{\cos(-6\theta) - i\sin(-6\theta)}$

$= \cos(-\theta - -6\theta) + i\sin(-\theta - -6\theta)$

$= \cos 5\theta - i\sin 5\theta$

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Exercise C, Question 2

Question:

Evaluate $\dfrac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \frac{4\pi}{13} - i \sin \frac{4\pi}{13}\right)^6}$.

Solution:

$$\frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \frac{4\pi}{13} - i \sin \frac{4\pi}{13}\right)^6}$$

$$= \frac{\left(\cos \frac{7\pi}{13} + i \sin \frac{7\pi}{13}\right)^4}{\left(\cos \left(-\frac{4\pi}{13}\right) - i \sin \left(-\frac{4\pi}{13}\right)\right)^6}$$

$$= \frac{\cos \left(\frac{28\pi}{13}\right) + i \sin \left(\frac{28\pi}{13}\right)}{\cos \left(-\frac{24\pi}{13}\right) - i \sin \left(-\frac{24\pi}{13}\right)}$$

$$= \cos \left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right) + i \sin\left(\frac{28\pi}{13} - -\frac{24\pi}{13}\right)$$

$$= \cos \left(\frac{52\pi}{13}\right) + i \sin \left(\frac{52\pi}{13}\right)$$

$$= \cos 4\pi + i \sin 4\pi$$

$$= \cos 0 + i \sin 0$$

$$= 1 + i(0)$$

$$= 1$$

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Exercise C, Question 3

Question:

Express the following in the form $x + iy$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

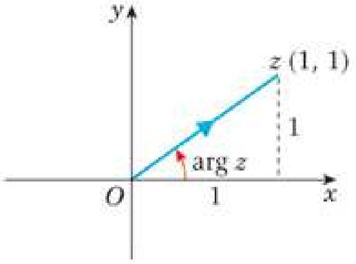
a $(1 + i)^5$ **b** $(-2 + 2i)^8$ **c** $(1 - i)^6$

d $(1 - \sqrt{3}i)^6$ **e** $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$ **f** $\left(-2\sqrt{3} - 2i\right)^5$

Solution:

a $(1 + i)^5$

If $z = 1 + i$, then



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

So, $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

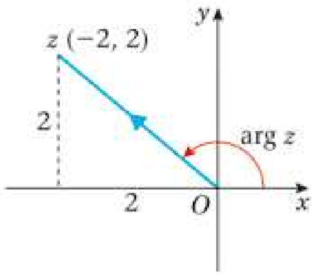
$$\therefore (1 + i)^5 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^5$$
$$= (\sqrt{2})^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$= 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$
$$= 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$
$$= -4 - 4i$$

$$(\sqrt{2}) = \sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2} = 4\sqrt{2}$$

Therefore, $(1 + i)^5 = -4 - 4i$

b $(-2 + 2i)^8$

If $z = -2 + i$, then



$r = \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{2}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

So, $-2 + 2i = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$$\therefore (-2 + 2i)^8 = \left[2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right]^8$$

$$= (2\sqrt{2})^8\left(\cos\left(\frac{24\pi}{4}\right) + i\sin\left(\frac{24\pi}{4}\right)\right)$$

$$= (256)(16)\,(\cos 6\pi + i\sin 6\pi)$$

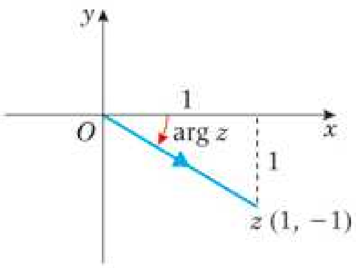
$$= 4096\,(1 + i(0))$$

$$= 4096$$

Therefore, $(-2 + 2i)^8 = 4096$

c $(1 - i)^6$

If $z = 1 - i$, then



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4}$

So, $1 - i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$$\therefore (1 - i)^6 = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right]^6$$

$$= (\sqrt{2})^6\left(\cos\left(-\frac{6\pi}{4}\right) + i\sin\left(-\frac{6\pi}{4}\right)\right)$$

$$= 8\left(\cos\left(-\frac{3\pi}{2}\right) + i\sin\left(-\frac{3\pi}{2}\right)\right)$$

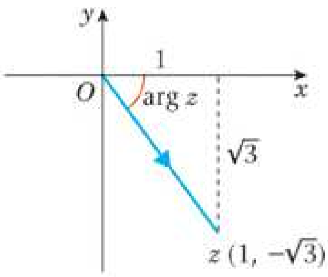
$$= 8\,(0 + i)$$

$$= 8i$$

Therefore, $(1 - i)^6 = 8i$

d $(1 - \sqrt{3}i)^6$

If $z = 1 - \sqrt{3}\,i$, then



$$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

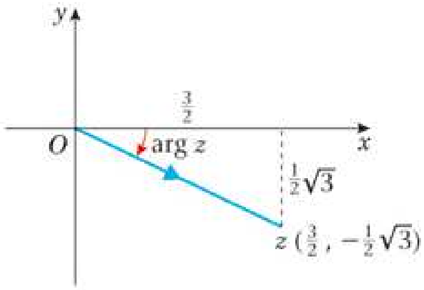
So, $1 - \sqrt{3}\,i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$$\begin{aligned}\therefore\ (1 - \sqrt{3}\,i)^6 &= \left[2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)\right]^6 \\ &= (2)^6\left(\cos\left(-\frac{6\pi}{3}\right) + i\sin\left(-\frac{6\pi}{3}\right)\right) \\ &= 64(\cos(-2\pi) + i\sin(-2\pi)) \\ &= 64(1 + i(0)) \\ &= 64\end{aligned}$$

Therefore, $(1 - \sqrt{3}\,i)^6 = 64$

e $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}i\right)^9$

If $z = \frac{3}{2} - \frac{1}{2}\sqrt{3}\,i$, then



$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\sqrt{3}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$\theta = \arg z = -\tan^{-1}\left(\frac{\frac{1}{2}\sqrt{3}}{\frac{3}{2}}\right) = -\tan^{-1}\frac{\sqrt{3}}{3} = -\frac{\pi}{6}$$

So, $\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i} = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$

$$\therefore \left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9 = \left[\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)\right]^9$$

$$= (\sqrt{3})^9\left(\cos\left(-\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{6}\right)\right)$$

$$= 81\sqrt{3}\left(\cos\left(-\frac{3\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{2}\right)\right)$$

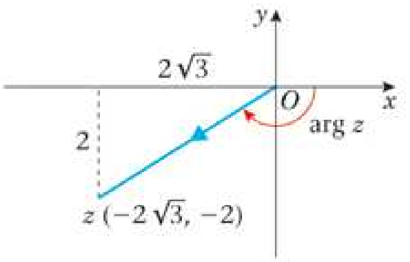
$$= 81\sqrt{3}(0 + \mathrm{i})$$

$$= 81\sqrt{3}\,\mathrm{i}$$

Therefore, $\left(\frac{3}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}\right)^9 = 81\sqrt{3}\,\mathrm{i}$

f $(-2\sqrt{3} - 2\mathrm{i})^5$

If $z = -2\sqrt{3} - 2\mathrm{i}$, then



$$r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\theta = \arg z = -\pi - \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

So, $-2\sqrt{3} - 2\mathrm{i} = 4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)$

$$\therefore (-2\sqrt{3} - 2\mathrm{i})^5 = \left[4\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right)\right]^5$$

$$= 4^5\left(\cos\left(-\frac{25\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{25\pi}{6}\right)\right)$$

$$= 1024\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 1024\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right)$$

$$= 512\sqrt{3} - 512\mathrm{i}$$

Therefore, $(-2\sqrt{3} - 2\mathrm{i})^5 = 512\sqrt{3} - 512\mathrm{i}$

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Exercise C, Question 4

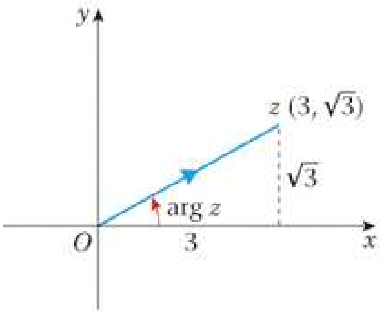
Question:

Express $(3+\sqrt{3}\mathrm{i})^5$ in the form $a+b\sqrt{3}\mathrm{i}$ where a and b are integers.

Solution:

$(3+\sqrt{3}\mathrm{i})^5$

If $z = 3+\sqrt{3}\,\mathrm{i}$, then



$$r = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

So, $3+\sqrt{3}\,\mathrm{i} = 2\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

$$\therefore\ (3+\sqrt{3}\,\mathrm{i})^5 = \left[2\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)\right]^5$$

$$= (2\sqrt{3})^5\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$$

$$= 32(9\sqrt{3})\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$$

$$= 288\sqrt{3}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$$

$$= -144\sqrt{3}\sqrt{3} + 144\sqrt{3}\,\mathrm{i}$$

$$= -432 + 144\sqrt{3}\,\mathrm{i}$$

Therefore, $(3+\sqrt{3}\,\mathrm{i})^5 = -432 + 144\sqrt{3}\,\mathrm{i}$

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Exercise D, Question 1

Question:

$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

Solution:

$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + i \sin^3 \theta$

de Moivre's Theorem.

$= \cos^3 \theta + {}^3C_1 \cos^2 \theta (i \sin \theta)$

$\quad + {}^3C_2 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3$

Binomial expansion.

$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta + 3i^2 \cos \theta \sin^2 \theta + i^3 \sin^3 \theta$

$= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$

Hence,

$\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$

Equating the imaginary parts gives,

$\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$

Applying $\cos^2 \theta = 1 - \sin^2 \theta$.

$= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta$

$= 3 \sin \theta (1 - \sin^2 \theta) - \sin^3 \theta$

$= 3 \sin \theta - 3\sin^3 \theta - \sin^3 \theta$

$= 3 \sin \theta - 4\sin^3 \theta$

Hence, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ (as required)

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Exercise D, Question 2

Question:

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Solution:

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5C_1\cos^4\theta(i\sin\theta) + {}^5C_2\cos^3\theta(i\sin\theta)^2$
$\quad + {}^5C_3\cos^2\theta(i\sin\theta)^3 + {}^5C_4\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

Binomial expansion.

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$
$\quad + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

Hence,

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$\quad + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating the imaginary parts gives,

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$= 5(\cos^2\theta)^2\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$= 5\sin\theta(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^3\theta(1 - \sin^2\theta) + \sin^5\theta$
$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

Hence, $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (as required)

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Exercise D, Question 3

Question:

$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Solution:

$$(\cos\theta + \mathrm{i}\sin\theta)^7 = \cos 7\theta + \mathrm{i}\sin 7\theta$$

de Moivre's Theorem.

$$= \cos^7\theta + {}^7C_1\cos^6\theta(\mathrm{i}\sin\theta) + {}^7C_2\cos^5\theta(\mathrm{i}\sin\theta)^2 + {}^7C_3\cos^4\theta(\mathrm{i}\sin\theta)^3 + {}^7C_4\cos^3\theta(\mathrm{i}\sin\theta)^4 + {}^7C_5\cos^2\theta(\mathrm{i}\sin\theta)^5 + {}^7C_6\cos\theta(\mathrm{i}\sin\theta)^6 + (\mathrm{i}\sin\theta)^7$$

Binomial expansion.

$$= \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta + 21\mathrm{i}^2\cos^5\theta\sin^2\theta + 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta + 7\mathrm{i}^6\cos\theta\sin^6\theta + \mathrm{i}^7\sin^7\theta$$

Hence,

$$\cos 7\theta + \mathrm{i}\sin 7\theta = \cos^7\theta + 7\mathrm{i}\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35\mathrm{i}^3\cos^4\theta\sin^3\theta + 35\mathrm{i}^4\cos^3\theta\sin^4\theta + 21\mathrm{i}^5\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - \mathrm{i}\sin^7\theta$$

Equating the imaginary parts gives,

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$

Applying $\cos^2\theta = 1 - \sin^2\theta$.

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta) - 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Hence, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos^5\theta$ (as required)

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Exercise D, Question 4

Question:

$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

Solution:

Let $z = \cos\theta + i\sin\theta$

$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$ — $z + \frac{1}{z} = 2\cos\theta$

$= z^4 + {}^4C_1 z^3\left(\frac{1}{z}\right) + {}^4C_2 z^2\left(\frac{1}{z}\right)^2 + {}^4C_3 z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$

$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z^2\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$

$= z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$

$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 4(2\cos 2\theta) + 6$ — $z^n + \frac{1}{z^n} = 2\cos n\theta$

So, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$16\cos^4\theta = 2(\cos 4\theta + 4\cos 2\theta + 3)$

$\cos^4\theta = \frac{2}{16}(\cos 4\theta + 4\cos 2\theta + 3)$

Therefore, $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (as required)

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Exercise D, Question 5

Question:

$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Solution:

Let $z = \cos\theta + \mathrm{i}\sin\theta$

$$\left(z - \frac{1}{z}\right)^5 = (2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}^5\sin^5\theta = 32\mathrm{i}\sin^5\theta$$

$z - \frac{1}{z} = 2\mathrm{i}\sin\theta$

$$= z^5 + {}^5C_1 z^4\left(-\frac{1}{z}\right) + {}^5C_2 z^3\left(-\frac{1}{z}\right)^2 + {}^5C_3 z^2\left(-\frac{1}{z}\right)^3 + {}^5C_4 z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2 + 10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$$

$$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$$

$z^n + \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$

So, $32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta \quad (\div 2\mathrm{i})$

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Therefore, $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

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Exercise D, Question 6

Question:

a Show that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are constants.

Solution:

Let $z = \cos\theta + i\sin\theta$

a $\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta$

$z - \frac{1}{z} = 2\cos\theta$

$$= z^6 + {}^6C_1 z^5\left(\frac{1}{z}\right) + {}^6C_2 z^4\left(\frac{1}{z}\right)^2 + {}^6C_3 z^3\left(\frac{1}{z}\right)^3 + {}^6C_4 z^2\left(\frac{1}{z}\right)^4 + {}^6C_5{}^2\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$$

$$= z^6 - 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 - \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\sin 2\theta) + 20$$

$z^n + \frac{1}{z^n} = 2\cos n\theta$

So, $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$ (as required)

b $\int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{1}{32}\int_0^{\frac{\pi}{6}} \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\theta$

$$= \frac{1}{32}\left[\frac{\sin^6\theta}{6} + \frac{6\sin^4\theta}{4} + \frac{15\sin^2\theta}{2} + 10\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}\left[\left(\frac{\sin(\pi)}{6} + \frac{6\sin\left(\frac{2\pi}{3}\right)}{4} + \frac{15\sin\left(\frac{\pi}{3}\right)}{2} + \frac{10\pi}{6}\right) - (0)\right]$$

$$= \frac{1}{32}\left[0 + \frac{3}{2}\frac{\sqrt{3}}{2} + \frac{15}{2}\frac{\sqrt{3}}{2} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{3}{4}\sqrt{3} + \frac{15}{4}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{1}{32}\left[\frac{9}{2}\sqrt{3} + \frac{5\pi}{3}\right]$$

$$= \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$$

$\therefore \int_0^{\frac{\pi}{6}} \cos^6\theta = \frac{5\pi}{96} + \frac{9}{64}\sqrt{3}$

$a = \frac{5}{96}, b = \frac{9}{64}$

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Exercise D, Question 7

Question:

a Use de Moivre's theorem to show that $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

b Hence, or otherwise, show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

c Use your answer to part **b** to find, to 2 d.p., the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

Solution:

a $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ | de Moivre's Theorem.

$$= \cos^4\theta + {}^4C_1\cos^3\theta(i\sin\theta) + {}^4C_2\cos^2\theta(i\sin\theta)^2 + {}^4C_3\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$

Binomial expansion.

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta$$

$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Hence,

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta \quad ①$$

Equating the imaginary parts of ① gives:

$$\sin^4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \text{ (as required)}$$

b Equating the real parts of ① gives:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \qquad \frac{(\cos 4\theta \div \cos^4\theta)}{(\cos 4\theta \div \cos^4\theta)}$$

$$= \frac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{\dfrac{4\cancel{\cos^3\theta}}{\cancel{\cos^3\theta}}\dfrac{\sin\theta}{\cos\theta} - \dfrac{4\cancel{\cos\theta}\sin^3\theta}{\cancel{\cos\theta}\cos^3\theta}}{\dfrac{\cancel{\cos^4\theta}}{\cancel{\cos^4\theta}} - \dfrac{6\cancel{\cos^2\theta}\sin^2\theta}{\cancel{\cos^2\theta}\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Therefore, $\tan^4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (as required)

c $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

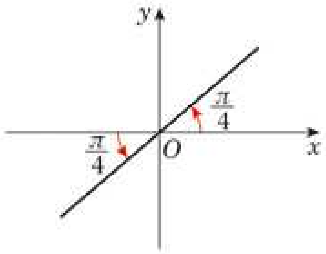
$1 = \dfrac{4x - 4x^3}{x^4 - 6x^2 + 1}$ ②

Let $x = \tan\theta$; then

② $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

$\tan 4\theta = 1$ — From part **b.**

$\alpha = \dfrac{\pi}{4}$



$4\theta = \left\{\dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}, \ldots\right\}$

$\theta = \left\{\dfrac{\pi}{16}, \dfrac{5\pi}{16}, \dfrac{9\pi}{16}, \dfrac{13\pi}{16}, \ldots\right\}$

$\therefore x = \tan\theta = \tan\dfrac{\pi}{16}, \tan\dfrac{5\pi}{16}, \tan\dfrac{9\pi}{16}, \tan\dfrac{13\pi}{16}$

$x = 0.19891\ldots, 1.49660\ldots, -5.02733\ldots, -0.66817\ldots,$

$x = 0.20, 1.50, -5.03, -0.67$ (2 d.p.)

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Exercise E, Question 1

Question:

Solve the following equations, expressing your answers for z in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

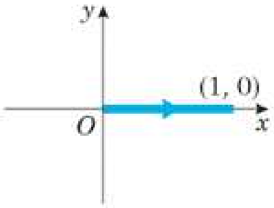
a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

Solution:

a $z^4 - 1 = 0$

$z^4 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^4 = 1\,(\cos 0 + i\sin 0)$

$z^4 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + i\sin(2k\pi)]^{\frac{1}{4}}$

$z = \cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right)$

de Moivre's Theorem.

$z = \cos\left(\frac{k\pi}{2}\right) + i\sin\left(\frac{k\pi}{2}\right)$

$k = 0,\ z = \cos 0 + i\sin 0 = 1$

$k = 1,\ z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

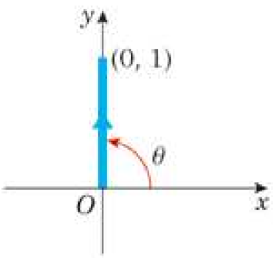
$k = 2,\ z = \cos\pi + i\sin\pi = -1$

$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i$

Therefore, $z = 1, i, -1, -i$

b $z^3 - i = 0$

$z^3 = i$



for i, $r = 1$ and $\theta = \frac{\pi}{2}$

So $z^3 = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z^3 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right), \quad k \in \mathbb{Z}$

Hence, $z = \left[\cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)\right]^{\frac{1}{3}}$

$z = \cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)$

de Moivre's Theorem.

$z = \cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)$

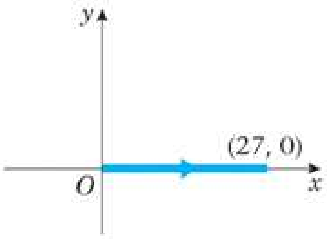
$\therefore\ k = 0,\ z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k = 1,\ z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k = -1,\ z = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = 0 - i$

Therefore, $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -i$

c $z^3 = 27$



for 27, $r = 27$ and $\theta = 0$

So $z^3 = 27(\cos 0 + i\sin 0)$

$z^3 = 27[\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)] \quad k \in \mathbb{Z}$

Hence, $z = [27(\cos(2k\pi) + i\sin(2k\pi))]^{\frac{1}{3}}$

de Moivre's Theorem.

$z = 3\left[\cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)\right]$

$k = 0;\ z = 3(\cos 0 + i\sin 0) = 3$

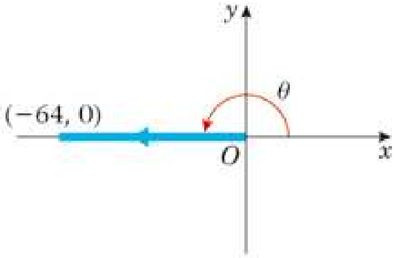
$k = 1;\ z = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$

$k = -1;\ z = 3\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right) = 3\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

Therefore, $z = 3,\ -\frac{3}{2} + \frac{3\sqrt{3}}{2}i,\ -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

d $z^4 + 64 = 0$

$z^4 = -64$



for -64, $r = 64$ and $\theta = \pi$

So $z^4 = 64(\cos\pi + i\sin\pi)$

$z^4 = 64(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [64(\pi + 2k\pi) + i\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 64^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0;\ z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2 + 2i$

$k = 1;\ z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -2 + 2i$

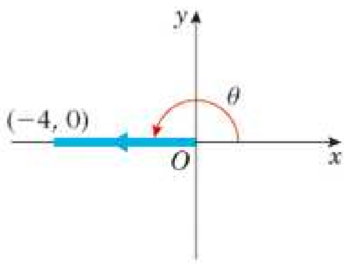
$k = -1;\ z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 2 - 2i$

$k = -2;\ z = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -2 - 2i$

Therefore, $z = 2 + 2i,\ -2 + 2i,\ 2 - 2i,\ -2 - 2i$

e $z^4 + 4 = 0$

$z^4 = -4$



for -4, $r = 4$ and $\theta = \pi$

So $z^4 = 4(\cos\pi + \mathrm{i}\sin\pi)$

$z^4 = 4(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [4(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi))]^{\frac{1}{4}}$

$$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{\pi + 2k\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{4}\right)\right)$$

de Moivre's Theorem.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + \mathrm{i}\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0;\ z = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = 1 + \mathrm{i}$

$k = 1;\ z = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) = -1 + \mathrm{i}$

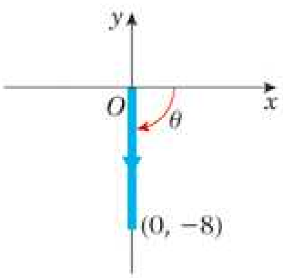
$k = -1;\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = 1 - \mathrm{i}$

$k = -2;\ z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = -1 - \mathrm{i}$

Therefore, $z = 1 + \mathrm{i}, -1 + \mathrm{i}, 1 - \mathrm{i}, -1 - \mathrm{i}$

f $z^3 + 8\mathrm{i} = 0$

$z^3 = -8\mathrm{i}$



for $-8\mathrm{i}$, $r = 8$, $\theta = -\frac{\pi}{2}$

So $z^3 = 8\left(\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right)$

$$z^4 = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

Hence, $z = \left[8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + \mathrm{i}\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$$z = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

$k = 0$; $z = 2\left(\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = \sqrt{3} - \mathrm{i}$

$k = 1$; $z = 2\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right) = 2(0 + \mathrm{i}) = 2\mathrm{i}$

$k = -1$; $z = 2\left(\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}\right) = -\sqrt{3} - \mathrm{i}$

Therefore, $z = \sqrt{3} - \mathrm{i},\ 2\mathrm{i},\ -\sqrt{3} - \mathrm{i}$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 2

Question:

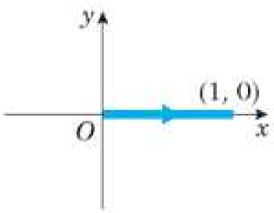
Solve the following equations, expressing your answers for z in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2\sqrt{3}i = 2$ **f** $z^3 + 32\sqrt{3} + 32i = 0$

Solution:

a $z^7 = 1$



for 1, $r = 1$ and $\theta = 0$

So $z^7 = 1\,(\cos 0 + \mathrm{i}\sin 0)$

$z^7 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = (\cos(2k\pi) + \mathrm{i}\sin(2k\pi))^{\frac{1}{7}}$

$$z = \cos\left(\frac{2k\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{7}\right)$$

de Moivre's Theorem.

$k = 0$, $z = \cos 0 + \mathrm{i}\sin 0$

$k = 1$, $z = \cos\left(\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(\frac{2\pi}{7}\right)$

$k = 2$, $z = \cos\left(\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(\frac{4\pi}{7}\right)$

$k = 3$, $z = \cos\left(\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(\frac{6\pi}{7}\right)$

$k = -1$, $z = \cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right)$

$k = -2$, $z = \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$

$k = -3$, $z = \cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$

Therefore, $z = \cos 0 + \mathrm{i}\sin 0,\ \cos\frac{2\pi}{7} + \mathrm{i}\sin\frac{2\pi}{7}$

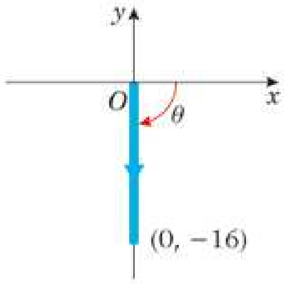
$$\cos\frac{4\pi}{7} + \mathrm{i}\sin\frac{4\pi}{7},\ \cos\frac{6\pi}{7} + \mathrm{i}\sin\frac{6\pi}{7}$$

$$\cos\left(-\frac{2\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{7}\right),\ \cos\left(-\frac{4\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{4\pi}{7}\right)$$

$$\cos\left(-\frac{6\pi}{7}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{7}\right)$$

b $z^4 + 16i = 0$

$z^4 = -16i$



for $-16i$, $r = 16$ and $\theta = -\frac{\pi}{2}$

So $z^4 = 16\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$z^4 = 16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[16\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(-\frac{\pi}{2} + 2k\pi\right)\right)\right]^{\frac{1}{4}}$

$z = 16^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{2} + 2k\pi}{4}\right)\right)$

de Moivre's Theorem.

$z = \left(\cos\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{8} + \frac{k\pi}{2}\right)\right)$

$k = 0, z = 2\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$

$k = 1, z = 2\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$

$k = 2, z = 2\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$

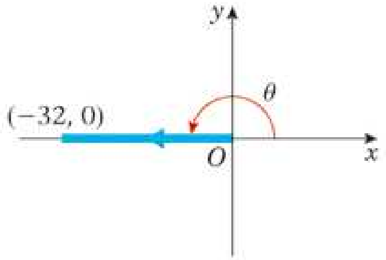
$k = -1, z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$

Therefore, $z = 2\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right), 2\left(\cos\left(\frac{3\pi}{8}\right) + i\sin\left(\frac{3\pi}{8}\right)\right)$

$2\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right), 2\left(\cos\left(-\frac{5\pi}{8}\right) + i\sin\left(-\frac{5\pi}{8}\right)\right)$

c $z^5 + 32 = 0$

$z^5 = -32$



for -32, $r = 32$ and $\theta = \pi$

So $z^5 = 32(\cos\pi + \mathrm{i}\sin\pi)$

$z^5 = 32(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi)) \quad k \in \mathbb{Z}$

Hence, $z = [32(\cos(\pi + 2k\pi) + \mathrm{i}\sin(\pi + 2k\pi))]^{\frac{1}{5}}$

$$z = 32^{\frac{1}{5}}\left(\cos\left(\frac{\pi + 2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{5}\right)\right)$$

$$z = 2\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

de Moivre's Theorem.

$k = 0,\ z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right)$

$k = 1,\ z = 2\left(\cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}\right)$

$k = 1,\ z = 2(\cos\pi + \mathrm{i}\sin\pi)$

$k = 2,\ z = 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$

$k = -1,\ z = 2\left(\cos\left(-\frac{5\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{8}\right)\right)$

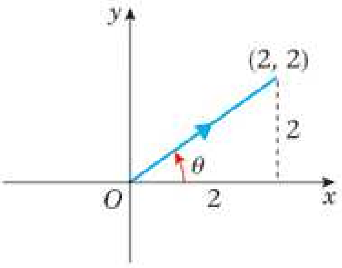
$k = -2,\ z = 2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

Therefore, $z = 2\left(\cos\frac{\pi}{5} + \mathrm{i}\sin\frac{\pi}{5}\right),\ 2\left(\cos\frac{3\pi}{5} + \mathrm{i}\sin\frac{3\pi}{5}\right),$

$2(\cos\pi + \mathrm{i}\sin\pi),\ 2\left(\cos\left(-\frac{\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right),$

$2\left(\cos\left(-\frac{3\pi}{5}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{5}\right)\right)$

d $z^3 = 2 + 2i$



$r = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$

So $z^3 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z^3 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$

$z = (2\sqrt{2})^{\frac{1}{3}}\left(\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right)$

de Moivre's Theorem.

$z = \sqrt{2}\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$

$k = 0, z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k = 1, z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

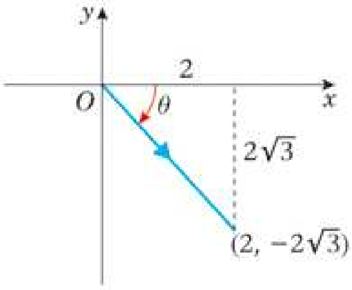
$k = -1, z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$

Therefore, $z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$

$\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right)$

e $z^4 + 2\sqrt{3}\,i = 2$

$z^4 = 2 - 2\sqrt{3}\,i$



$r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$

$\theta = -\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

So $z^4 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

$z^4 = 4\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$

Hence, $z = \left[4\left[\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right]\right]^{\frac{1}{4}}$

$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right)\right)$ de Moivre's Theorem.

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right) + i\sin\left(-\frac{\pi}{12} + \frac{k\pi}{2}\right)\right)$

$k = 0,\ z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$

$k = 1,\ z = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right)$

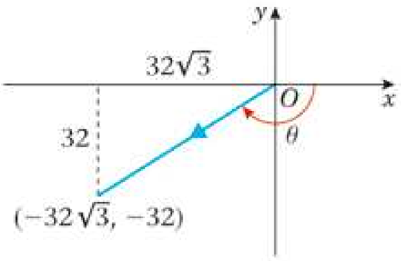
$k = -1,\ z = \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

Therefore, $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right),$

$\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right), \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

f $z^3 + 32\sqrt{3} + 32i = 0$

$z^3 = -32\sqrt{3} - 32i$



$$r = \sqrt{(-32\sqrt{3})^2 + (-32)^2} = \sqrt{3072 + 1024} = \sqrt{4096} = 64$$

$$\theta = -\pi + \tan^{-1}\left(\frac{32}{32\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

$$\text{So } z^4 = 64\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$z^3 = 64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right) \quad k \in \mathbb{Z}$$

$$\text{Hence, } z = \left[64\left(\cos\left(-\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(-\frac{5\pi}{6} + 2k\pi\right)\right)\right]^{\frac{1}{3}}$$

$$z = 64^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{5\pi}{6} + 2k\pi}{3}\right)\right)$$

de Moivre's Theorem.

$$z = 4\left(\cos\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{5\pi}{18} + \frac{2k\pi}{3}\right)\right)$$

$$k = 0,\ z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{18}\right)\right)$$

$$k = 1,\ z = 4\left(\cos\left(\frac{7\pi}{18}\right) + i\sin\left(\frac{7\pi}{18}\right)\right)$$

$$k = -1,\ z = 4\left(\cos\left(-\frac{17\pi}{18}\right) + i\sin\left(-\frac{17\pi}{18}\right)\right)$$

$$\text{Therefore, } z = 4\left(\cos\left(-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{18}\right)\right), 4\left(\cos\left(\frac{7\pi}{18}\right) + i\sin\left(\frac{7\pi}{18}\right)\right),$$

$$4\left(\cos\left(-\frac{17\pi}{18}\right) + i\sin\left(-\frac{17\pi}{18}\right)\right)$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 3

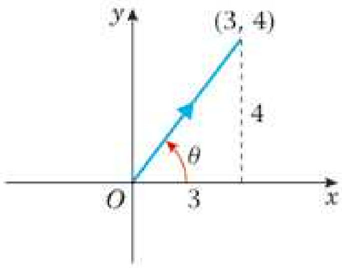
Question:

Solve the following equations, expressing your answers for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give θ to 2 d.p.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

Solution:

a $z^4 = 3 + 4\text{i}$



$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 0.927295...$

So, $z^4 = 5e^{i(0.927295...)}$

$z^4 = 5e^{i(0.927295... + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [5e^{i(0.927295... + 2k\pi)}]^{\frac{1}{4}}$

$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295... + 2k\pi}{4}\right)}$

$= 5^{\frac{1}{4}}e^{i\left(\frac{0.927295...}{4} + \frac{k\pi}{2}\right)}$

de Moivre's Theorem.

$k = 0,\ z = 5^{\frac{1}{4}}e^{i(0.2318...)}$

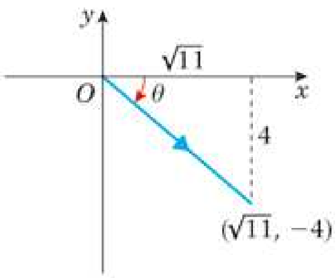
$k = 1,\ z = 5^{\frac{1}{4}}e^{i(1.8026...)}$

$k = -1,\ z = 5^{\frac{1}{4}}e^{i(-1.3389...)}$

$k = -2,\ z = 5^{\frac{1}{4}}e^{i(-2.9097...)}$

Therefore, $z = 5^{\frac{1}{4}}e^{0.23i}, 5^{\frac{1}{4}}e^{1.80i}, 5^{\frac{1}{4}}e^{-1.34i}, 5^{\frac{1}{4}}e^{-2.91i}$

b $z^3 = \sqrt{11} + 4\text{i}$



$r = \sqrt{(\sqrt{11})^2 + (-4^2)} = \sqrt{11 + 16} = \sqrt{27}$

$\theta = -\tan^{-1}\left(\frac{4}{\sqrt{11}}\right) = 0.878528\ldots$

So, $z^3 = \sqrt{27}\,e^{i(-0.878528\ldots)}$

$z^3 = \sqrt{27}\,e^{i(-0.878528\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [\sqrt{27}\,e^{i(-0.878528\ldots + 2k\pi)}]^{\frac{1}{3}}$

$= (\sqrt{27})^{\frac{1}{3}} e^{i\left(\frac{-0.878528\ldots + 2k\pi}{3}\right)}$ — de Moivre's Theorem.

$= \sqrt{3}\,e^{i\frac{-0.878528\ldots}{3} + \frac{2k\pi}{3}}$

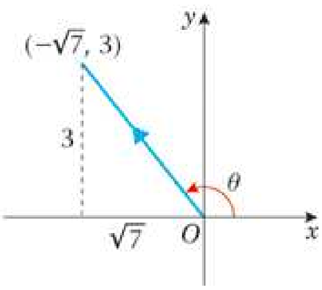
$k = 0,\ z = \sqrt{3}\,e^{i(-0.2928\ldots)}$

$k = 1,\ z = \sqrt{3}\,e^{i(1.8015\ldots)}$

$k = -1,\ z = \sqrt{3}\,e^{i(-2.3872\ldots)}$

Therefore, $z = \sqrt{3}\,e^{-0.29i}, \sqrt{3}\,e^{1.80i}, \sqrt{3}\,e^{-2.39i}$

c $z^4 = -\sqrt{7} + 3i$



$r = \sqrt{(-\sqrt{7})^2 + 3^2} = \sqrt{7+9} = \sqrt{16} = 4$

$\theta = \pi - \tan^{-1}\left(\frac{3}{\sqrt{7}}\right) = 2.293530\ldots$

So, $z^4 = 4e^{i(2.293530\ldots)}$

$z^4 = 4e^{i(2.293530\ldots + 2k\pi)}, \quad k \in \mathbb{Z}$

Hence, $z = [4e^{i(2.293530\ldots + 2k\pi)}]^{\frac{1}{4}}$

$= 4^{\frac{1}{4}} e^{i\left(\frac{2.293530\ldots + 2k\pi}{4}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\, e^{i\left(\frac{2.293530\ldots}{4} + \frac{k\pi}{2}\right)}$

$k = 0,\ z = \sqrt{2}\, e^{i(0.5733\ldots)}$

$k = 1,\ z = \sqrt{2}\, e^{i(2.1441\ldots)}$

$k = -1,\ z = \sqrt{2}\, e^{i(-0.9974\ldots)}$

$k = -2,\ z = \sqrt{2}\, e^{i(-2.5682\ldots)}$

Therefore, $z = \sqrt{2}\, e^{0.57i},\ z = \sqrt{2}\, e^{2.14i},\ z = \sqrt{2}\, e^{-1.00i},\ z = \sqrt{2}\, e^{-2.57i}$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 4

Question:

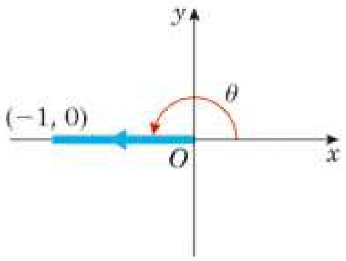
a Find the three roots of the equation $(z + 1)^3 = -1$.
Give your answers in the form $x + iy$, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

Solution:

a $(z + 1)^3 = -1$



For -1, $r = 1$ and $\theta = \pi$

So, $(z + 1)^3 = 1(\cos\pi + i\sin\pi)$

$(z + 1)^3 = (\pi + 24\pi) + i\sin(\pi + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z + 1 = [\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]^{\frac{1}{3}}$

$$z + 1 = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

de Moivre's Theorem.

$$z + 1 = \cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)$$

$k = 0$, $z + 1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\Rightarrow z = -\frac{1}{2} + \sqrt{\frac{3}{2}}i$

$k = 1$, $z + 1 = \cos\pi + i\sin\pi = -1 + 0i$

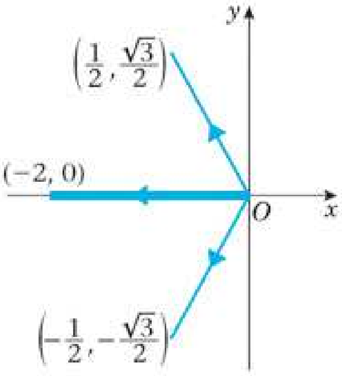
$\Rightarrow z = -2$

$k = -1$, $z + 1 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Therefore, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -2, \frac{1}{2} - \frac{\sqrt{3}}{2}i$

b



c The solutions to $w^3 = -1$, lie on a circle centre (0, 0), radius 1.

As $w = z + 1$, then the three solutions for z are the three solutions for w translated by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Hence the three points (the solutions for z), lie on a circle centre $(-1, 0)$, radius 1.

Solutionbank FP2
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Exercise E, Question 5

Question:

a Find the five roots of the equation $z^5 - 1 = 0$.
Give your answers in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

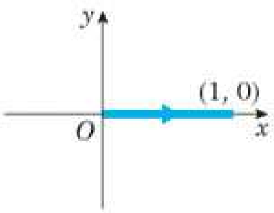
b Given that the sum of all five roots of $z^5 - 1 = 0$ is zero, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

Solution:

a $z^5 - 1 = 0$

$z^5 = 1$



For 1, $r = 1$ and $\theta = 0$

So, $z^5 = 1(\cos 0 + \text{i}\sin 0)$

$z^5 = \cos(0 + 2k\pi) + \text{i}\sin(0 + 2k\pi) \quad k \in \mathbb{Z}$

Hence, $z = [\cos(2k\pi) + \text{i}\sin(2k\pi)]^{\frac{1}{5}}$

$$z = \cos\left(\frac{2k\pi}{5}\right) + \text{i}\sin\left(\frac{2k\pi}{5}\right)$$ de Moivre's Theorem.

$k = 0,\ z_1 = \cos 0 + \text{i}\sin 0 = 1 + \text{i}(0) = 1$

$k = 1,\ z_2 = \cos\left(\frac{2\pi}{5}\right) + \text{i}\sin\left(\frac{2\pi}{5}\right)$

$k = 2,\ z_3 = \cos\left(\frac{4\pi}{5}\right) + \text{i}\sin\left(\frac{4\pi}{5}\right)$

$k = -1,\ z_4 = \cos\left(-\frac{2\pi}{5}\right) + \text{i}\sin\left(-\frac{2\pi}{5}\right)$

$k = -2,\ z_5 = \cos\left(-\frac{4\pi}{5}\right) + \text{i}\sin\left(-\frac{4\pi}{5}\right)$

Therefore $z = 1,\ \cos\left(\frac{2\pi}{5}\right) + \text{i}\sin\left(\frac{2\pi}{5}\right),\ \cos\left(\frac{4\pi}{5}\right) + \text{i}\sin\left(\frac{4\pi}{5}\right),$

$\cos\left(-\frac{2\pi}{5}\right) + \text{i}\sin\left(-\frac{2\pi}{5}\right),\ \cos\left(-\frac{4\pi}{5}\right) + \text{i}\sin\left(-\frac{4\pi}{5}\right)$

b So, $z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$$1 + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$+ \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right) = 0$$

$$\Rightarrow 1 + \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

$$+ \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right) = 0$$

$$1 + 2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = 0$$

$$2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right) = -1$$

$$2\left(\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{4\pi}{5}\right)\right) = -1$$

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2} \text{ (as required)}$$

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Exercise E, Question 6

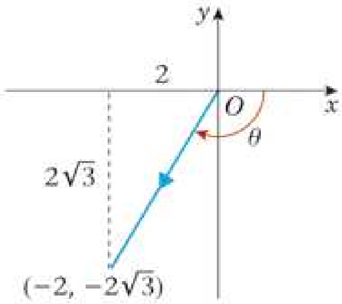
Question:

a Find the modulus and argument of $-2 - 2\sqrt{3}i$.

b Hence find all the solutions of the equation $z^4 + 2 + 2\sqrt{3}i = 0$.
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Solution:

a $-2-2\sqrt{3}\text{i}$.



modulus $= r\sqrt{(-2)^2+(-2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$

argument $=\theta=2\pi+\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)=-\pi+\frac{\pi}{3}=-\frac{2\pi}{3}$

Therefore, $r=4,\ \theta=-\frac{2\pi}{3}$

b $z^4+2+2\sqrt{3}\,\text{i}=0$

$z^4=-2-2\sqrt{3}\,\text{i}$

and $r=4,\ \theta=-\frac{2\pi}{3}$ for $-2-2\sqrt{3}\,\text{i}$

So $z^4=4\text{e}^{\text{i}\left(-\frac{2\pi}{3}\right)}$

$z^4=4\text{e}^{\text{i}\left(-\frac{2\pi}{3}+2k\pi\right)},\quad k\in\mathbb{Z}$

Hence, $z=\left[4\text{e}^{\text{i}\left(-\frac{2\pi}{3}+2k\pi\right)}\right]^{\frac{1}{4}}$

$=4^{\frac{1}{4}}\text{e}^{\text{i}\left(\frac{-\frac{2\pi}{3}+2k\pi}{4}\right)}$

de Moivre's Theorem.

$=\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{6}+\frac{k\pi}{2}\right)}$

$k=0,\ z=\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}$

$k=1,\ z=\sqrt{2}\,\text{e}^{\text{i}\left(\frac{\pi}{3}\right)}$

$k=2,\ z=\sqrt{2}\,\text{e}^{\text{i}\left(\frac{5\pi}{6}\right)}$

$k=-1,\ z=\sqrt{2}\,\text{e}^{\text{i}\left(-\frac{2\pi}{3}\right)}$

Therefore, $z=\sqrt{2}\,\text{e}^{-\frac{\pi\text{i}}{6}},\ \sqrt{2}\,\text{e}^{\frac{\pi\text{i}}{3}},\ \sqrt{2}\,\text{e}^{\frac{5\pi\text{i}}{6}},\ \sqrt{2}\,\text{e}^{-\frac{2\pi\text{i}}{3}}$

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Exercise E, Question 7

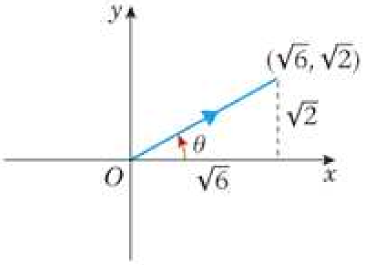
Question:

a Find the modulus and argument of $\sqrt{6} + \sqrt{2}\mathrm{i}$.

b Solve the equation $z^{\frac{3}{4}} = \sqrt{6} + \sqrt{2}\mathrm{i}$.
Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

Solution:

a $\sqrt{6} + \sqrt{2}\text{i}$.



modulus $= r\sqrt{(\sqrt{6})^2 + (\sqrt{2})^2} = \sqrt{6+2} = \sqrt{8}$

argument $= \theta = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Therefore, $r = \sqrt{8},\ \theta = \frac{\pi}{6}$

b $z^{\frac{3}{4}} = \sqrt{6} + \sqrt{2}\,\text{i}$

For $\sqrt{6} + \sqrt{2}\,\text{i},\ r = \sqrt{8},\ \theta = \frac{\pi}{6}$

So, $z^{\frac{3}{4}} = \sqrt{8}\,e^{i\left(\frac{\pi}{6}\right)}$

$z^3 = \left[\sqrt{8}\,e^{i\left(\frac{\pi}{6}\right)}\right]^4$

$z^3 = (\sqrt{8})^4 e^{i\left(\frac{4\pi}{6}\right)}$ — de Moivre's Theorem.

$(\sqrt{8})^4 = \left(8^{\frac{1}{2}}\right)^4 = 8^2 = 64$

$z^3 = 64e^{i\left(\frac{2\pi}{3}\right)}$

$z^3 = 64e^{i\left(\frac{2\pi}{3} + 2k\pi\right)},\quad k \in \mathbb{Z}$

Hence, $z = \left[64e^{i\left(\frac{2\pi}{3} + 2k\pi\right)}\right]^{\frac{1}{3}}$

$= (64)^{\frac{1}{3}} e^{i\left(\frac{\frac{2\pi}{3} + 2k\pi}{3}\right)}$

$= 4e^{i\left(\frac{2\pi}{9} + \frac{2k\pi}{3}\right)}$

$k = 0,\ z = 4e^{i\left(\frac{2\pi}{9}\right)}$

$k = 1,\ z = 4e^{i\left(\frac{8\pi}{9}\right)}$

$k = -1,\ z = 4e^{i\left(-\frac{4\pi}{9}\right)}$

Therefore, $z = 4e^{\frac{2\pi i}{9}},\ z = 4e^{\frac{8\pi i}{9}},\ z = 4e^{-\frac{4\pi i}{9}}$

Solutionbank FP2

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Exercise F, Question 1

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z| = 6$

b $|z| = 10$

c $|z - 3| = 2$

d $|z + 3i| = 3$

e $|z - 4i| = 5$

f $|z + 1| = 1$

g $|z - 1 - i| = 5$

h $|z + 3 + 4i| = 4$

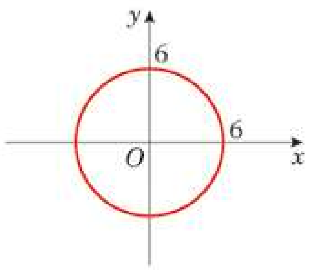
i $|z - 5 + 6i| = 5$

j $|2z + 6 - 4i| = 6$

k $|3z - 9 - 6i| = 12$

Solution:

a $|z| = 6$

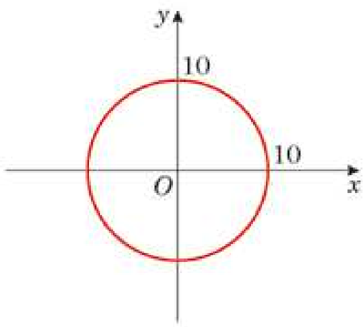


circle centre (0, 0), radius 6

equation: $x^2 + y^2 = 6^2$

$x^2 + y^2 = 36$

b $|z| = 10$

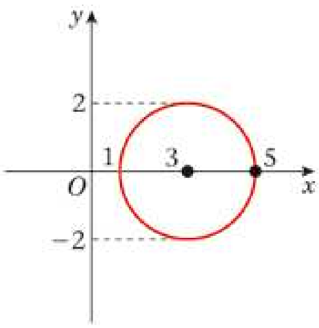


circle centre (0, 0), radius 10

equation: $x^2 + y^2 = 10^2$

$x^2 + y^2 = 100$

c $|z - 3| = 2$

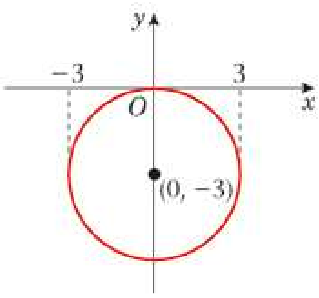


circle centre (3, 0), radius 2

equation: $(x + 3)^2 + y^2 = 2^2$

$(x + 3)^2 + y^2 = 4$

d $|z + 3\text{i}| = 3 \Rightarrow |z - (-3\text{i})| = 3$

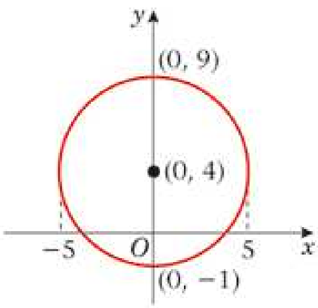


circle centre (0, −3), radius 3

equation: $x^2 + (y - 3)^2 = 3^2$

$x^2 + (y - 3)^2 = 9$

e $|z - 4i| = 5$

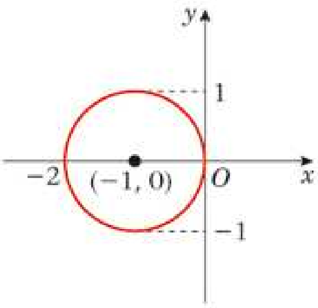


circle centre (0, 4), radius 5

equation: $x^2 + (y - 4)^2 = 5^2$

$x^2 + (y - 4)^2 = 25$

f $|z + 1| = 1 \Rightarrow |z - (-1)| = 1$

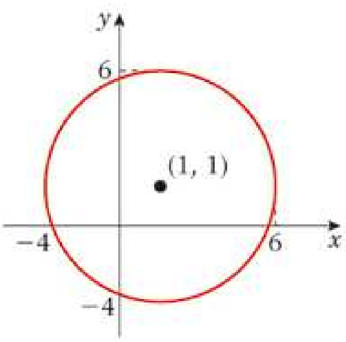


circle centre (−1, 0), radius 1

equation: $(x + 1)^2 + y^2 = 1^2$

$(x + 1)^2 + y^2 = 1$

g $|z - 1 - i| = 5 \Rightarrow |z - (1 + i)| = 5$

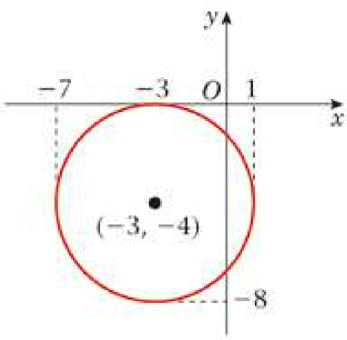


circle centre (1, 1), radius 5

equation: $(x - 1)^2 + (y - 1)^2 = 5^2$

$(x - 1)^2 + (y - 1)^2 = 25$

h $|z + 3 + 4i| = 4 \Rightarrow |z - (-3 - 4i)| = 4$

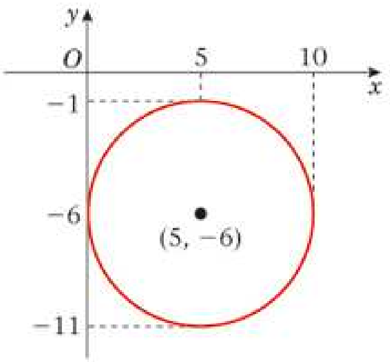


circle centre (−3, −4), radius 4

equation: $(x + 3)^2 + (y + 4)^2 = 4^2$

$(x + 3)^2 + (y + 4)^2 = 16$

i $|z - 5 + 6i| = 5 \Rightarrow |z - (5 - 6i)| = 4$



circle centre $(5, -6)$, radius 5

equation: $(x - 5)^2 + (y + 6)^2 = 5^2$

$(x - 5)^2 + (y + 6)^2 = 25$

j $|2z + 6 - 4i| = 6$

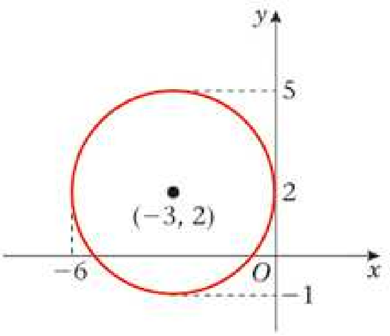
$\Rightarrow |2(z + 3 - 2i)| = 6$

$\Rightarrow |2|\,|z + 3 - 2i| = 6$

$\Rightarrow 2|z + 3 - 2i| = 6$

$\Rightarrow |z + 3 - 2i| = 3$

$\Rightarrow |z - (-3 + 2i)| = 3$



circle centre $(-3, 2)$, radius 3

equation: $(x + 3)^2 + (y - 2)^2 = 3^2$

$(x + 3)^2 + (y - 2)^2 = 9$

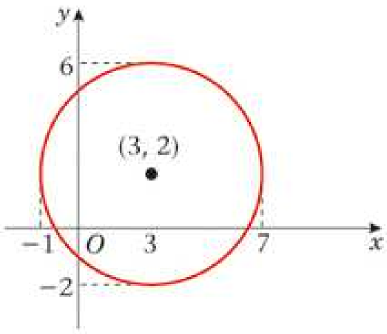
k $|3z - 9 - 6i| = 12$

$\Rightarrow |3(z - 3 - 2i)| = 12$

$\Rightarrow |3|\,|z - 3 - 2i| = 12$

$\Rightarrow 3|z - (3 + 2i)| = 12$

$\Rightarrow |z - (3 + 2i)| = 4$



circle centre (3, 2), radius 4

equation: $(x-3)^2+(y-2)^2=4^2$

$(x-3)^2+(y-2)^2=16$

l $|3z-9-6\text{i}|=9$

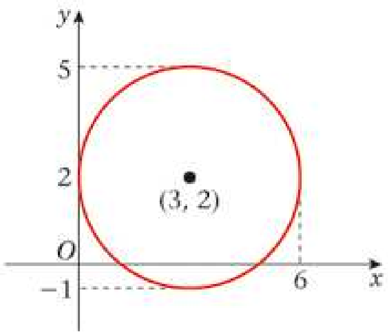
$\Rightarrow |3(z-3-2\text{i})|=9$

$\Rightarrow |3|\,|z-3-2\text{i}|=9$

$\Rightarrow 3|z-3-2\text{i}|=9$

$\Rightarrow |z-3-2\text{i}|=3$

$\Rightarrow |z-(3+2\text{i})|=3$



circle centre (3, 2), radius 3

equation: $(x-3)^2+(y-2)^2=3^2$

$(x-3)^2+(y-2)^2=9$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 2

Question:

Sketch the locus of z when:

a $\arg z = \frac{\pi}{3}$

b $\arg(z + 3) = \frac{\pi}{4}$

c $\arg(z - 2) = \frac{\pi}{2}$

d $\arg(z + 2 + 2i) = -\frac{\pi}{4}$

e $\arg(z - 1 - i) = \frac{3\pi}{4}$

f $\arg(z + 3i) = \pi$

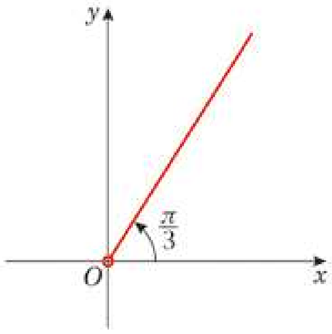
g $\arg(z - 1 + 3i) = \frac{2\pi}{3}$

h $\arg(z - 3 + 4i) = -\frac{\pi}{2}$

i $\arg(z - 4i) = -\frac{3\pi}{4}$

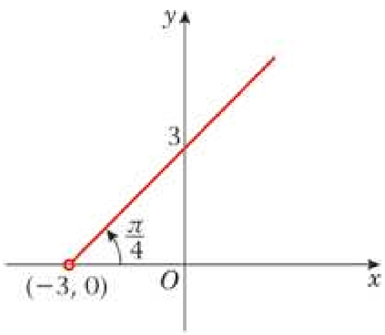
Solution:

a $\arg z = \frac{\pi}{3}$

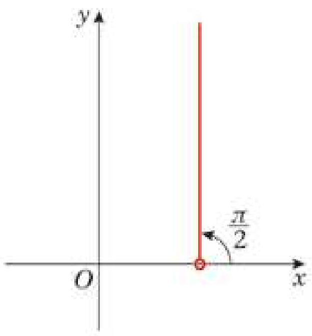


b $\arg(z + 3) = \frac{\pi}{4}$

$\Rightarrow \arg(z - (-3)) = \frac{\pi}{4}$

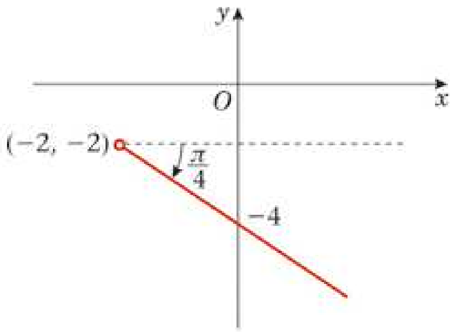


c $\arg(z - 2) = \frac{\pi}{2}$



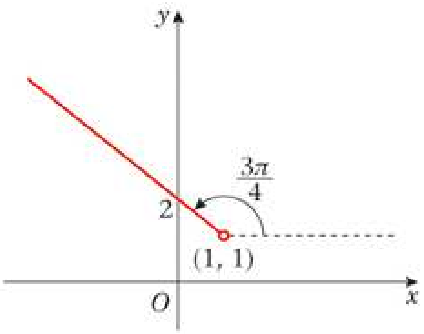
d $\arg(z + 2 + 2\text{i}) = -\frac{\pi}{4}$

$\Rightarrow \arg(z - (-2 - 2\text{i})) = -\frac{\pi}{4}$



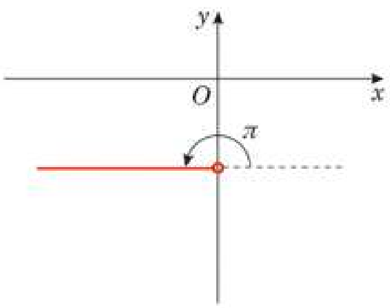
e $\arg(z - 1 - \text{i}) = \frac{3\pi}{4}$

$\Rightarrow \arg(z - (1 + \text{i})) = \frac{3\pi}{4}$



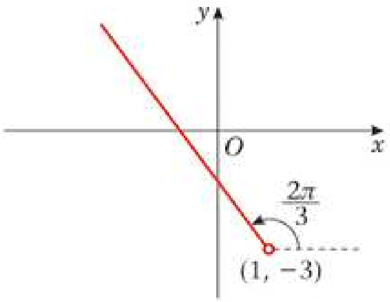
f $\arg(z + 3\text{i}) = \pi$

$\Rightarrow \arg(z - (-3\text{i})) = \pi$



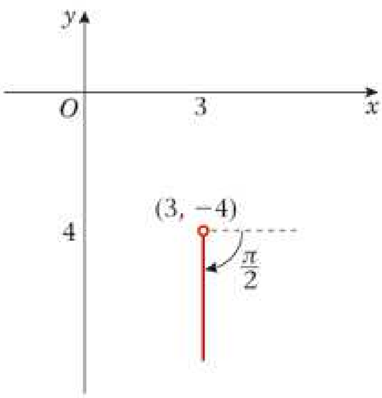
g $\arg(z - 1 + 3i) = \frac{2\pi}{3}$

$\Rightarrow \arg(z - (1 - 3i)) = \frac{2\pi}{3}$

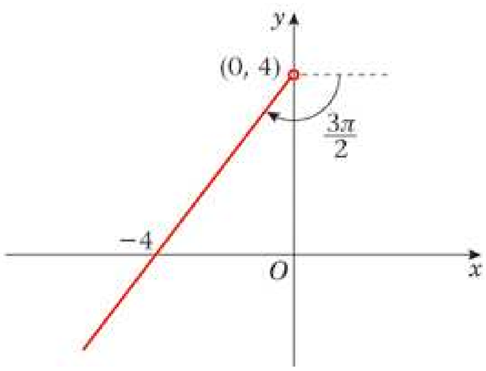


h $\arg(z - 3 + 4i) = -\frac{\pi}{2}$

$\Rightarrow \arg(z - (3 - 4i)) = -\frac{\pi}{2}$



i $\arg(z - 4i) = -\frac{3\pi}{4}$



Solutionbank FP2
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Exercise F, Question 3

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z-6|=|z-2|$

b $|z+8|=|z-4|$

c $|z|=|z+6\text{i}|$

d $|z+3\text{i}|=|z-8\text{i}|$

e $|z-2-2\text{i}|=|z+2+2\text{i}|$

f $|z+4+\text{i}|=|z+4+6\text{i}|$

g $|z+3-5\text{i}|=|z-7-5\text{i}|$

h $|z+4-2\text{i}|=|z-8+2\text{i}|$

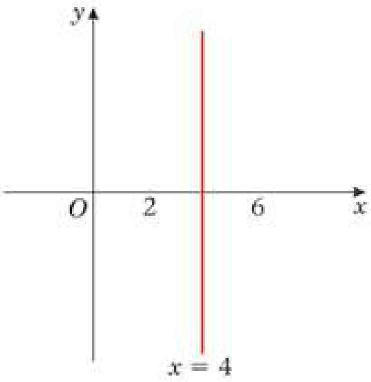
i $\dfrac{|z+3\text{i}|}{|z-6\text{i}|}=1$

j $|z+7+2\text{i}|=|z-4-3\text{i}|$

k $|z+1-6\text{i}|=|2+3\text{i}-z|$

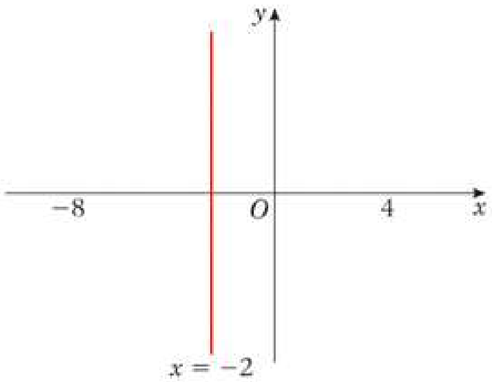
Solution:

a $|z - 6| = |z - 2|$
perpendicular bisector of the line joining (6, 0) and (2, 0).



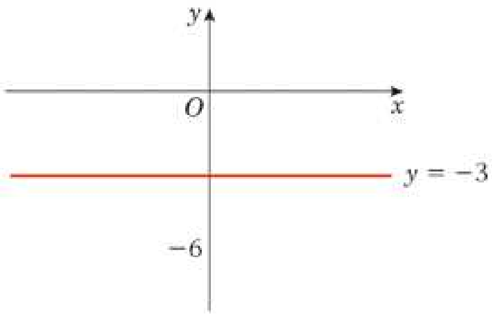
Equation: $x = 4$

b $|z + 8| = |z - 4|$
$\Rightarrow |z - (-8)| = |z - 4|$
perpendicular bisector of the line joining (−8, 0) and (4, 0).



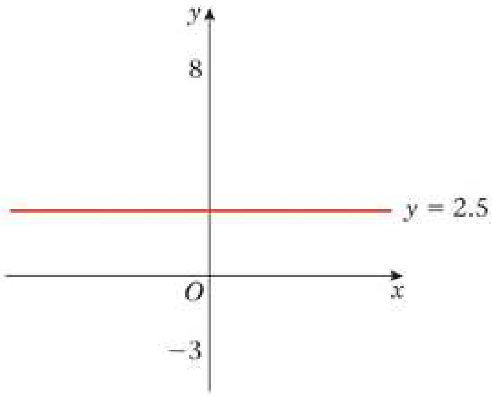
Equation: $x = -2$

c $|z| = |z + 6\text{i}|$
$\Rightarrow |z| = |z - (-6\text{i})|$
perpendicular bisector of the line joining (0, 0) to (0, −6).



Equation: $y = -3$

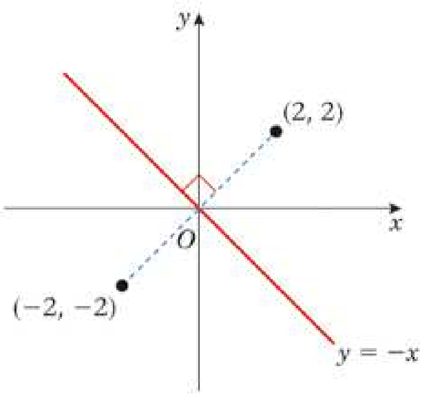
d $|z + 3i| = |z - 8i|$
$\Rightarrow |z - (-3i)| = |z - 8i|$
perpendicular bisector of the line joining (0, −3) to (0, 8).



Equation: $y = 2.5$

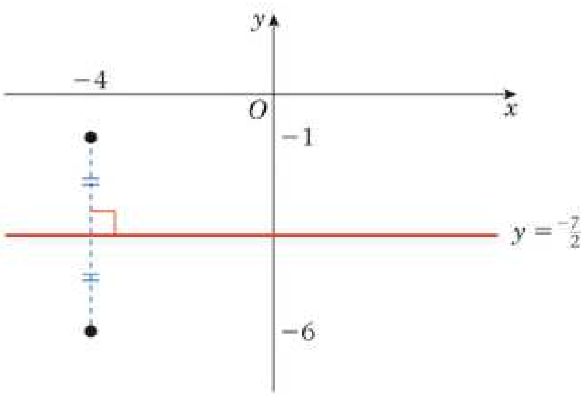
Equation: $y = 2.5$

e $|z - 2 - 2i| = |z + 2 + 2i|$
$\Rightarrow |z - (2 + 2i)| = |z - (-2 - 2i)|$
perpendicular bisector of the line joining (2, 2) to (−2, −2).
So, $|x + iy - 2 - 2i| = |x + iy + 2 + 2i|$
$\Rightarrow |(x - 2) + i(y - 2)| = |(x + 2) + i(y + 2)|$
$\Rightarrow (x - 2)^2 + (y - 2)^2 = (x + 2)^2 + (y + 2)^2$
$\Rightarrow \cancel{x^2} - 4x + 4 + \cancel{y^2} - 4y + 4 = \cancel{x^2} + 4x + 4 + \cancel{y^2} + 4y + 4$
$\Rightarrow -4x - 4y^2 + 8 = 4x + 4y + 8$
$\Rightarrow 0 = 8x + 8y$
$\Rightarrow -8x = 8y$
$\Rightarrow y = -x$



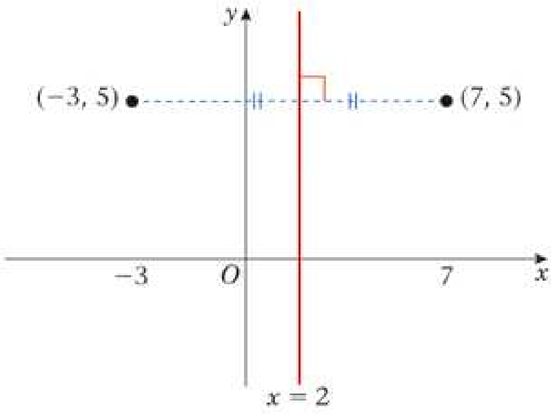
Equation: $y = -x$

f $|z + 4 + i| = |z + 4 + 6i|$
$\Rightarrow |z - (-4 - i)| = |z + (-4 - 6i)|$
perpendicular bisector of the line joining (−4, −1) to (−4, −6).



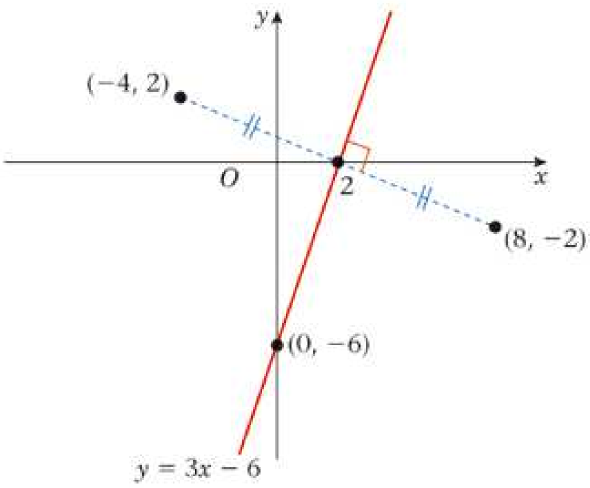
Equation: $y = -\frac{7}{2}$

g $|z + 3 - 5i| = |z - 7 - 5i|$
$\Rightarrow |z - (-3 + 5i)| = |z - (7 + 5i)|$
perpendicular bisector of the line joining $(-3, 5)$ to $(7, 5)$.



Equation: $x = 2$

h $|z + 4 - 2i| = |z - 8 + 2i|$
$\Rightarrow |z - (-4 + 2i)| = |z - (8 - 2i)|$
perpendicular bisector of the line joining $(-4, 2)$ to $(8, -2)$.
So, $|x + iy + 4 - 2i| = |x + iy - 8 + 2i|$
$\Rightarrow |(x + 4) + i(y - 2)| = |(x - 8) + i(y + 2)|$
$\Rightarrow (x + 4)^2 + (y - 2)^2 = (x - 8)^2 + (y + 2)^2$
$\Rightarrow \cancel{x^2} + 8x + 16 + \cancel{y^2} - 4y + 4 = \cancel{x^2} - 16x + 64 + \cancel{y^2} + 4y + 4$
$\Rightarrow 8x - 4y + 20 = -16x + 4y + 68$
$\Rightarrow 0 = -24x + 8y + 48$
$\Rightarrow 0 = -3x + y + 6$
$\Rightarrow 3x - 6 = y$



$y = 0$
$\Rightarrow 3x - 6 = 0$
$\Rightarrow 3x = 6$
$\Rightarrow x = 2$

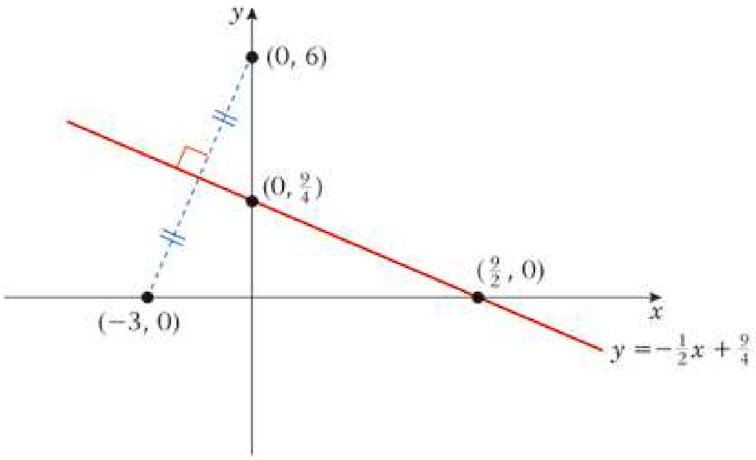
$x = 0, y = -6$

Equation: $y = 3x - 6$

i $\frac{|z+3|}{|z-6i|} = 1$

$\Rightarrow |z+3| = |z-6i|$
$\Rightarrow |z-(-3)| = |z-6i|$
perpendicular bisector of the line joining $(-3, 0)$ to $(0, 6)$.

So, $|x+iy+3| = |x+iy-6i|$
$\Rightarrow |(x+3)+iy| = |x+i(y-6)|$
$\Rightarrow (x+3)^2 + y^2 = x^2 + (y-6)^2$
$\Rightarrow x^2 + 6x + 9 + y^2 = x^2 + y^2 - 12y + 36$
$\Rightarrow 6x + 12y = 36 - 9$
$\Rightarrow 6x + 12y = 27$
$\Rightarrow 2x + 4y = 9$
$\Rightarrow 4y = 9 - 2x$
$\Rightarrow y = -\frac{1}{2}x + \frac{9}{4}$



$y = 0$
$\Rightarrow 0 = 9 - 2x$
$\Rightarrow 2x = 9$
$\Rightarrow x = \frac{9}{2}$

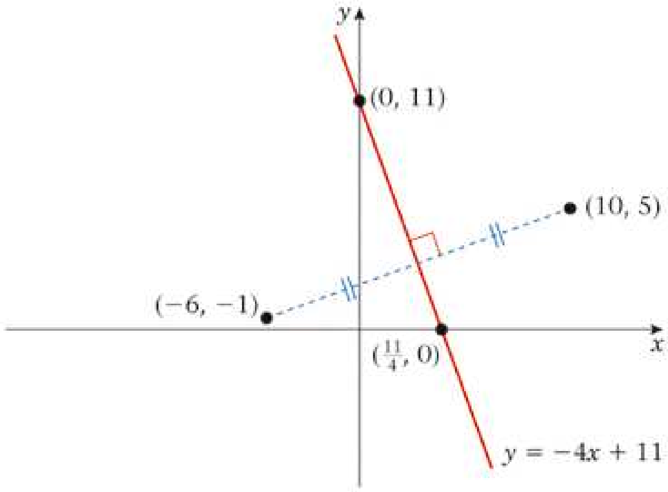
$x = 0, y = \frac{9}{4}$

Equation: $y = -\frac{1}{2}x + \frac{9}{4}$

j $\frac{|z+6-i|}{|z-10-5i|} = 1$

$\Rightarrow |z+6-i| = |z-10-5i|$
$\Rightarrow |z-(-6+i)| = |z-(10+5i)|$
perpendicular bisector of the line joining $(-6, 1)$ to $(10, 5)$.

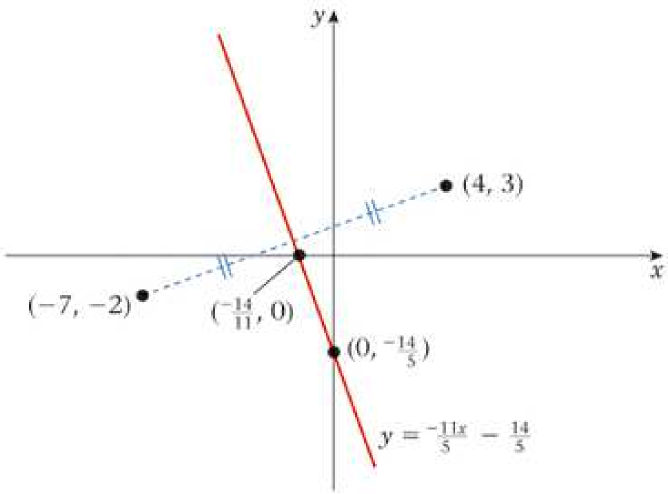
So, $|x+iy+6-i| = |x+iy-10-5i|$
$\Rightarrow |(x+6)+i(y-1)| = |(x-10)+i(y-5)|$
$\Rightarrow (x+6)^2 + (y-1)^2 = (x-10)^2 + (y-5)^2$
$\Rightarrow x^2 + 12x + 36 + y^2 - 2y + 1 = x^2 - 20x + 100 + y^2 - 10y + 25$
$\Rightarrow 12x - 2y + 37 = -20x - 10y + 125$
$\Rightarrow 32x + 8y + 37 - 125 = 0$
$\Rightarrow 32x + 8y - 88 = 0$
$\Rightarrow 4x + y - 11 = 0$
$\Rightarrow y = -4x + 11$



$y = 0$
$\Rightarrow 0 = -4x + 11$
$\Rightarrow 4x = 11$
$\Rightarrow x = \frac{11}{4}$

Equation: $y = -4x + 11$

k $|z + 7 + 2i| = |z - 4 - 3i|$
$\Rightarrow |z - (-7 - 2i)| = |z - (4 + 3i)|$
perpendicular bisector of the line joining (−7, −2) to (4, 3).
So, $|x + iy + 7 + 2i| = |x + iy - 4 - 3i|$
$\Rightarrow |(x + 7) + i(y + 2)| = |(x - 4) + i(y - 3)|$
$\Rightarrow (x + 7)^2 + (y + 2)^2 = (x - 4)^2 + (y - 3)^2$
$\Rightarrow x^2 + 14x + 49 + y^2 + 4y + 4 = x^2 - 8x + 16 + y^2 - 6y + 9$
$\Rightarrow 14x + 4y + 53 = -8x - 6y + 25$
$\Rightarrow 22x + 10y + 28 = 0$
$\Rightarrow 11x + 5y + 14 = 0$
$\Rightarrow 5x = -11x - 14$
$\Rightarrow y = -\frac{11x}{5} - \frac{14}{5}$

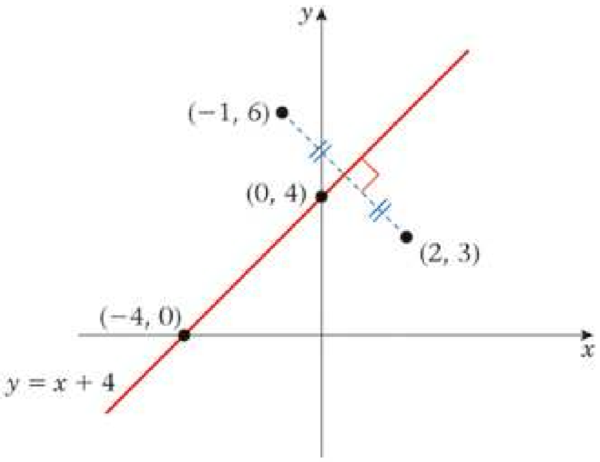


when $x = 0$, $y = -\frac{14}{5}$
when $y = 0$; $0 = -11x - 14$
$14 = -11x$
$-\frac{14}{11} = x$

Equation: $y = -\frac{11x}{5} - \frac{14}{5}$

I $|z + 1 - 6i| = |2 + 3i - z|$
$\Rightarrow |z + 1 - 6i| = |(-1)(z - 2 - 3i)|$
$\Rightarrow |z + 1 - 6i| = |(-1)||z - 2 - 3i|$
$\Rightarrow |z - (-1 + 6i)| = |z - (2 + 3i)|$
perpendicular bisector of the line joining $(-1, 6)$ to $(2, 3)$.

So, $|x + iy + 1 - 6i| = |x + iy - 2 - 3i|$
$\Rightarrow |(x + 1) + i(y - 6)| = |(x - 2) + i(y - 3)|$
$\Rightarrow (x + 1)^2 + (y - 6)^2 = (x - 2)^2 + (y - 3)^2$
$\Rightarrow \cancel{x^2} + 2x + 1 + \cancel{y^2} - 12y + 36 = \cancel{x^2} - 4x + 4 + \cancel{y^2} - 6y + 9$
$\Rightarrow 2x - 12y + 37 = -4x - 6y + 13$
$\Rightarrow 6x - 6y + 24 = 0$
$\Rightarrow x - y + 4 = 0$
$\Rightarrow y = x + 4$



$y = 0$
$x = -4$

$x = 0, y = 4$

Equation: $y = x + 4$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 4

Question:

Find the Cartesian equation of the locus of z when:

a $z - z^* = 0$

b $z + z^* = 0$

Solution:

a $z - z^* = 0$

$\Rightarrow (x + iy) - (x - iy) = 0$

$\Rightarrow 2iy = 0 \quad (\times i)$

$\Rightarrow -2y = 0$

$\Rightarrow y = 0$

The Cartesian equation of the locus of $z - z^* = 0$ is $y = 0$.

$z = x + iy$ $z^* = x - iy$

b $z + z^* = 0$

$\Rightarrow (x + iy) + (x - iy) = 0$

$\Rightarrow 2x = 0$

$x = 0$

The Cartesian equation of the locus of $z + z^* = 0$ is $x = 0$.

$z = x + iy$ $z^* = x - iy$

Solutionbank FP2
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Exercise F, Question 5

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|2 - z| = 3$ **b** $|5i - z| = 4$ **c** $|3 - 2i - z| = 3$

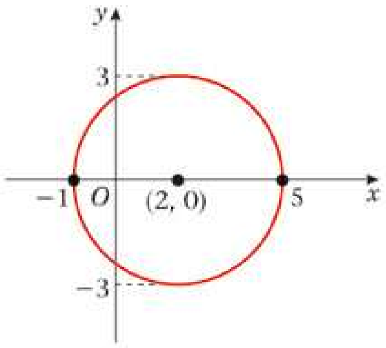
Solution:

a $|2 - z| = 3$

$\Rightarrow |(-1)(z - 2)| = 3$

$\Rightarrow |(-1)||(z - 2)| = 3$ — $|-1| = 1$

$\Rightarrow |(z - 2)| = 3$



circle centre (2, 0), radius 3

equation: $(x - 2)^2 + y^2 = 3^2$

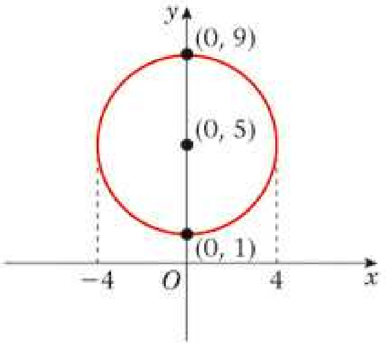
$(x - 2)^2 + y^2 = 9$

b $|5i - z| = 4$

$\Rightarrow |(-1)(z - 5i)| = 4$

$\Rightarrow |(-1)||(z - 5i)| = 4$

$\Rightarrow |(z - 5i)| = 4$ — $|-1| = 1$



circle centre (0, 5), radius 4

equation: $x^2 + (y - 5)^2 = 4^2$

$x^2 + (y - 5)^2 = 16$

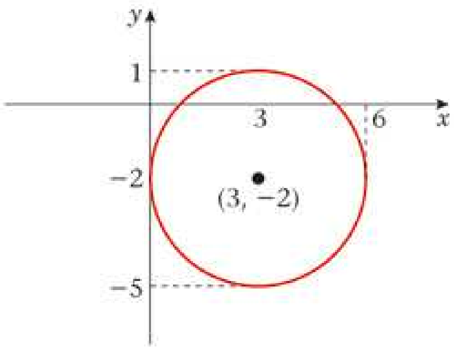
c $|3 - 2i - z| = 3$

$\Rightarrow |(-1)(z - 3 + 2i)| = 3$

$\Rightarrow |(-1)||(z - 3 + 2i)| = 3$

$\Rightarrow |z - 3 + 2i| = 3$ — $|-1| = 1$

$\Rightarrow |z - (3 - 2i)| = 3$



circle centre (3, −2), radius 3

equation: $(x - 3)^2 + (y + 2)^2 = 3^2$

$(x - 3)^2 + (y + 2)^2 = 9$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 6

Question:

Sketch the locus of z and give the Cartesian equation of the locus of z when:

a $|z + 3| = 3|z - 5|$

b $|z - 3| = 4|z + 1|$

c $|z - \text{i}| = 2|z + \text{i}|$

d $|z + 2 - 7\text{i}| = 2|z - 10 + 2\text{i}|$

e $|z + 4 - 2\text{i}| = 2|z - 2 - 5\text{i}|$

f $|z| = 2|2 - z|$

Solution:

a $|z + 3| = 3|z - 5|$

$\Rightarrow |x + \text{i}y + 3| = 3|x + \text{i}y - 3|$

$\Rightarrow |(x + 3) + \text{i}y| = 3|(x - 5) + \text{i}y|$

$\Rightarrow |(x + 3) + \text{i}y|^2 = 3^2|(x - 5) + \text{i}y|^2$

$\Rightarrow (x + 3)^2 + y^2 = 9[(x - 5)^2 + y^2]$

$\Rightarrow x^2 + 6x + 9 + y^2 = 9[(x^2 - 10x + 25 + y^2]$

$\Rightarrow x^2 + 6x + 9 + y^2 = 9x^2 - 90x + 225 + 9y^2$

$\Rightarrow 0 = 8x^2 - 96x + 8y^2 + 216 \quad (\div 8)$

$\Rightarrow x^2 - 12x + y^2 + 27 = 0$

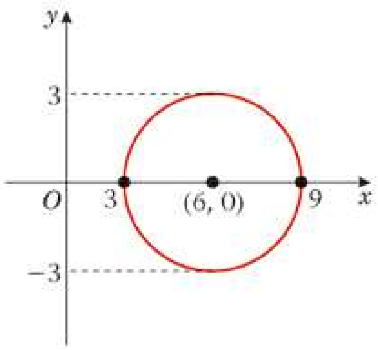
$\Rightarrow (x - 6)^2 - 36 + y^2 + 27 = 0$

$\Rightarrow (x - 6)^2 + y^2 - 9 = 0$

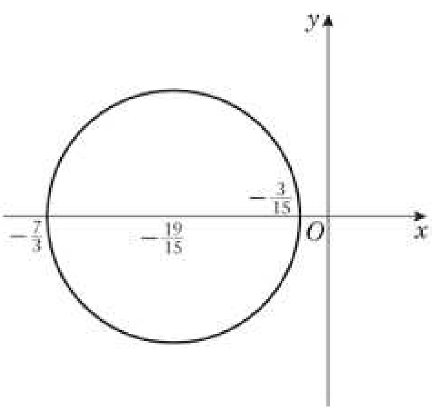
$\Rightarrow (x - 6)^2 + y^2 = 9$

The Cartesian equation of the locus of z is $(x - 6)^2 + y^2 = 9$

This is a circle centre (6, 0), radius = 3



b



$|z - 3| = 4|z + 1|$

$|x + \text{i}y - 3| = 4|x + \text{i}y + 1|$

$|x - 3 + \text{i}y|^2 = 16|x + 1 + \text{i}y|^2$

$(x - 3)^2 + y^2 = 16((x + 1)^2 + y)^2$

$x^2 - 6x + 9 + y^2 = 16(x^2 + 2x + 1 + y^2)$

$= 16x^2 + 32x + 16 + 16y^2$

$15x^2 + 38x + 15y^2 + 7 = 0$

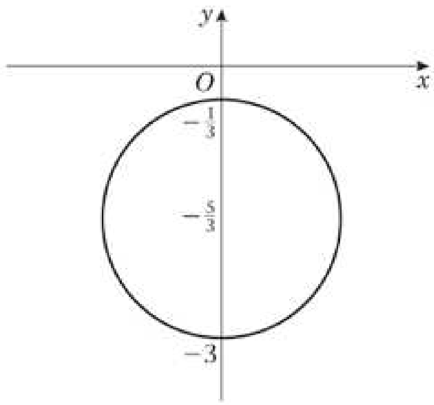
$x^2 + \frac{38}{15}x + y^2 + \frac{7}{15} = 0$

$\left(x + \frac{19}{15}\right)^2 - \frac{19^2}{15^2} + y^2 + \frac{7}{15} = 0$

$\left(x + \frac{19}{15}\right)^2 + y^2 = \frac{256}{225}$

Circle centre $\left(-\frac{19}{15}, 0\right)$ radius $\frac{16}{15}$

c



$|z - i| = 2|z + i|$

$|x + iy - i| = 2|x + iy + i|$

$|x + i(y - 1)|^2 = 4|x + i(y + 1)|^2$

$x^2 + (y - 1)^2 = 4[x^2 + (y + 1)^2]$

$$x^2 + y^2 - 2y + 1 = 4(x^2 + y^2 + 2y)$$
$$= 4x^2 + 4y^2 + 8y$$

$3x^2 + 3y^2 + 10y + 3 = 0$

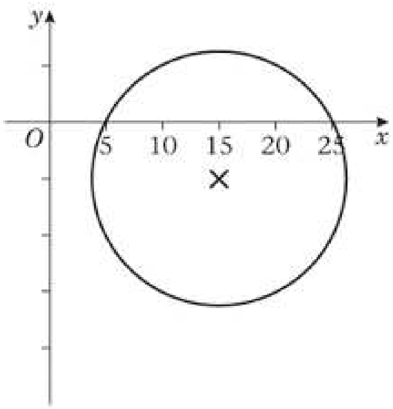
$$x^2 + y^2 + \frac{10}{3}y + 1 = 0$$

$$x^2 + \left(y + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$x^2 + \left(y + \frac{5}{3}\right)^2 = \frac{16}{9}$$

Circle centre $\left(0, -\frac{5}{3}\right)$ radius $\frac{4}{3}$

d



$|z + 2 - 7i| = 2|z - 10 + 2i|$

$|x + iy + 2 - 7i| = 2|x + iy - 10 + 2i|$

$|(x + 2) + i(y - 7)|^2 = 4|(x - 10) + i(y + 2)|^2$

$(x + 2)^2 + (y - 7)^2 = 4[(x - 10)^2 + (y + 2)^2]$

$$x^2 + 4x^2 + 4 + y^2 - 14y + 49 = [x^2 - 20x + 100$$
$$+ y^2 + 4y + 4]$$

$3x^2 - 84x + 3y^2 + 30y + 363 = 0$

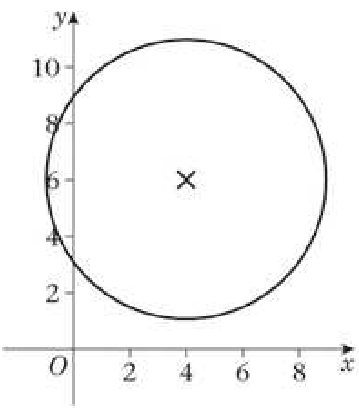
$x^2 - 28x + y^2 + 10y + 121 = 0$

$(x - 14)^2 - 14^2 + (y + 5)^2 - 5^2 + 121 = 0$

$(x - 14)^2 + (y + 5)^2 = 100$

Circle centre $(14, -5)$ radius 10

e



$|z + 4 - 2i| = 2|z - 2 - 5i|$

$|x + iy + 4 - 2i| = 2|x + iy - 2 - 5i|$

$|(x + 4) + i(y - 2)|^2 = 4|(x - 2) + i(y - 5)|^2$

$(x + 4)^2 + (y - 2)^2 = 4[(x - 2)^2 + (y - 5)^2]$

$$x^2 + 8x^2 + 16 + y^2 - 4y + 4 = [x^2 - 4x + 4$$
$$+ y^2 + 10y + 25]$$

$3x^2 - 24x + 3y^2 - 36y + 96 = 0$

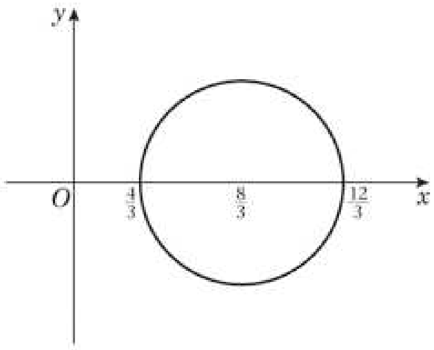
$x^2 - 8x + y^2 - 12y + 32 = 0$

$(x - 4)^2 - 16 + (y - 6)^2 - 36 + 32 = 0$

$(x - 4)^2 + (y - 6)^2 = 20$

Circle centre $(4, 6)$ radius $\sqrt{20} = 2\sqrt{5}$

f



$|z| = 2|2 - z|$

$\quad = 2|-1||z - 2|$

$|x + iy| = 2 \times 1 \times |x + iy - 2|$

$x^2 + y^2 = 4((x - 2)^2 + y^2)$

$x^2 + y^2 = 4(x^2 - 4x + 4 + y^2)$

$3x^2 - 16x + 3y^2 + 16 = 0$

$x^2 - \frac{16}{3}x + y^2 + \frac{16}{3} = 0$

$\left(x - \frac{8}{3}\right)^2 - \frac{64}{9} + y^2 + \frac{16}{3} = 0$

$\left(x - \frac{8}{3}\right)^2 + y^2 = \frac{16}{9}$

Circle centre $\left(\frac{8}{3}, 0\right)$ radius $\frac{4}{3}$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 7

Question:

Sketch the locus of z when:

a $\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$

b $\arg\left(\frac{z-3i}{z+4}\right) = \frac{\pi}{6}$

c $\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

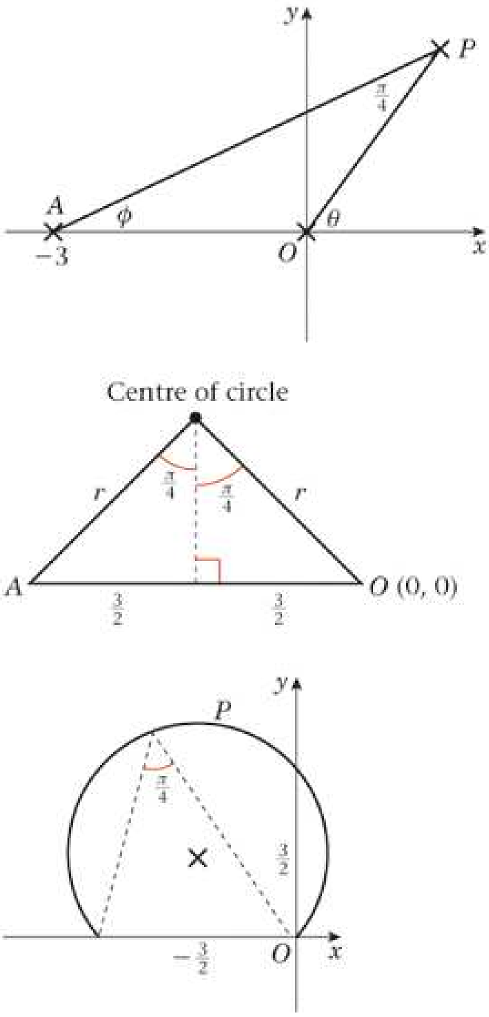
d $\arg\left(\frac{z-3i}{z-5}\right) = \frac{\pi}{4}$

e $\arg z - \arg(z-2+3i) = \frac{\pi}{3}$

f $\arg\left(\frac{z-4i}{z+4}\right) = \frac{\pi}{2}$

Solution:

a



$\arg\left(\frac{z}{z+3}\right) = \frac{\pi}{4}$

$\arg z - \arg(z+3) = \frac{\pi}{4}$

$\arg z - \arg(z-(-3)) = \frac{\pi}{4}$

$\arg z = \theta$

$\arg(z-(-3)) = \phi$

$\theta - \phi = \frac{\pi}{4}$

$\theta = \phi + \frac{\pi}{4}$

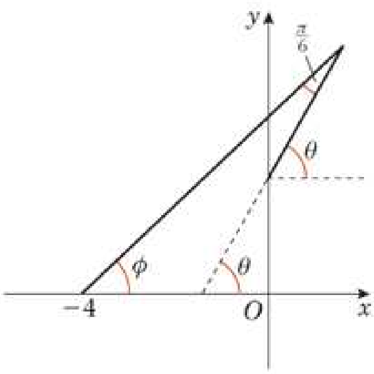
P lies on an arc of a circle cut off at $A(-3, 0)$ and $O(0, 0)$

Angle at the centre is twice the angle at the circumference $\therefore \frac{\pi}{2}$

It follows that the centre is at $\left(-\frac{3}{2}, \frac{3}{2}\right)$

and the radius is $\frac{3}{2}\sqrt{2}$

b



$\arg\left(\frac{z-3\mathrm{i}}{z+4}\right) = \frac{\pi}{6}$

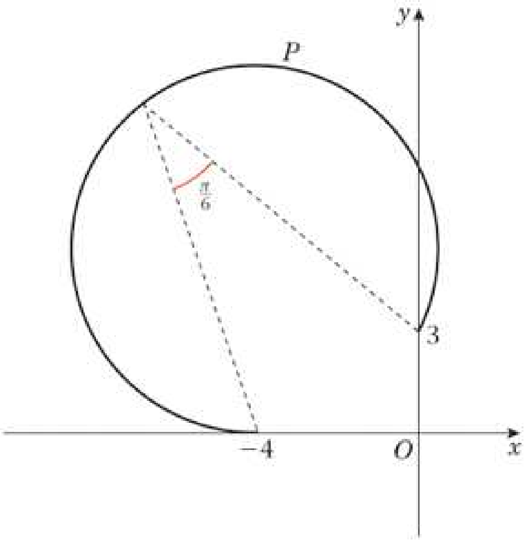
$\arg(z-3\mathrm{i}) - \arg(z-(-4)) = \frac{\pi}{6}$

$\arg(z-3\mathrm{i}) = \theta.$

$\arg(z-(-4)) = \phi$

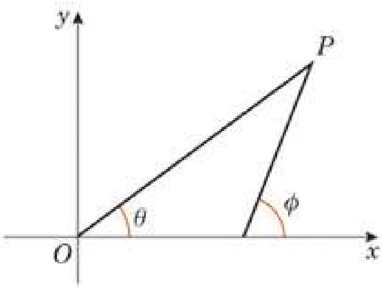
$\theta - \phi = \frac{\pi}{6}$

Arc of a circle from $(-4, 0)$ to $(0, 3)$



(The centre is at $\left(-\frac{4+3\sqrt{3}}{2}, \frac{3+4\sqrt{3}}{2}\right)$ you do not need to calculate this for a sketch!)

c



$\arg\left(\frac{z}{z-2}\right) = \frac{\pi}{3}$

$\arg z = \theta$

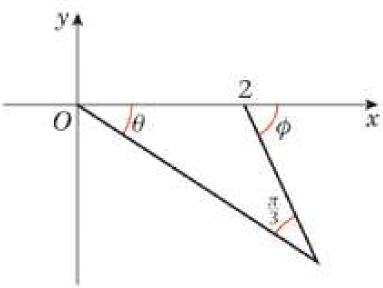
$\arg(z-2) = \phi$

$\theta - \phi = \frac{\pi}{3}$

As our diagram has $\phi > \theta$, we have P on the wrong side of the line joining O or ϕ.

We want the arc below the x-axis.

Redrawing:



$\arg z = -\theta$

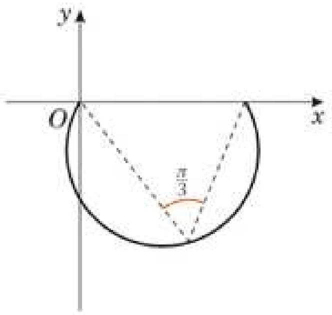
$\arg(z-2) = -\phi$

Hence $\arg z - \arg(z-2) = \frac{\pi}{3}$

becomes $-\theta - (-\phi) = \frac{\pi}{3}$

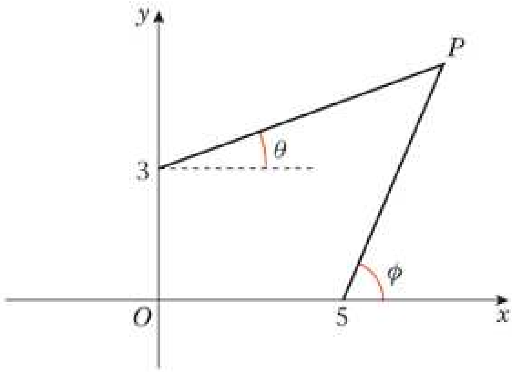
$\phi = \theta + \frac{\pi}{3}$

Arc of a circle, ends 0 and 2, subtending angle $\frac{\pi}{3}$



[The centre is at $\left(1, -\frac{1}{\sqrt{3}}\right)$ radius $\frac{2\sqrt{3}}{3}$ not needed to be calculated for a sketch]

d



$\arg\left(\frac{z - 3i}{z - 5}\right) = \frac{\pi}{4}$

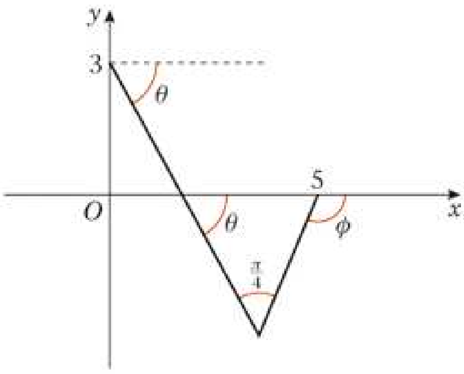
$\arg(z - 3i) - \arg(z - 5) = \frac{\pi}{4}$

$\arg(z - 3i) = \theta$

$\arg(z - 5) = \phi$

$\theta - \phi = \frac{\pi}{4}$

But $\phi > \theta$, we have P on the wrong side of the line joining 3i and 5.

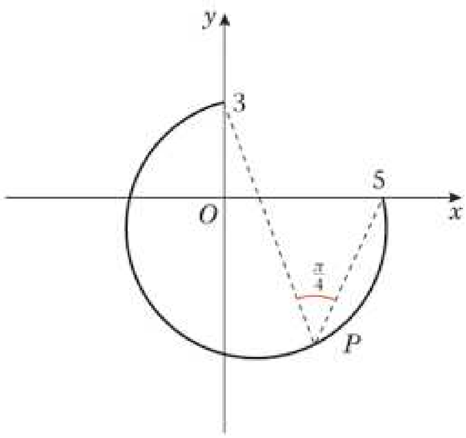


$\arg(z - 3i) = -\theta$

$\arg(z - 5) = -\phi$

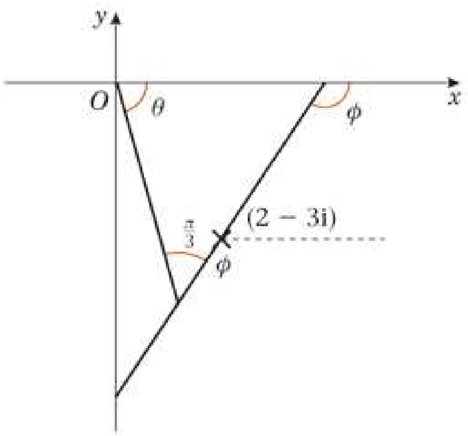
$-\theta - (-\phi) = \frac{\pi}{4}$

$\phi = \theta + \frac{\pi}{4}$



(Arc of Circle centre $(1, -1)$ radius $\sqrt{17}$ not needed for sketch)

e



$\arg z - \arg(z - 2 + 3\text{i}) = \frac{\pi}{3}$

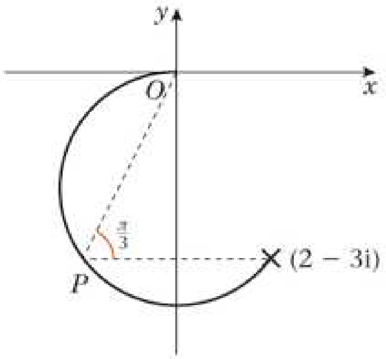
$\arg z - \arg(z - (2 - 3\text{i})) = \frac{\pi}{3}$

$\arg z = -\theta$

$\arg(z - (2 - 3\text{i})) = -\phi$

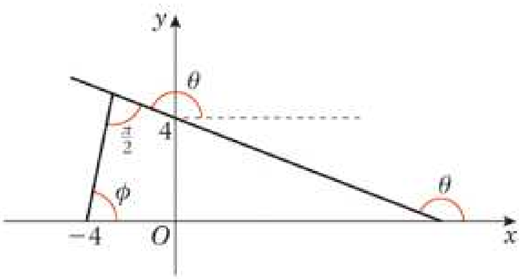
$-\theta - (-\phi) = \frac{\pi}{3}$

$\phi = \theta + \frac{\pi}{3}$



Arc of circle, centre at $\left(\frac{2-\sqrt{3}}{2}, -\frac{9+2\sqrt{3}}{6}\right)$, this need not be calculated for your sketch.

f



$\arg\left(\frac{z - 4\text{i}}{z + 4}\right) = \frac{\pi}{2}$

$\arg(z - 4\text{i}) - \arg(z + 4) = \frac{\pi}{2}$

$\arg(z - 4\text{i}) = \theta$

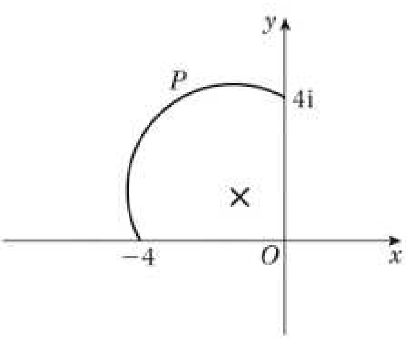
$\arg(z + 4) = \phi = \arg(z - (-4\text{i}))$

$\theta - \phi = \frac{\pi}{2}$

$\theta = \phi + \frac{\pi}{2}$

The locus is an arc of a circle, ends at -4 and 4i, angle subtended being $\frac{\pi}{2}$.

$\therefore$ It is a semi-circle.



(Circle arc has centre $(-2, 2)$, radius $2\sqrt{2}$)

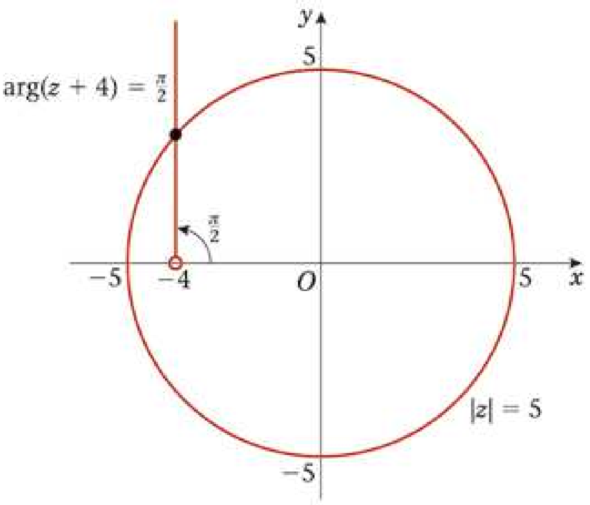
Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 8

Question:

Use the Argand diagram to find the value of z that satisfies the equations $|z| = 5$ and $\arg(z + 4) = \frac{\pi}{2}$.

Solution:



$|z| = 5$
is a circle centre (0, 0), radius 5

$\arg(z + 4) = \frac{\pi}{2}$ is a half-line from (−4, 0) making an angle of $\frac{\pi}{2}$ with the positive x-axis.

$|z| = 5 \Rightarrow x^2 + y^2 = 25$ ①

$\arg(z + 4) = \frac{\pi}{2} \Rightarrow x = -4$ and $y > 0$ ②

Substituting ② into ① gives
$$(-4)^2 + y^2 = 25$$
$$16 + y^2 = 25$$
$$y^2 = 9$$
$$y = 3$$

Therefore, $z = -4 + 3i$

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Edexcel AS and A Level Modular Mathematics

Exercise F, Question 9

Question:

Given that the complex number z satisfies $|z - 2 - 2\mathrm{i}| = 2$,

a sketch, on an Argand diagram, the locus of z.

Given further that $\arg(z - 2 - 2\mathrm{i}) = \frac{\pi}{6}$,

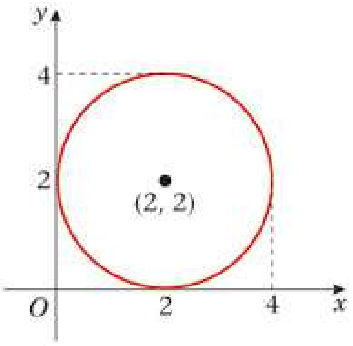
b find the value of z in the form $a + \mathrm{i}b$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

Solution:

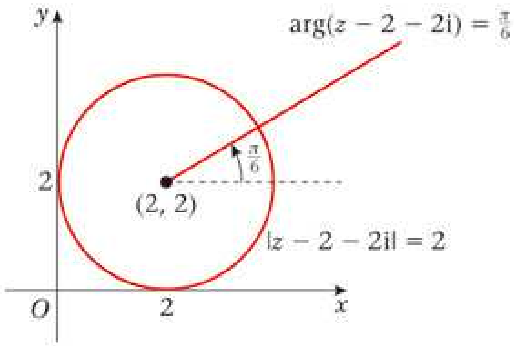
a $|z - 2 - 2\text{i}| = 2$

$\Rightarrow |z - (2 + 2\text{i})| = 2$

The locus of z is a circle centre $(2, 2)$, radius 2.



b $\arg(z - 2 - 2\text{i}) = \frac{\pi}{6}$, is a half-line from $(2, 2)$, as shown below.



$|z - 2 - 2\text{i}| = 2 \Rightarrow (x - 2)^2 + (y - 2)^2 = 4$ ①

$\arg(z - 2 - 2\text{i}) = \frac{\pi}{6} \Rightarrow \arg(x + \text{i}y - 2 - 2\text{i}) = \frac{\pi}{6}$

$\Rightarrow \arg((x - 2) + \text{i}(y - 2)) = \frac{\pi}{6}$

$\Rightarrow \frac{y - 2}{x - 2} = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(x - 2)$

$\Rightarrow (y - 2)^2 = \left[\frac{1}{\sqrt{3}}(x - 2)\right]^2$

$\Rightarrow (y - 2)^2 = \frac{1}{3}(x - 2)^2$ ②

Substituting ② into ① gives $(x-2)^2 + \frac{1}{3}(x-2)^2 = 4$

$\Rightarrow \frac{4}{3}(x-2)^2 = 4$

$\Rightarrow 4(x-2)^2 = 12$

$\Rightarrow (x-2)^2 = 3$

$\Rightarrow x - 2 = \pm\sqrt{3}$

$\Rightarrow x = 2 \pm \sqrt{3}$

From the Argand diagram, $x > 2$.

So $x = 2 + \sqrt{3}$ ③

As $y - 2 = \frac{1}{\sqrt{3}}(x - 20$ ④

Substituting ③ into ④ gives $y - 2 = \frac{1}{\sqrt{3}}(2 + \sqrt{3} - 2)$

$\Rightarrow y - 2 = \frac{1}{\sqrt{3}}(\sqrt{3})$

$\Rightarrow y - 2 = 1$

$\Rightarrow y = 3$

Therefore, $z = (2 + \sqrt{3}) + 3\text{i}$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise F, Question 10

Question:

Sketch on the same Argand diagram the locus of points satisfying

a $|z-2i| = |z-8i|$, **b** $\arg(z-2-i) = \frac{\pi}{4}$.

The complex number z satisfies both $|z-2i| = |z-8i|$ and $\arg(z-2-i) = \frac{\pi}{4}$.

c Use your answers to parts **a** and **b** to find the value of z.

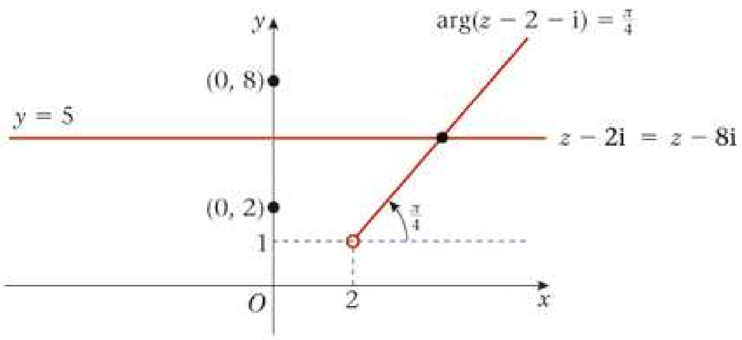
Solution:

a $|z-2i| = |z-8i|$

perpendicular bisector of the line joining (0, 2) to (0, 8), having equation $y = 5$.

b $\arg(z-2-i) = \frac{\pi}{4}$

is a half-line from (1, 1), as shown below.



c $|z-2i| = |z-8i| \Rightarrow y = 5$ ①

$\arg(z-2-i) = \frac{\pi}{4} \Rightarrow \arg(x+iy-2-i) = \frac{\pi}{4}$

$\Rightarrow \arg((x-2)+i(y-1)) = \frac{\pi}{4}$

$\Rightarrow \frac{y-1}{x-2} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{y-1}{x-2} = 1$

$\Rightarrow y-1 = x-2$

$\Rightarrow y - x - 1$ ②

Substituting ① into ② gives $5 = x - 1$

$\Rightarrow 6 = x$

Therefore, $z = 6 + 5i$

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Exercise F, Question 11

Question:

Sketch on the same Argand diagram the locus of points satisfying

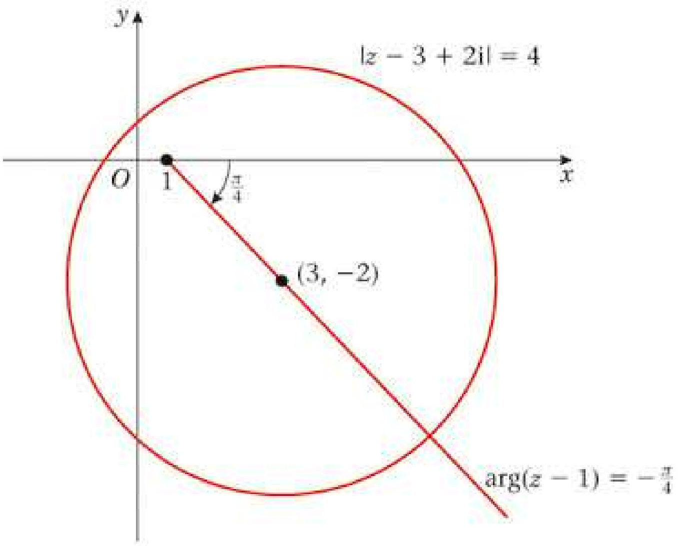
a $|z - 3 + 2i| = 4$ **b** $\arg(z - 1) = -\frac{\pi}{4}$.

The complex number z satisfies both $|z - 3 + 2i| = 4$ and $\arg(z - 1) = -\frac{\pi}{4}$.

Given that $z = a + ib$ where $a \in \mathbb{R}$ and $b \in \mathbb{R}$,

c find the exact value of a and the exact value of b.

Solution:



a $|z - 3 + 2i| = 4$ is a circle centre $(3, -2)$ radius 4.

b $\arg(z - 1) = -\frac{\pi}{4}$ is a half-line from $(1, 0)$ making an angle of $-\frac{\pi}{4}$ with the positive x-axis.

c $|z - 3 + 2i| = 4 \Rightarrow (x - 3)^2 + (y + 2)^2 = 16$ ①

$\arg(z - 1) = -\frac{\pi}{4} \Rightarrow \arg(x + iy - 1) = -\frac{\pi}{4}$

$\Rightarrow \arg((x - 1) + iy) = -\frac{\pi}{4}$

$\Rightarrow \frac{y}{x - 1} = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \frac{y}{x - 1} = -1$

$\Rightarrow y = -1(x - 1)$

$\Rightarrow y = -x + 1$ ② for $x > 1, y < 0$

Substituting ② into ① gives $(x-3)^2 + (-x+1+2)^2 = 16$

$\Rightarrow (x-3)^2 + (-x+3)^2 = 16$

$\Rightarrow x^2 - 6x + 9 + x^2 - 6x + 9 = 16$

$\Rightarrow 2x^2 - 12x + 18 = 16$

$\Rightarrow 2x^2 - 12x + 2 = 0$

$\Rightarrow x^2 - 6x + 1 = 0$

$$\Rightarrow x = \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{32}}{2}$$

$$\Rightarrow x = \frac{6 \pm \sqrt{16}\sqrt{2}}{2}$$

$$\Rightarrow x = \frac{6 \pm 4\sqrt{2}}{2}$$

$\Rightarrow x = 3 \pm 2\sqrt{2}$

as $x > 1$ then $x = 3 \pm 2\sqrt{2}$

② $\Rightarrow y = -(3 + 2\sqrt{2}) + 1$

$\Rightarrow y = -3 - 2\sqrt{2} + 1$

$\Rightarrow y = -2 - 2\sqrt{2}$

Therefore, $z = (3 + 2\sqrt{2}) + (-2 - 2\sqrt{2})\text{i}$

Note: $z = a + \text{i}b$

So $a = 3 + 2\sqrt{2}$, $b = -2 - 2\sqrt{2}$

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Exercise F, Question 12

Question:

On an Argand diagram the point P represents the complex number z.
Given that $|z - 4 - 3\text{i}| = 8$,

a find the Cartesian equation for the locus of P,

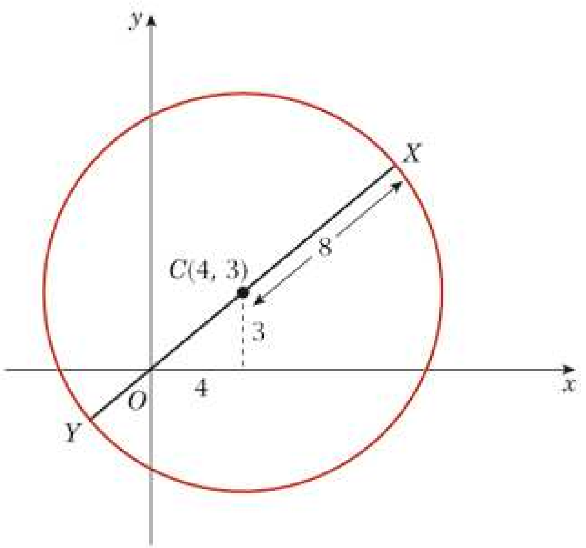
b sketch the locus of P,

c find the maximum and minimum values of $|z|$ for points on this locus.

Solution:

a $|z - 4 - 3\text{i}| = 8 \Rightarrow |z - (4 + 3\text{i})| = 8$
circle centre (4, 3), radius 8
Hence the Cartesian equation of the locus of P is $(x - 4)^2 + (y - 3)^2 = 64$

b



c $|z|$ is the distance from (0, 0) to the locus of points.
$|z|_{max}$ is the distance OX.
$|z|_{min}$ is the distance OY.
Note radius $= CY = CX = 8$
and $OC = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$
$|z|_{max} = OC + CX = 5 + 8 = 13$
$|z|_{min} = CY - OC = 8 - 5 = 3$

The maximum value of $|z|$ is 13 and the minimum value of $|z|$ is 3.

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Exercise F, Question 13

Question:

On an Argand diagram the point P represents the complex number z.
Given that $|z - 4 - 3i| = 8$,

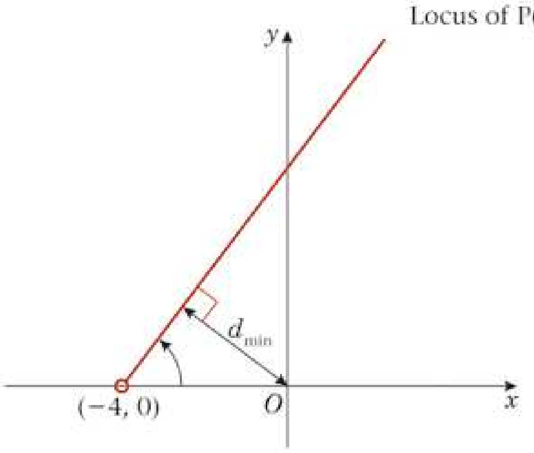
a find the Cartesian equation for the locus of P,

b sketch the locus of P,

c find the maximum and minimum values of $|z|$ for points on this locus.

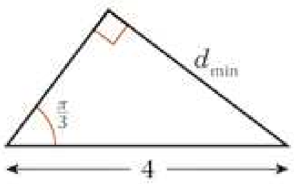
Solution:

a



b $|z|$ is the distance from (0, 0) to the locus of points.
Marked as d_{min} on the Argand diagram is the minimum value of $|z|$.

Hence,



$$\frac{d_{min}}{4} = \sin\left(\frac{\pi}{3}\right)$$

$$d_{min} = 4\sin\left(\frac{\pi}{3}\right)$$

$$d_{min} = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$$

Hence the minimum value of $|z|$ is $|z|_{min} = 2\sqrt{3}$.

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Exercise F, Question 14

Question:

The complex number $z = x + \text{i}y$ satisfies the equation $|z + 1 + \text{i}| = 2|z + 4 - 2\text{i}|$.

The complex number z is represented by the point P on the Argand diagram.

a Show that the locus of P is a circle with centre $(-5, 3)$.

b Find the exact radius of this circle.

Solution:

a $|z + 1 + \text{i}| = 2|z + 4 - 2\text{i}|$

$\Rightarrow |x + \text{i}y + 1 + \text{i}| = 2|x + \text{i}y + 4 - 2\text{i}|$

$\Rightarrow |(x + 1) + \text{i}(y + 1)| = 2|(x + 4) + \text{i}(y - 2)|$

$\Rightarrow |(x + 1) + \text{i}(y + 1)|^2 = 2^2|(x + 4) + \text{i}(y - 2)|^2$

$\Rightarrow (x + 1)^2 + (y + 1)^2 = 4[(x + 4)^2 + (y - 2)^2]$

$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4[x^2 + 8x + 16 + y^2 - 4y + 4]$

$\Rightarrow x^2 + 2x + 1 + y^2 + 2y + 1 = 4x^2 + 32x + 64 + 4y^2 - 16y + 16$

$\Rightarrow 0 = 3x^2 + 30x + 3y^2 - 18y + 64 + 16 - 1 - 1$

$\Rightarrow 3x^2 + 30x + 3y^2 - 18y + 78 = 0$

$\Rightarrow x^2 + 10x + y^2 - 6y + 26 = 0$

$\Rightarrow (x + 5)^2 - 25 + (y - 3)^2 - 9 + 26 = 0$

$\Rightarrow (x + 5)^2 + (y - 3)^2 = 25 + 9 - 26$

$\Rightarrow (x + 5)^2 + (y - 3)^2 = 8$

Therefore the locus of P is a circle centre $(-5, 3)$. (as required)

b radius $= \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

The exact radius is $2\sqrt{2}$.

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Exercise F, Question 15

Question:

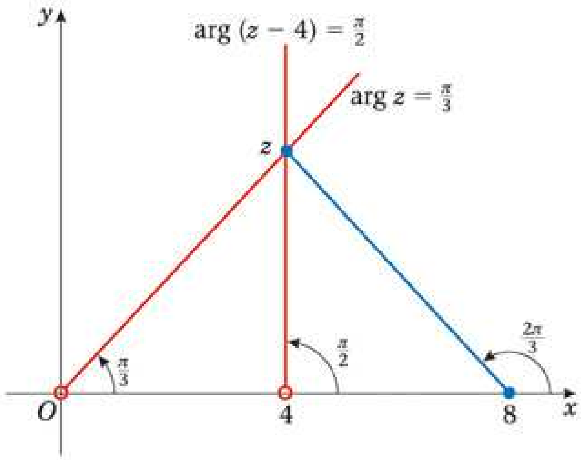
If the complex number z satisfies both $\arg z = \frac{\pi}{3}$ and $\arg(z - 4) = \frac{\pi}{2}$,

a find the value of z in the form $a + ib$, where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

b Hence, find $\arg(z - 8)$.

Solution:

a



From part **b** $\arg(z - 8) = \frac{2\pi}{3}$.

$\arg z = \frac{\pi}{3} \Rightarrow \arg(x + iy) = \frac{\pi}{3}$

$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{3}$

$\Rightarrow \frac{y}{x} = \sqrt{3}$

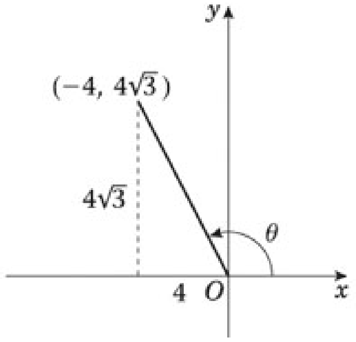
$\Rightarrow y = \sqrt{3}x$ (for $x > 0, y > 0$) ①

$\arg(z - 4) = \frac{\pi}{2} \Rightarrow x = 4$ (for $y > 0$) ②

Substituting ② and ① gives $y = \sqrt{3}(4) = 4\sqrt{3}$

The value of z satisfying both equations is $z = 4 + 4\sqrt{3}\,i$.

b $\arg(z - 8\text{i}) = \arg(4 + 4\sqrt{3}\,\text{i} - 8)$
$= \arg(-4 + 4\sqrt{3}\,\text{i}) = \theta$



$\therefore \theta = \pi - \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \pi - \frac{\pi}{3}$

$\theta = \frac{2\pi}{3}$

Therefore, $\arg(z - 8) = \frac{2\pi}{3}$.

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Exercise F, Question 16

Question:

The point P represents a complex number z in an Argand diagram.

Given that $|z + 2 - 2\sqrt{3}i| = 2$,

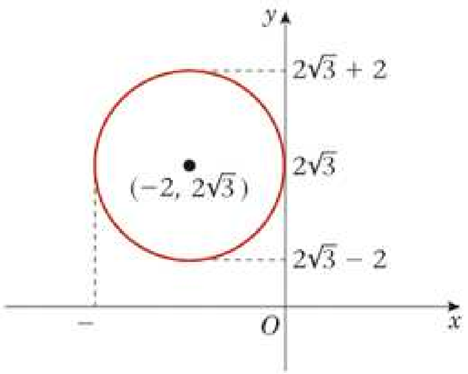
a sketch the locus of P on an Argand diagram.

b Write down the minimum value of arg z.

c Find the maximum value of arg z.

Solution:

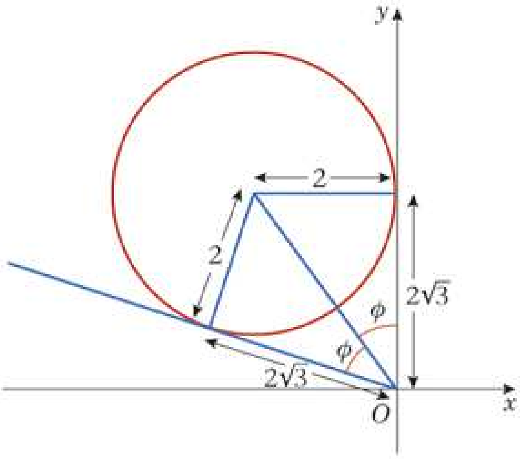
a



$|z + 2 - 2\sqrt{3}\,i| = 2$ is a circle centre $(-2, 2\sqrt{3})$, radius 2.

b From the diagram, the minimum value of arg(z) is $\frac{\pi}{2}$.

c



The maximum value of arg z is $\frac{\pi}{2} + \phi + \phi = \frac{\pi}{2} + 2\phi$.

$\tan\phi = \frac{2}{2\sqrt{3}}$

$\Rightarrow \tan\phi = \frac{1}{\sqrt{3}}$

$\Rightarrow \phi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\arg(z)_{\max} = \frac{\pi}{2} + 2\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$.

The maximum value of arg(z) is $\frac{5\pi}{6}$.

Solutionbank FP2

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Exercise F, Question 17

Question:

The point P represents a complex number z in an Argand diagram.

Given that $\arg z - \arg(z+4) = \frac{\pi}{4}$ is a locus of points P lying on an arc of a circle C,

a sketch the locus of points P,

b find the coordinates of the centre of C,

c find the radius of C,

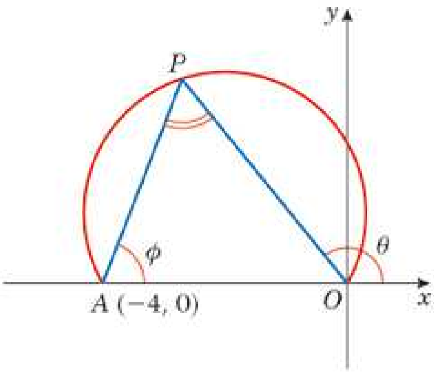
d find a Cartesian equation for the circle C,

e find the finite area bounded by the locus of P and the x-axis.

Solution:

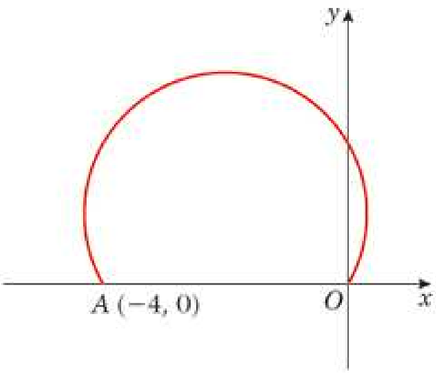
a $\arg(z) - \arg(z + 4) = \frac{\pi}{4}$

$\Rightarrow \theta - \phi = \frac{\pi}{4}$, where $\arg(z) = \theta$ and $\arg(z + 4) = \phi$

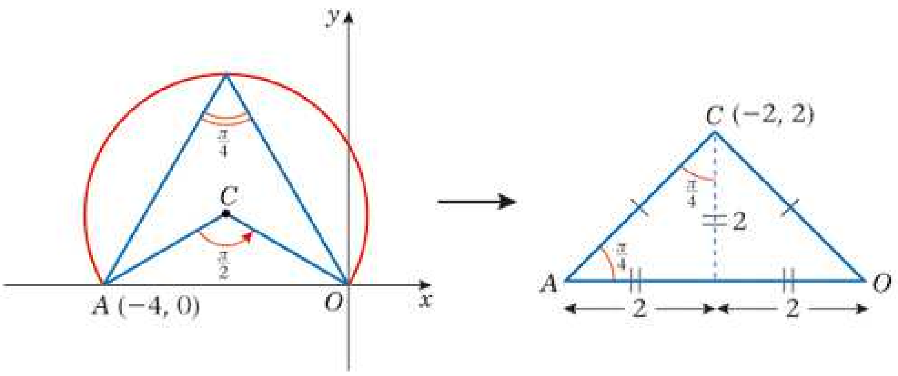


from $\triangle AOP$,
$A\hat{P}O + \phi = \theta$
$\Rightarrow A\hat{P}O = \theta - \phi$
$\Rightarrow A\hat{P}O = \frac{\pi}{4}$

The locus of points P is an arc of a circle cut off at $(-4, 0)$ and $(0, 0)$, as shown below.



b



Therefore the centre of the circle has coordinates $(-2, 2)$.

c $r = \sqrt{2^2 + 2^2} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

Therefore, the radius of C is $2\sqrt{2}$.

d The Cartesian equation of C is $(x + 2)^2 + (y - 2)^2 = 8$.

e Finite area = Area of major sector ACO + Area $\triangle ACO$

$$= \frac{1}{2}(\sqrt{8})^2\left(2\pi - \frac{\pi}{2}\right) + \frac{1}{2}(4)(2)$$

$$= \frac{1}{2}(8)\left(2\pi - \frac{\pi}{2}\right) + 4$$

$$= 4\left(\frac{3\pi}{2}\right) + 4$$

$$= 6\pi + 4$$

Finite area bounded by the locus of P and the x-axis is $6\pi + 4$.

b, **c**, **d** Method ②:

$$\arg z - \arg(z + 4) = \arg\left(\frac{z}{z+4}\right)$$

$$= \arg\left(\frac{x + \mathrm{i}y}{x + \mathrm{i}y + 4}\right)$$

$$= \arg\left[\frac{x + \mathrm{i}y}{(x+4) + \mathrm{i}y}\right]$$

$$= \arg\left[\frac{(x + \mathrm{i}y)}{(x+4) + \mathrm{i}y} \times \frac{(x+4) - \mathrm{i}y}{(x+4) - \mathrm{i}y}\right]$$

$$= \arg\left[\frac{x(x+4) - \mathrm{i}yx + \mathrm{i}y(x+4) + y^2}{(x+4)^2 + y^2}\right]$$

$$= \arg\left[\left(\frac{x(x+4) + y^2}{(x+4)^2 + y^2}\right) + \mathrm{i}\left(\frac{y(x+4) - yx}{(x+4)^2 + y^2}\right)\right]$$

$$= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x+4)^2 + y^2}\right) + \mathrm{i}\left(\frac{xy + 4y - xy}{(x+4)^2 + y^2}\right)\right]$$

$$= \arg\left[\left(\frac{x^2 + 4x + y^2}{(x+4)^2 + y^2}\right) + \mathrm{i}\left(\frac{4y}{(x+4)^2 + y^2}\right)\right]$$

Applying $\arg\left(\frac{z}{z+4}\right) = \frac{\pi}{4} \Rightarrow \frac{\left(\frac{4y}{(x+4)^2 + y^2}\right)}{\left(\frac{x^2 + 4x + y^2}{(x+4)^2 + y^2}\right)} = \tan\left(\frac{\pi}{4}\right) = 1$

$$\Rightarrow \frac{4y}{x^2 + 4x + y^2} = 1$$

$$\Rightarrow 4y = x^2 + 4x + y^2$$

$$\Rightarrow 0 = x^2 + 4x + y^2 - 4y$$

$$\Rightarrow (x+2)^2 - 4 + (y-2)^2 - 4 = 0$$

$$\Rightarrow (x+2)^2 + (y-2)^2 = 8$$

$$\Rightarrow (x+2)^2 + (y-2)^2 = (2\sqrt{2})^2$$

C is a circle with centre $(-2, 2)$, radius $2\sqrt{2}$ and has Cartesian equation $(x+2)^2 + (y-2)^2 = 8$.

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Edexcel AS and A Level Modular Mathematics

Exercise G, Question 1

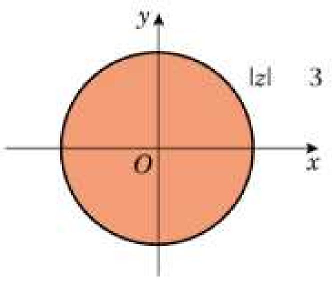
Question:

On an Argand diagram shade in the regions represented by the following inequalities:

a $|z| < 3$ **b** $|z - 2i| > 2$ **c** $|z + 7| \geqslant |z - 1|$ **d** $|z + 6| > |z + 2 + 8i|$
e $2 \leqslant |z| \leqslant 3???$ **f** $1 \leqslant |z + 4i| \leqslant 4$ **g** $3 \leqslant |z - 3 + 5i| \leqslant 5$ **h** $2|z| . |z - 3|$

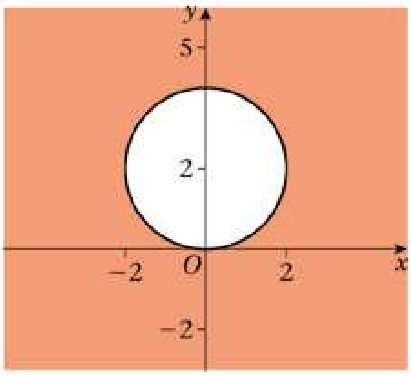
Solution:

a

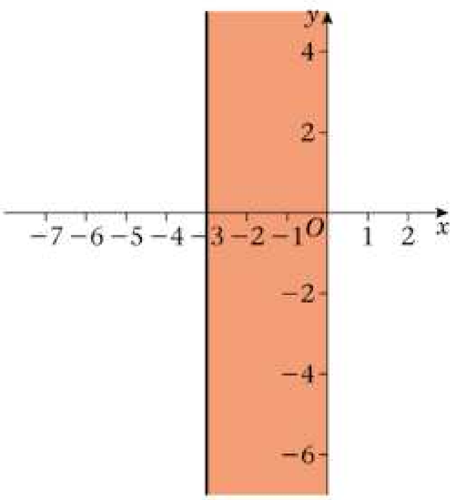


$|z| = 3$ represents a circle centre (0, 0), radius 3

b $|z - (2i)| > 2$

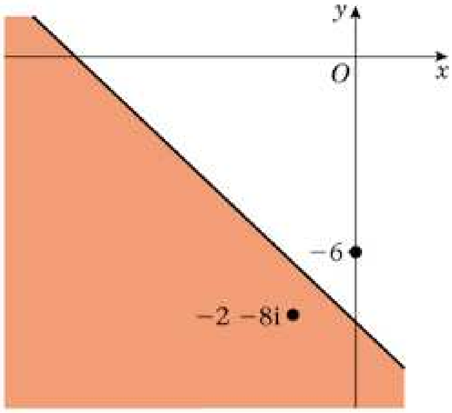


c $|z + 7| \leqslant |z - 1|$



$|z + 7| = |z - 1|$ represents a perpendicular bisector of the line joining $(-7, 0)$ to $(1, 0)$ which has equation $x = -3$.

d $|z + 6| > |z + 2 + 8i|$



$|z + 6| = |z + 2 + 8i|$ represents a perpendicular bisector of the line joining $(-6, 0)$ to $(-2, -8)$.

$|x + iy + 6| = |x + iy + 2 + 8i|$

$\Rightarrow |x + 6 + iy| = |(x + 2) + i(y + 8)|$

$\Rightarrow |(x + 6) + iy|^2 = |(x + 2) + i(y + 8)|^2$

$\Rightarrow (x + 6)^2 + y^2 = (x + 2)^2 + (y + 8)^2$

$\Rightarrow x^2 + 12x + 36 + y^2 = x^2 + 4x + 4 + y^2 + 16y + 64$

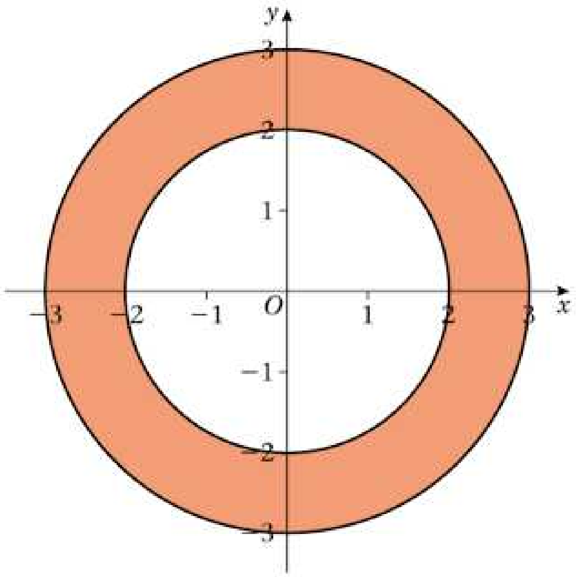
$\Rightarrow (2x + 36 = 4x + 16y + 68$

$\Rightarrow 8x + 36 - 68 = 16y$

$\Rightarrow 8x - 32 = 16y$

$\Rightarrow y = \frac{1}{2}x - 2$

e

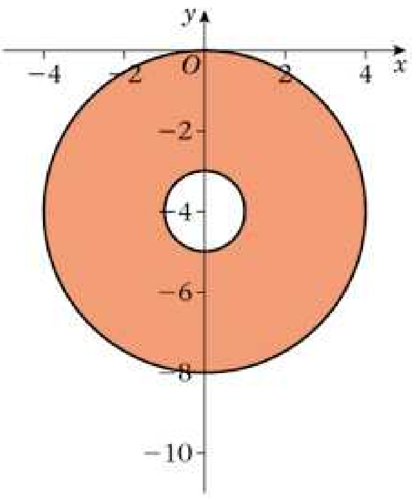


$2 \leqslant |z| \leqslant 3$

$|z| = 2$ represents a circle centre $(0, 0)$, radius 2

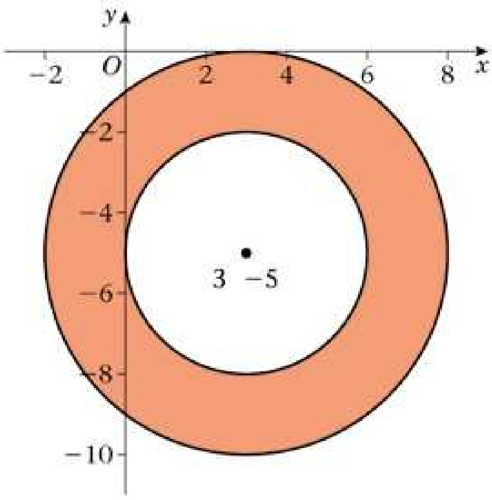
$|z| = 3$ represents a circle centre $(0, 0)$, radius 3

f $1 \leq |z + 4i|$



$|z + 4i| = 1$ represents a circle centre (0, 4), radius 1.
$|z + 4i| = 4$ represents a circle centre (0, −4), radius 4.

g $3 \leq |z - 3 + 5i| \leq 5$



h $2|z| \geqslant |z - 3|$

Consider $2|z| = |z - 3|$ let $z = x + iy$

$2|x + iy| = |x + iy - 3|$

$4(x^2 + y^2) = (x - 3)^2 + y^2$

$4x^2 + 4y^2 = x^2 - 6x + 9 + y^2$

$3x^2 + 6x + 3y^2 - 9 = 0$

$(x + 1)^2 - 1 + y^2 - 3 = 0$

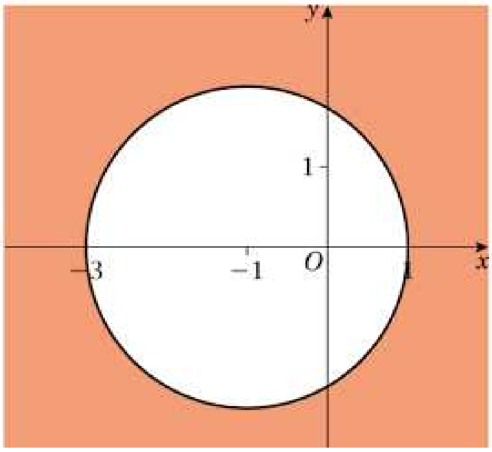
$(x + 1)^2 + y^2 = 4$

Circle centre $(-1, 0)$ radius 2.

Consider $z = 0$ in $2|z| \geqslant |z - 3|$

$2 \times 0 \geqslant 3$

So $z = 0$ is not in the region.



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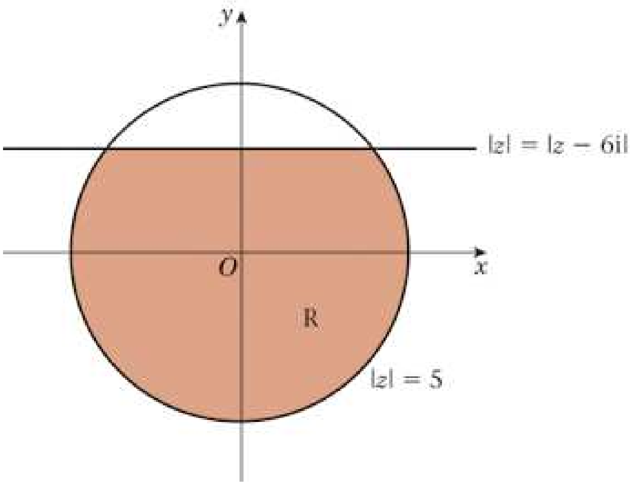
Exercise G, Question 2

Question:

The region R in an Argand diagram is satisfied by the inequalities $|z| \leq 5$ and $|z| \leq |z - 6\text{i}|$. Draw an Argand diagram and shade in the region R.

Solution:



$|z| \leq 5$
$|z| \leq |z - 6\text{i}|$

$|z| = 5$ represents a circle centre (0, 0), radius 5

$|z| = |z - 6\text{i}|$ represents a perpendicular bisector of the line joining (0, 0), to (0, 6) and has the equation $y = 3$.

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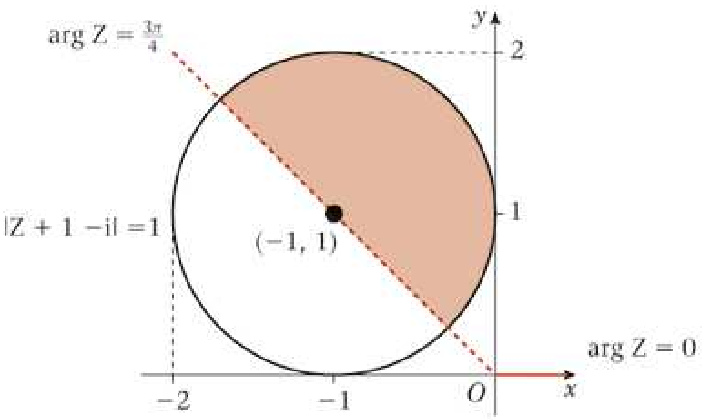
Exercise G, Question 3

Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z + 1 - i| \leqslant 1$ and $0 \leqslant \arg z < \frac{3\pi}{4}$.

Solution:

$|z + 1 - i| \leqslant 1 \qquad |z - (-1 - i)| \leqslant 1$



Inside of a circle centre $(-1, 1)$ radius 1

$\arg z = \frac{3\pi}{4}$ is a half-line with equation $y = -x$, which goes through the centre of the circle, $(-1, 1)$.

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Exercise G, Question 4

Question:

Shade in on an Argand diagram the region satisfied by the set of points $P(x, y)$, where $|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi$.

Solution:

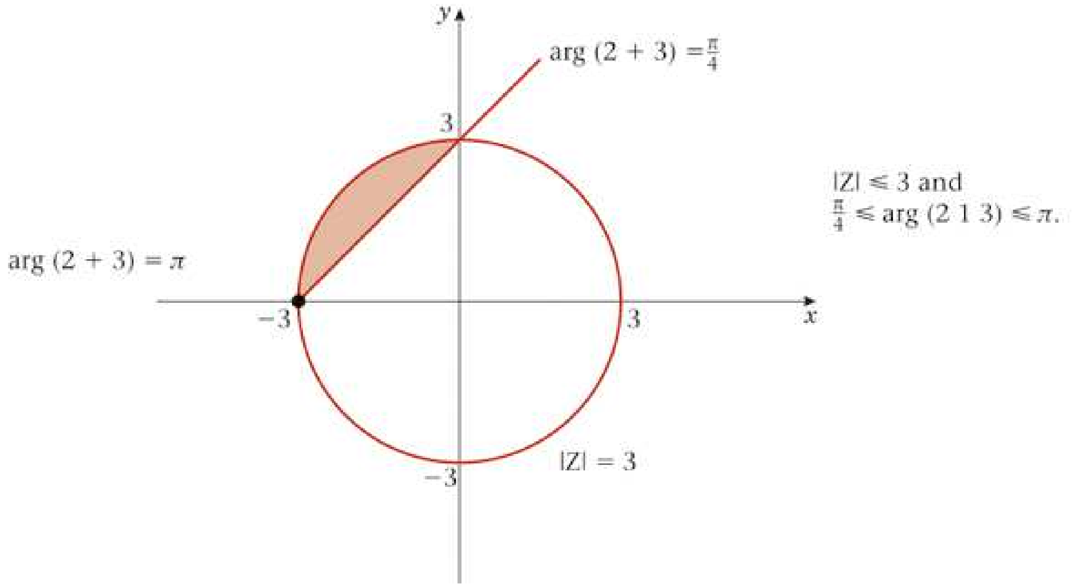
$|z| \leqslant 3$ and $\frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi$

$|z| = 3$ represents a circle centre (0, 0) radius 3.

$\arg(z + 3) = \frac{\pi}{4}$ is a half-line with equation $y - 0 = 1\,(x + 3) \Rightarrow y = x + 3, x > 0$.

Note it passes through the points (−3, 0) and (0, 3).

$\arg(z + 3) = \pi$ is a half-line with equation $y = 0, x < -3$.



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Exercise G, Question 5

Question:

a Sketch on the same Argand diagram:

i the locus of points representing $|z-2| = |z-6-8\text{i}|$,

ii the locus of points representing $\arg(z-4-2\text{i}) = 0$,

iii the locus of points representing $\arg(z-4-2\text{i}) = \frac{\pi}{2}$.

The region R is defined by the inequalities $|z-2| \leqslant |z-6-8\text{i}|$ and $0 \leqslant \arg(z-4-2\text{i}) \leqslant \frac{\pi}{2}$.

b On your sketch in part **a**, identify, by shading, the region R.

Solution:

a $|z-2| = |z-6-8\text{i}|$ represents a perpendicular bisector of the line joining (2, 0) to (6, 8).

$|x+\text{i}y-2| = |x+\text{i}y-6-8\text{i}|$

$\Rightarrow |(x-2)+\text{i}y| = |(x-6)+\text{i}(y-8)|$

$\Rightarrow |(x-2)+\text{i}y|^2 = |(x-6)+\text{i}(y-8)|^2$

$\Rightarrow (x-2)^2 + y^2 = (x-6)^2 + (y-8)^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = x^2 - 12x + 36 + y^2 - 16y + 64$

$\Rightarrow -4x + 4 = -12x - 16y + 100$

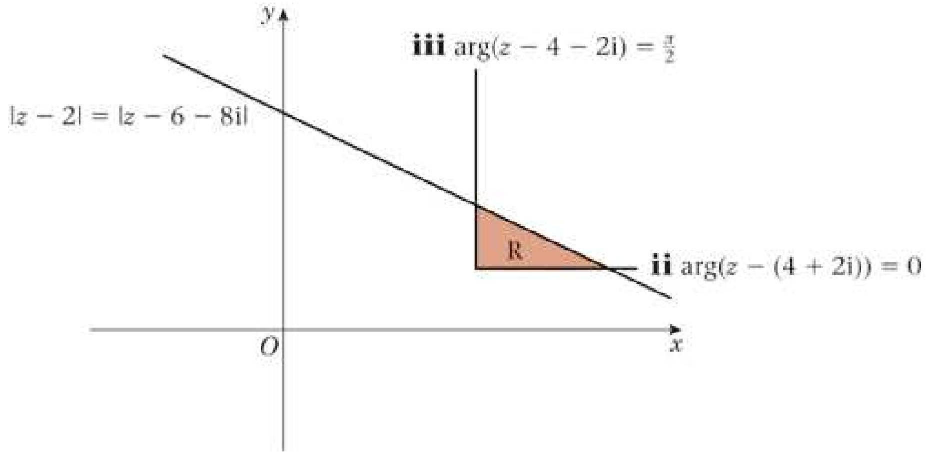
$\Rightarrow 8x + 16y - 96 = 0 \qquad (\div 8)$

$\Rightarrow x + 2y - 12 = 0$

$\Rightarrow 2y = -x + 12$

$\Rightarrow y = \frac{-1}{2}x + 6$

i $|z-2| = |z-(6-8\text{i})|$



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Exercise G, Question 6

Question:

a Find the Cartesian equations of:

i the locus of points representing $|z + 10| = |z - 6 - 4\sqrt{2}i|$,

ii the locus of points representing $|z + 1| = 3$.

b Find the two values of z that satisfy both $|z + 10| = |z - 6 - 4\sqrt{2}i|$ and $|z + 1| = 3$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 10| \leqslant |z - 6 - 4\sqrt{2}i|$ and $|z + 1| \leqslant 3$.

Solution:

a i $|x + iy + 10| = |x + iy - 6 - 4\sqrt{2}i|$

so $(x + 10)^2 + y^2 = (x - 6)^2 + (y - 4\sqrt{2})^2$

$\cancel{x^2} + 20x + 100 + \cancel{y^2} = \cancel{x^2} + 12x + 36 + \cancel{y^2} - 8\sqrt{2}y + 32$

$32x = -8\sqrt{2}y - 32$

$8\sqrt{2}y + (x + 1)32 = 0$

$y + (x + 1)2\sqrt{2} = 0$

$y = -2\sqrt{2}(x + 1)$

ii $(x + 1)^2 + y^2 = 9$

$(x^2 + 2x + y^2 = 8)$

b Substitute $y = -2\sqrt{2}(x + 1)$ into $(x + 1)^2 + y^2 = 9$

$(x + 1)^2 + 8(x + 1)^2 = 9$

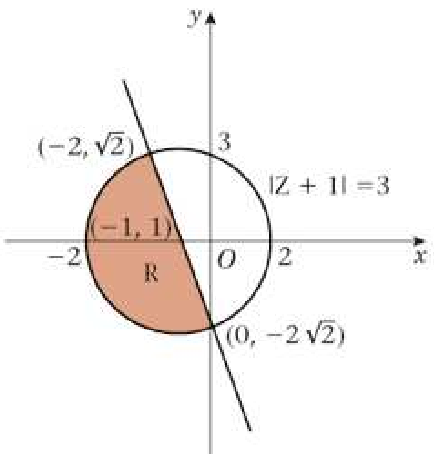
$9(x + 1)^2 = 9$

$x + 1 = \pm 1$

$x = 0, -2$ $\quad (0, -2\sqrt{2})$ and $(-2, 2\sqrt{2})$

$z = -2\sqrt{2}i$ and $z = -2 + 2\sqrt{2}i$

c



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Exercise H, Question 1

Question:

For the transformation $w = z + 4 + 3\text{i}$, sketch on separate Argand diagrams the locus of w when

a z lies on the circle $|z| = 1$,

b z lies on the half-line $\arg z = \frac{\pi}{2}$,

c z lies on the line $y = x$.

Solution:

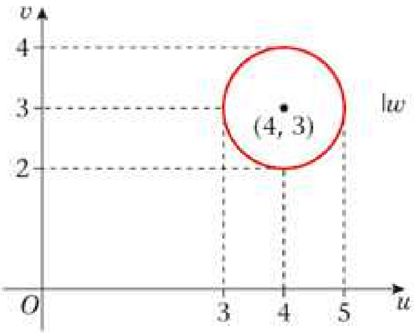
$w = z + 4 + 3\mathrm{i}$

a $|z| = 1$ is a circle, centre (0, 0), radius 1

METHOD ① $|z|$ is translated by a translation vector $\begin{pmatrix}4\\3\end{pmatrix}$ to give a circle, centre (4, 3), radius 1, in the w plane.

METHOD ② $w = z + 4 + 3\mathrm{i}$
$\Rightarrow w - 4 - 3\mathrm{i} = z$
$\Rightarrow |w - 4 - 3\mathrm{i}| = |z|$
$\Rightarrow |w - 4 - 3\mathrm{i}| = 1$

The locus of w is a circle centre (4, 3), radius 1.

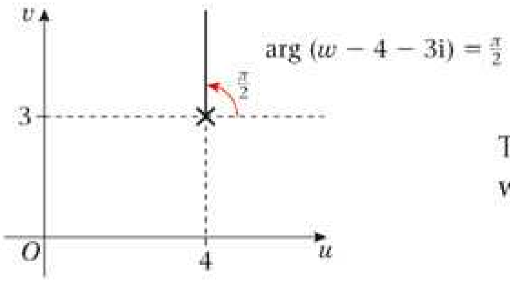


b $\arg z = \frac{\pi}{2}$

METHOD ① $\arg z = \frac{\pi}{2}$ is translated by a translation vector $\begin{pmatrix}4\\3\end{pmatrix}$ to give a half-line from (4, 3) at $\frac{\pi}{2}^c$ with the positive real axis.

METHOD ② $w = z + 4 + 3\mathrm{i}$
$\Rightarrow w - 4 - 3\mathrm{i} = z$

So $\arg z = \frac{\pi}{2} \Rightarrow \arg(w - 4 - 3\mathrm{i}) = \frac{\pi}{2}$



The locus of w is the half-line with equation $u = 4$, $v > 3$.

c $y = x$

$w = z + 4 + 3\text{i}$

$\Rightarrow z = w - 4 - 3\text{i}$

$\Rightarrow x + \text{i}y = u + \text{i}v - 4 - 4\text{i}$

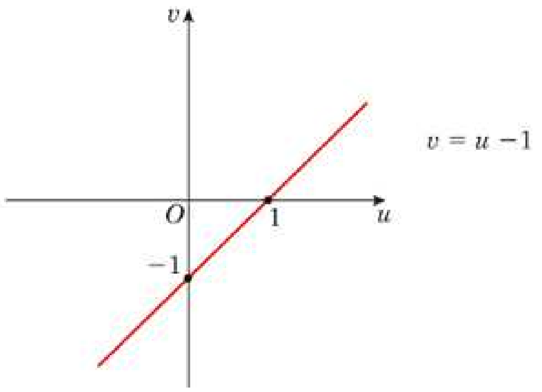
$\Rightarrow x + \text{i}y = (u - 4) + \text{i}(v - 3)$

$y = x \Rightarrow v - 3 = u - 4$

$\Rightarrow v = u - 4 + 3$

$\Rightarrow v = u - 1$

The locus of w is a line with equation $v = u - 1$.



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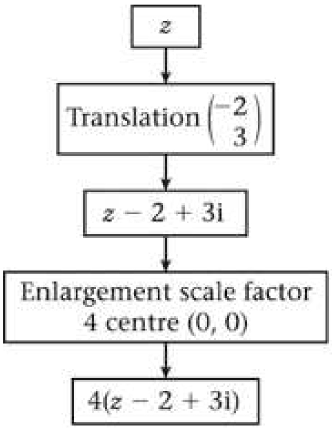
Exercise H, Question 2

Question:

A transformation T from the z-plane to the w-plane is a translation with translation vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ followed by an enlargement scale factor 4, centre O. Write down the transformation T in the form $w = a\,z + b$, where $a, b \in \mathbb{C}$.

Solution:



Hence T: $w = 4(z - 2 + 3i)$
$\quad\quad = 4z - 8 + 12i$

The transformation T is $w = 4z - 8 + 12i$

Note: $a = 4$, $b = -8 + 12i$.

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Exercise H, Question 3

Question:

For the transformation $w = 3z + 2 - 5\text{i}$, find the equation of the locus of w when z lies on a circle centre O, radius 2.

Solution:

$w = 3z + 2 - 5\text{i}$

METHOD ① z lies on a circle, centre 0, radius 2.

$\Rightarrow |z| = 2$
$w = 3z + 2 - 5\text{i}$
$\Rightarrow w - 2 + 5\text{i} = 3z$
$\Rightarrow |w - 2 + 5\text{i}| = |3z|$
$\Rightarrow |w - 2 - 5\text{i}| = |3|\,|z|$
$\Rightarrow |w - 2 - 5\text{i}| = 3|z|$
$\Rightarrow |w - 2 - 5\text{i}| = 3(2)$
$\Rightarrow |w - 2 - 5\text{i}| = 6$
$\Rightarrow |w - (2 - 5\text{i})| = 6$

So the locus of w is a circle centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

METHOD ② z lies on a circle, centre 0, radius 2.

↓ enlargement scale factor 3, centre 0.

$3z$ lies on a circle, centre 0, radius 6.

↓ translation by a translation vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

$3z + 2 - 5\text{i}$ lies on a circle centre $(2, -5)$, radius 6.

So the locus of w is a circle, centre $(2, -5)$, radius 6 with equation $(u - 2)^2 + (v + 5)^2 = 36$.

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Exercise H, Question 4

Question:

For the transformation $w = 2z - 5 + 3i$, find the equation of the locus of w as z moves on the circle $|z - 2| = 4$.

Solution:

z moves on a circle $|z - 2| = 4$

METHOD ①

$$w = 2z - 5 + 3i$$
$$\Rightarrow w + 5 - 3i = 2z$$
$$\Rightarrow \frac{w + 5 - 3i}{2} = z$$
$$\Rightarrow \frac{w + 5 - 3i}{2} - 2 = z - 2$$
$$\Rightarrow \frac{w + 5 - 3i - 4}{2} = z - 2$$
$$\Rightarrow \frac{w + 1 - 3i}{2} = z - 2$$
$$\Rightarrow \left|\frac{w + 1 - 3i}{2}\right| = |z - 2|$$
$$\Rightarrow \frac{|w + 1 - 3i|}{|2|} = |z - 2|$$
$$\Rightarrow |w + 1 - 3i| = 2|z - 2|$$
$$\Rightarrow |w + 1 - 3i| = 2(4)$$
$$\Rightarrow |w + 1 - 3i| = 8$$
$$\Rightarrow |w - (-1 + 3i)| = 8$$

So the locus of w is a circle centre $(-1, 3)$, radius 8 with equation $(u + 1)^2 + (v - 3)^2 = 8$.

METHOD ②

$|z - 2| = 4$

z lies on a circle, centre $(2, 0)$, radius 4

↓ enlargement scale factor 2, centre 0.

$2z$ lies on a circle, centre $(4, 0)$, radius 8.

↓ translation by a translation vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

$w = 2z - 5 + 3i$ lies on a circle centre $(-1, 3)$, radius 8.

So the locus of w is a circle, centre $(-1, 3)$, radius 8 with equation $(u - 1)^2 + (v - 3)^2 = 8$.

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Exercise H, Question 5

Question:

For the transformation $w = z - 1 + 2\text{i}$ sketch on separate Argand diagrams the locus of w when:

a z lies on the circle $|z - 1| = 3$,

b z lies on the half-line $\arg(z - 1 + \text{i}) = \frac{\pi}{4}$,

c z lies on the line $y = 2x$.

Solution:

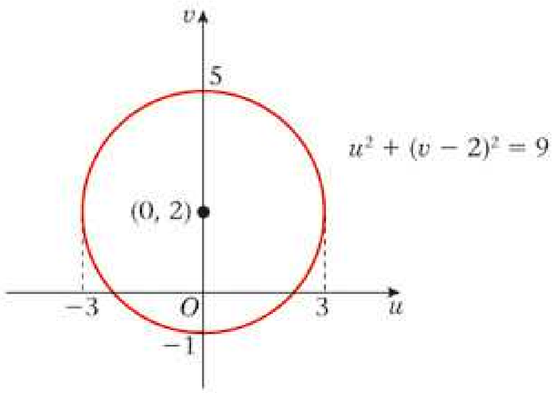
$w = z - 1 + 2i$

a $|z - 1| = 3$ circle centre (1, 0) radius 3.

METHOD ① $|z - 1| = 3$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a circle, centre (0, 2), radius 3, in the w-plane.

METHOD ② $w = z - 1 + 2i$
$\Rightarrow w - 2i = z - 1$
$\Rightarrow |w - 2i| = |z - 1|$
$\Rightarrow |w - 2i| = 3$

The locus of w is a circle, centre (0, 2), radius 3.



b $\arg(z - 1 + i) = \frac{\pi}{4}$ half-line from (1, −1) at $\frac{\pi}{4}^c$ with the positive real axis.

METHOD ① $\arg(z - 1 + i) = \frac{\pi}{4}$ is translated by a translation vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to give a half-line from (0, 1) at $\frac{\pi}{4}^c$ with the positive real axis.

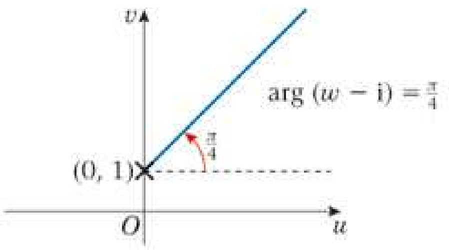
METHOD ② $w = z - 1 + 2i$
$\Rightarrow w + 1 - 2i = z$

So $\arg(z - 1 + i) = \frac{\pi}{4}$

becomes $\arg(w + 1 - 2i - 1 + i) = \frac{\pi}{4}$

$\Rightarrow \arg(w - i) = \frac{\pi}{4}$

Therefore, the locus of w is a half-line from (0, 1) at $\frac{\pi}{4}^c$ with the positive real axis.



c $y = 2x$

$w = z - 1 + 2i$

$\Rightarrow z = w + 1 - 2i$

$\Rightarrow x + iy = u + iv + 1 - 2i$

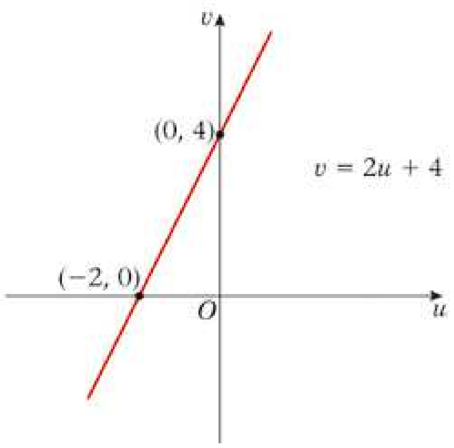
$\Rightarrow x + iy = u + 1 + i(v - 2)$

So $y = 2x \Rightarrow v - 2 = 2(u + 1)$

$\Rightarrow v - 2 = 2u + 2$

$\Rightarrow v = 2u + 4$

The locus of w is a line with equation $v = 2u + 4$.



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Exercise H, Question 6

Question:

For the transformation $w = \frac{1}{z}$, $z \neq 0$, find the locus of w when:

a z lies on the circle $|z| = 2$,

b z lies on the half-line with equation $\arg z = \frac{\pi}{4}$,

c z lies on the line with equation $y = 2x + 1$.

Solution:

$w = \frac{1}{z}$, $z \neq 0$

a z lies on a circle, $|z| = 2$

$w = \frac{1}{z}$

$\Rightarrow |w| = \left|\frac{1}{z}\right|$

$\Rightarrow |w| = \frac{|1|}{|z|}$

$\Rightarrow |w| = \frac{1}{2}$ — apply $|z| = 2$

Therefore the locus of w is a circle, centre (0, 0), radius $\frac{1}{2}$, with equation $u^2 + v^2 = \frac{1}{4}$.

b z lies on the half-line, $\arg z = \frac{\pi}{4}$

$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$

So $\arg z = \frac{\pi}{4}$, becomes $\arg\left(\frac{1}{w}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(1) - \arg(w) = \frac{\pi}{4}$

$\Rightarrow -\arg w = \frac{\pi}{4}$ — $\arg 1 = 0$

$\Rightarrow \arg w = -\frac{\pi}{4}$

Therefore the locus of w is a half-line from (0, 0) at $-\frac{\pi}{4}^c$ with the positive x-axis.

The locus of w has equation, $v = -u$, $u > 0$, $v < 0$.

c z lies on the line $y = 2x + 1$

$w = \frac{1}{z} \Rightarrow wz = 1 \Rightarrow z = \frac{1}{w}$.

$\Rightarrow x + \mathrm{i}y = \frac{1}{u + \mathrm{i}v}$

$\Rightarrow x + \mathrm{i}y = \frac{1}{(u + \mathrm{i}v)} \frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$

$\Rightarrow x + \mathrm{i}y = \frac{u - \mathrm{i}v}{u^2 + v^2}$

$\Rightarrow x + \mathrm{i}y = \frac{u}{u^2 + v^2} + \mathrm{i}\left(\frac{-v}{u^2 + v^2}\right)$

So $x = \frac{u}{u^2 + v^2}$ and $y = \frac{-v}{u^2 + v^2}$

Hence $y = 2x + 1$ becomes $\frac{-v}{u^2 + v^2} = \frac{2u}{u^2 + v^2} + 1 \qquad \times (u^2 + v^2)$

$$\Rightarrow -v = 2u + u^2 + v^2$$
$$\Rightarrow 0 = u^2 + 2u + v^2 + v$$
$$\Rightarrow (u + 1)^2 - 1 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$
$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$
$$\Rightarrow (u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{4}\right)^2$$

Therefore, the locus of w is a circle, centre $\left(-1, -\frac{1}{2}\right)$, radius $\frac{\sqrt{5}}{2}$, with equation

$(u + 1)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 7

Question:

For the transformation $w = z^2$,

a show that as z moves once round a circle centre (0, 0), radius 3, w moves twice round a circle centre (0, 0), radius 9,

b find the locus of w when z lies on the real axis, with equation $y = 0$,

c find the locus of w when z lies on the imaginary axis.

Solution:

$w = z^2$

a z moves once round a circle, centre (0, 0), radius 3.
The equation of the circle, $|z| = 3$ is also $r = 3$.
The equation of the circle can be written as $z = 3e^{i\theta}$
or $z = 3(\cos\theta + i\sin\theta)$

$\Rightarrow w = z^2 = (3(\cos\theta + i\sin\theta))^2$
$= 3^2(\cos 2\theta + i\sin 2\theta)$ — de Moivre's Theorem.
$= 9(\cos 2\theta + i\sin 2\theta)$

So, $w = 9(\cos 2\theta + i\sin 2\theta)$ can be written as $|w| = 9$
Hence, as $|w| = 9$ and $\arg w = 2\theta$ then w moves twice round a circle, centre (0, 0), radius 9.

b z lies on the real-axis $\Rightarrow y = 0$
So $z = x + iy$ becomes $z = x$ (as $y = 0$)
$\Rightarrow w = z^2 = x^2$
$\Rightarrow u + iv = x^2 + i(0)$
$\Rightarrow u = x^2$ and $v = 0$
As $v = 0$ and $u = x^2 \geqslant 0$ then w lies on the positive real-axis including the origin, 0.

c z lies on the imaginary axis $\Rightarrow x = 0$
So $z = x + iy$ becomes $z = iy$ (as $x = 0$)
$\Rightarrow w = z^2 = (iy)^2 = -y^2$
$\Rightarrow u + iv = -y^2 + i(0)$
$\Rightarrow u = -y^2$ and $v = 0$
As $v = 0$ and $u = -y^2 \leqslant 0$ then w lies on the negative real-axis including the origin, 0.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 8

Question:

If z is any point in the region R for which $|z + 2i| < 2$,

a shade in on an Argand diagram the region R.

Sketch on separate Argand diagrams the corresponding regions for $\mathbb{R}$ where:

b $w = z - 2 + 5i$,

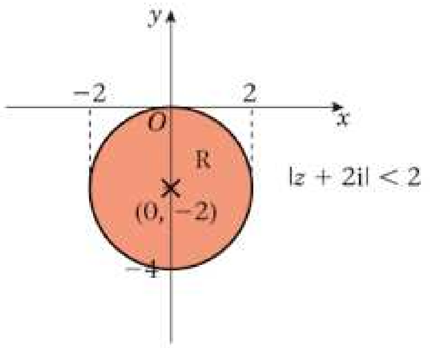
c $w = 4z + 2 + 4i$,

d $|zw + 2iw| = 1$.

Solution:

$|z + 2i| < 2$

a $|z + 2i| = 2$ is a circle, centre $(0, -2)$, radius 2.



b $w = z - 2 + 5i$

$\Rightarrow w + 2 - 5i = z$

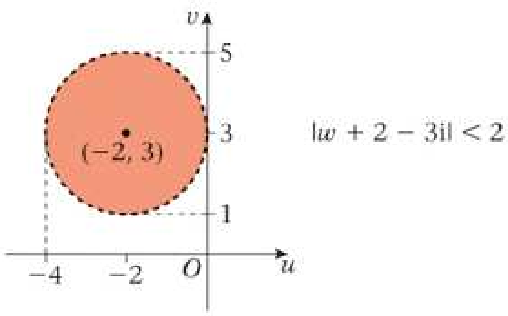
$\Rightarrow z + 2i = w + 2 - 5i + 2i$

$\Rightarrow z + 2i = w + 2 - 3i$

$\Rightarrow |z + 2i| = |w + 2 - 3i|$

As $|z + 2i| < 2$, then $|z + 2i| = |w + 2 - 3i| < 2$

Note that $|w + 2 - 3i| = 2$ is a circle, centre $(-2, 3)$, radius 2.



c $w = 4z + 2 + 4i$

$\Rightarrow w - 2 - 4i = 4z$

$\Rightarrow \dfrac{w - 2 - 4i}{4} = z$

$\Rightarrow z + 2i = \dfrac{w - 2 - 4i}{4} + 2i$

$\Rightarrow z + 2i = \dfrac{w - 2 - 4i + 8i}{4}$

$\Rightarrow z + 2i = \dfrac{w - 2 + 4i}{4}$

$\Rightarrow |z + 2i| = \left|\dfrac{w - 2 + 4i}{4}\right|$

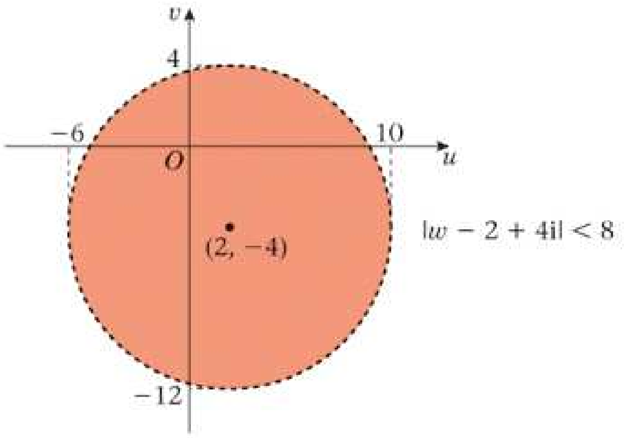
$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{|4|}$$

$$\Rightarrow |z + 2i| = \frac{|w - 2 + 4i|}{4}$$

As $|z + 2i| < 2$, then $|z + 2i| = \frac{|w - 2 + 4i|}{4} < 2$

$\Rightarrow |w - 2 + 4i| < 8$

Note that $|w - 2 + 4i| = 8$ is a circle, centre $(2, -4)$, radius 8.



d $|zw + 2iw| = 1$

$\Rightarrow |w(z + 2i)| = 1$

$\Rightarrow |w|\ |z + 2i| = 1$

$$\Rightarrow |z + 2i| = \frac{1}{|w|}$$

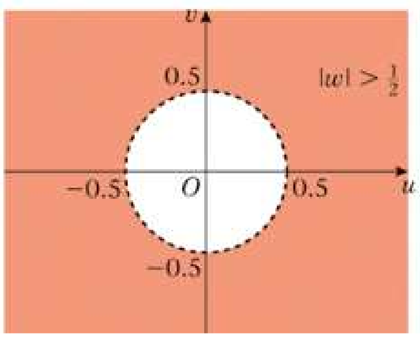
As $|z + 2i| < 2$, then $|z + 2i| = \frac{1}{|w|} < 2$

$\Rightarrow 1 < 2|w|$

$\Rightarrow \frac{1}{2} < |w|$

$\Rightarrow |w| > \frac{1}{2}$

Note that $|w| = \frac{1}{2}$ is a circle, centre $(0, 0)$ radius $\frac{1}{2}$.



Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 9

Question:

For the transformation $w = \frac{1}{2 - z}$, $z \neq 2$, show that the image, under T, of the circle centre O, radius 2 in the z-plane is a line l in the w-plane. Sketch l on an Argand diagram.

Solution:

Circle, centre 0, radius 2 in the z-plane $\Rightarrow |z| = 2$

T: $w = \frac{1}{2 - z}$

$\Rightarrow w(2 - z) = 1$

$\Rightarrow 2w - wz = 1$

$\Rightarrow 2w - 1 = wz$

$\Rightarrow \frac{2w - 1}{w} = z$

$\Rightarrow \left|\frac{2w - 1}{w}\right| = |z|$

$\Rightarrow \frac{|2w - 1|}{|w|} = |z|$

Applying $|z| = 2$ gives $\frac{|2w - 1|}{|w|} = 2$

$\Rightarrow |2w - 1| = 2|w|$

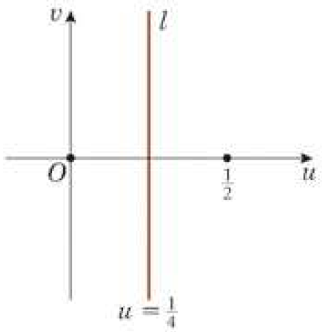
$\Rightarrow |2(w - \frac{1}{2})| = 2|w|$

$\Rightarrow |2|\,|(w - \frac{1}{2})| = 2|w|$

$\Rightarrow 2|w - \frac{1}{2}| = 2|w|$

$\Rightarrow |w - \frac{1}{2}| = |w|$

The image under T of $|z| = 2$ is the perpendicular bisector of the line segment joining (0, 0) and $(\frac{1}{2}, 0)$. Therefore the line l has equation $u = \frac{1}{4}$.

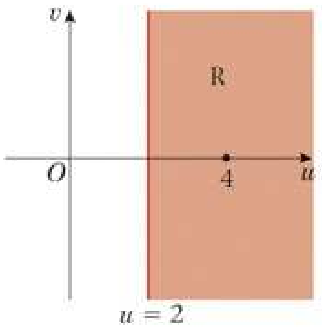


$|z - 4| < 4$ gives $\frac{|16 - 4w|}{|w|} < 4$

$\Rightarrow |16 - 4w| < 4|w|$

which leads to $|w - 4| < |w|$

$\Rightarrow |w| > |w - 4|$



Solutionbank FP2
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Exercise H, Question 10

Question:

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{16}{z}$, $z \neq 0$.

a The transformation T maps the points on the circle $|z - 4| = 4$, in the z-plane, to points on a line l in the w-plane. Find the equation of l.

b Hence, or otherwise, shade and label on an Argand diagram the region R which is the image of $|z - 4| < 4$ under T.

Solution:

T: $w = \frac{16}{z}$

$|z - 4| = 4$

$w = \frac{16}{z}$

$\Rightarrow wz = 16$

$\Rightarrow z = \frac{16}{w}$

$\Rightarrow z - 4 = \frac{16}{w} - 4$

$\Rightarrow z - 4 = \frac{16 - 4w}{w}$

$\Rightarrow |z - 4| = \left|\frac{16 - 4w}{w}\right|$

$\Rightarrow |z - 4| = \frac{|16 - 4w|}{|w|}$

Applying $|z - 4| = 4$ gives $\frac{|16 - 4w|}{|w|} = 4$

$\Rightarrow \frac{|-4(w - 4)|}{|w|} = 4|w|$

$\Rightarrow |-4||w - 4| = 4|w|$

$\Rightarrow 4|w - 4| = 4|w|$

$\Rightarrow |w - 4| = |w|$

The image under T of $|z - 4| = 4$ is the perpendicular bisector of the line segment joining $(0, 0)$ to $(4, 0)$. Therefore the line l has equation $u = 2$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 11

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{3}{2 - z}$, $z \neq 2$.
Show that under T the straight line with equation $2y = x$ is transformed to a circle in the w-plane with centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3}{4}\sqrt{5}$.

Solution:

T: $w = \frac{3}{2 - z}$, $z \neq 2$

$\Rightarrow w(2 - z) = 3$

$\Rightarrow 2w - wz = 3$

$\Rightarrow 2w = 3 + wz$

$\Rightarrow 2w - 3 = wz$

$\Rightarrow \frac{2w - 3}{w} = z$

$\Rightarrow z = \frac{2w - 3}{w}$

$\Rightarrow z = \frac{2(u + \mathrm{i}v) - 3}{u + \mathrm{i}v}$

$\Rightarrow z = \frac{(2u - 3) + 2\mathrm{i}v}{u + \mathrm{i}v}$

$\Rightarrow z = \frac{[(2u - 3) + 2\mathrm{i}v]}{[u + \mathrm{i}v]} \times \frac{[u - \mathrm{i}v]}{[u - \mathrm{i}v]}$

$\Rightarrow z = \frac{(2u - 3)u - \mathrm{i}v(2u - 3) + 2\mathrm{i}uv + 2v^2}{u^2 + v^2}$

$\Rightarrow z = \frac{2u^2 - 3u - \cancel{2uv\mathrm{i}} + 3\mathrm{i}v + \cancel{2uv\mathrm{i}} + 2v^2}{u^2 + v^2}$

$\Rightarrow z = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + \mathrm{i}\left[\frac{3v}{u^2 + v^2}\right]$

So, $x + \mathrm{i}y = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2} + \mathrm{i}\left[\frac{3v}{u^2 + v^2}\right]$

$\Rightarrow x = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$

and $y = \frac{3v}{u^2 + v^2}$

As, $2y = x \Rightarrow 2\left(\frac{3v}{u^2 + v^2}\right) = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$

$$\Rightarrow \frac{6v}{u^2 + v^2} = \frac{2u^2 - 3u + 2v^2}{u^2 + v^2}$$

$$\Rightarrow 6v = 2u^2 - 3u + 2v^2$$

$$\Rightarrow 0 = 2u^2 - 3u + 2v^2 - 6v$$

$$\Rightarrow 2u^2 - 3u + 2v^2 - 6v = 0 \quad (\div 2)$$

$$\Rightarrow u^2 - \frac{3}{2}u + v^2 - 3v = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 - \frac{9}{16} + \left(v - \frac{3}{2}\right)^2 - \frac{9}{4} = 0$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{9}{16} + \frac{9}{4}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \frac{45}{16}$$

$$\Rightarrow \left(u - \frac{3}{4}\right)^2 + \left(v - \frac{3}{2}\right)^2 = \left(\frac{3\sqrt{5}}{4}\right)^2$$

The image under T of $2y = x$ is a circle centre $\left(\frac{3}{4}, \frac{3}{2}\right)$, radius $\frac{3}{4}\sqrt{5}$, as required.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 12

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \dfrac{-\mathrm{i}z + \mathrm{i}}{z + 1}$, $z \neq -1$.

a The transformation T maps the points on the circle with equation $x^2 + y^2 = 1$ in the z-plane, to points on a line l in the w-plane. Find the equation of l.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane which is the image of $|z| \leqslant 1$ under T.

c Show that the image, under T, of the circle with equation $x^2 + y^2 = 4$ in the z-plane is a circle C in the w-plane. Find the equation of C.

Solution:

$T: w = \dfrac{-iz + i}{z + 1}, z \neq -1$

a Circle with equation $x^2 + y^2 = 1 \Rightarrow |z| = 1$

$w = \dfrac{-iz + i}{z + 1}$

$\Rightarrow w(z + 1) = -iz + i$

$\Rightarrow wz + w = -iz + i$

$\Rightarrow wz + iz = -i - w$

$\Rightarrow z(w + i) = i - w$

$\Rightarrow z = \dfrac{i - w}{w + i}$

$\Rightarrow |z| = \left|\dfrac{i - w}{w + i}\right|$

$\Rightarrow |z| = \dfrac{|i - w|}{|w + i|}$

Applying $|z| = 1 \Rightarrow 1 = \dfrac{|i - w|}{|w + i|}$

$\Rightarrow |w + i| = |i - w|$

$\Rightarrow |w + i| = |(-1)(w - i)|$

$\Rightarrow |w + i| = |(-1)|\,|(w - i)|$

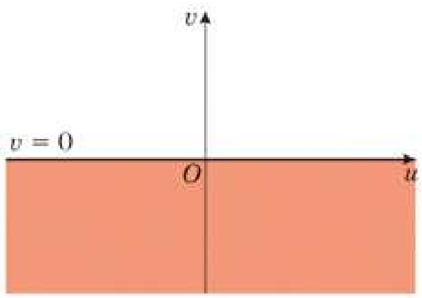
$\Rightarrow |w + i| = |w - i|$

The image under T of $x^2 + y^2 = 1$ is the perpendicular bisector of the line segment joining $(0, -1)$ to $(0, 1)$. Therefore the line l, has equation $v = 0$. (i.e. the u-axis.)

b $|z| \leqslant 1 \Rightarrow 1 \leqslant \dfrac{|i - w|}{|w + i|}$

$\Rightarrow |w + i| \leqslant |i - w|$

$\Rightarrow |w + i| \leqslant |w - i|$



c Circle with equation $x^2 + y^2 = 4 \Rightarrow |z| = 2$

from part **a** $\quad w = \dfrac{-\text{i}z + \text{i}}{z + 1}$

$$\Rightarrow z = \frac{\text{i} - w}{w + \text{i}}$$

$$\Rightarrow |z| = \frac{|\text{i} - w|}{|w + \text{i}|}$$

Applying $|z| = 2 \Rightarrow 2 = \dfrac{|\text{i} - w|}{|w + \text{i}|}$

$$\Rightarrow 2|w + \text{i}| = |\text{i} - w|$$

$$\Rightarrow 2|w + \text{i}| = |(-1)(w - \text{i})|$$

$$\Rightarrow 2|w + \text{i}| = |(-1)|\,|(w - \text{i})|$$

$$\Rightarrow 2|w + \text{i}| = |w - \text{i}|$$

$$\Rightarrow 2|u + \text{i}v + \text{i}| = |u + \text{i}v - \text{i}|$$

$$\Rightarrow 2|u + \text{i}(v + 1)| = |u + \text{i}(v - 1)|$$

$$\Rightarrow 2^2|u + \text{i}(v + 1)|^2 = |u + \text{i}(v - 1)|^2$$

$$\Rightarrow 4[u^2 + (v + 1)^2] = u^2 + (v - 1)^2$$

$$\Rightarrow 4[u^2 + v^2 + 2v + 1] = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 4u^2 + 4v^2 + 8v + 4 = u^2 + v^2 - 2v + 1$$

$$\Rightarrow 3u^2 + 3v^2 + 10v + 3 = 0$$

$$\Rightarrow u^2 + v^2 + \frac{10}{3}v + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{25}{9} - 1$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$$

$$\Rightarrow u^2 + \left(v + \frac{5}{3}\right)^2 = \left(\frac{4}{3}\right)^2$$

The image under T of $x^2 + y^2 = 4$ is a circle C with centre $\left(0, -\frac{5}{3}\right)$, radius $\frac{4}{3}$.

Therefore, the equation of C is $u^2 + \left(v + \frac{5}{3}\right)^2 = \frac{16}{9}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 13

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{4z - 3\mathrm{i}}{z - 1}$, $z \neq 1$.

Show that the circle $|z| = 1$ is mapped by T onto a circle C.

Find the centre and radius of C.

Solution:

$T: w = \frac{4z - 3\mathrm{i}}{z - 1}, z \neq 1$

Circle with equation $|z| = 3$

$w = \frac{4z - 3\mathrm{i}}{z - 1},$

$\Rightarrow w(z - 1) = 4z - 3\mathrm{i}$

$\Rightarrow wz - w = 4z - 3\mathrm{i}$

$\Rightarrow wz + 4z = w - 3\mathrm{i}$

$\Rightarrow z(w - 4) = w - 3\mathrm{i}$

$\Rightarrow z = \frac{w - 3\mathrm{i}}{w - 4}$

$\Rightarrow |z| = \left|\frac{w - 3\mathrm{i}}{w - 4}\right|$

$\Rightarrow |z| = \frac{|w - 3\mathrm{i}|}{|w - 4|}$

Applying $|z| = 3 \Rightarrow 3 = \frac{|w - 3\mathrm{i}|}{|w - 4|}$

$$\Rightarrow 3|w - 4| = |w - 3i|$$

$$\Rightarrow 3|u + iv - 4| = |u + iv - 3i|$$

$$\Rightarrow 3|(u - 4) + iv| = |u + i(v - 3)|$$

$$\Rightarrow 3^2|(u - 4) + iv|^2 = |u + i(v - 3)|^2$$

$$\Rightarrow 9[(u - 4)^2 + v^2] = u^2 + (v - 3)^2$$

$$\Rightarrow 9[u^2 - 8u + 16 + v^2] = u^2 + v^2 - 6v + 9$$

$$\Rightarrow 9u^2 - 72u + 144 + 9v^2 = u^2 + v^2 - 6v + 9$$

$$\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 144 - 9 = 0$$

$$\Rightarrow 8u^2 - 72u + 8v^2 + 6v + 135 = 0 \quad (\div 8)$$

$$\Rightarrow u^2 - 9u + v^2 + \frac{3}{4}v + \frac{135}{8} = 0$$

$$\Rightarrow \left(u - \frac{9}{2}\right)^2 - \frac{81}{4} + \left(v + \frac{3}{8}\right)^2 - \frac{9}{64} + \frac{135}{8} = 0$$

$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{81}{4} + \frac{9}{64} - \frac{135}{8}$$

$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \frac{225}{64}$$

$$\Rightarrow \left(u - \frac{9}{2}\right)^2 + \left(v + \frac{3}{8}\right)^2 = \left(\frac{15}{8}\right)^2$$

Therefore, the circle with equation $|z| = 1$ is mapped onto a circle C with centre $\left(\frac{9}{2}\ -\frac{3}{8}\right)$, radius $\frac{15}{8}$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 14

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{1}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

a Show that the image, under T, of the real axis in the z-plane is a circle C_1 in the w-plane. Find the equation of C_1.

b Show that the image, under T, of the line $x = 4$ in the z-plane is a circle C_2 in the w-plane. Find the equation of C_2.

Solution:

T: $w = \frac{1}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$

a Real axis in the z-plane $\Rightarrow y = 0$

$$w = \frac{1}{z - \mathrm{i}}$$

$$\Rightarrow w(z + \mathrm{i}) = 1$$

$$\Rightarrow wz + \mathrm{i}w = 1$$

$$\Rightarrow wz = 1 - \mathrm{i}w$$

$$\Rightarrow z = \frac{1 - \mathrm{i}w}{w}$$

$$\Rightarrow z = \frac{1 - \mathrm{i}(u + \mathrm{i}v)}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{1 - \mathrm{i}u + v}{u + \mathrm{i}v}$$

$$\Rightarrow z = \frac{((1 + v) - \mathrm{i}u)}{(u + \mathrm{i}v)} \times \frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$$

$$\Rightarrow z = \frac{(1 + v)u - \mathrm{i}v(1 + v) - \mathrm{i}u^2 - uv}{u^2 + v^2}$$

$$\Rightarrow z = \frac{(1 + v)u - uv}{u^2 + v^2} + \frac{\mathrm{i}(-v(1 + v) - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u + uv - uv}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$$

So $x + \mathrm{i}y = \frac{u}{u^2 + v^2} + \frac{\mathrm{i}(-v - v^2 - u^2)}{u^2 + v^2}$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \quad \textit{and} \quad y = \frac{-v - v^2 - u^2}{u^2 + v^2}$$

As $y = 0$, $\dfrac{-v - v^2 - u^2}{u^2 + v^2} = 0$

$\Rightarrow -v - v^2 - u^2 = 0$

$\Rightarrow u^2 + v^2 + v = 0$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$

Therefore, the image under T of the real axis in the z-plane is a circle C_1 with centre $\left(0, -\frac{1}{2}\right)$, radius $\frac{1}{2}$. The equation of C_1 is $u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$.

b As $x = 4$, $\dfrac{u}{u^2 + v^2} = 2$

$\Rightarrow u = 2(u^2 + v^2)$

$\Rightarrow u = 2u^2 + 2v^2$

$\Rightarrow 0 = 2u^2 - u + 2v^2 \quad (\div 2)$

$\Rightarrow 0 = u^2 - \frac{1}{2}u + v^2$

$\Rightarrow 0 = \left(u - \frac{1}{4}\right)^2 - \frac{1}{16} + v^2$

$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$

$\Rightarrow \left(u - \frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$

Therefore, the image under T of the line $x = 2$ is a circle C_2 with centre $\left(\frac{1}{4}, 0\right)$, radius $\frac{1}{4}$. The equation of C_2 is $\left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$.

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise H, Question 15

Question:

The transformation T from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$, is given by $w = z + \frac{4}{z}$, $z \neq 0$.

Show that the transformation T maps the points on a circle $|z| = 2$ to points in the interval $[-k, k]$ on the real axis. State the value of the constant k.

Solution:

$T: w = z + \frac{4}{z}, z \neq 0$

Circle with equation $|z| = 2 \Rightarrow x^2 + y^2 = 4$

$$w = z + \frac{4}{z}$$

$$\Rightarrow w = \frac{z^2 + 4}{z}$$

$$\Rightarrow w = \frac{(x + iy)^2 + 4}{x + iy}$$

$$\Rightarrow w = \frac{x^2 + 2xyi - y^2 + 4}{x + iy}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + i(2xy)]}{x + iy}$$

$$\Rightarrow w = \frac{[(x^2 - y^2 + 4) + i(2xy)]}{(x + iy)} \times \frac{(x - iy)}{(x - iy)}$$

$$\Rightarrow w = \frac{x^3 - xy^2 + 4x + 2xy^2 + i(2x^2y - x^2y + y^3 - 4y)}{x^2 + y^2}$$

$$\Rightarrow w = \left(\frac{x^3 - xy^2 + 4x}{x^2 + y^2}\right) + i\left(\frac{y^3 - x^2y - 4y}{x^2 + y^2}\right)$$

$$\Rightarrow w = \frac{x(x^2 + y^2 + 4)}{x^2 + y^2} + \frac{iy(x^2 + y^2 - 4)}{x^2 + y^2}$$

Apply $x^2 + y^2 + 4 \Rightarrow w = \frac{x(4 + 4)}{4} + \frac{iy(4 - 4)}{4}$

$$\Rightarrow w = \frac{8x}{4} + \frac{iy(0)}{4}$$

$$\Rightarrow w = 2x + 0i$$

$$\Rightarrow u + iv = 2x + 0i$$

$$\Rightarrow u = 2x, v = 0$$

As $|z| = 2 \Rightarrow -2 \leqslant x \leqslant 2$

So $\quad -4 \leqslant 2x \leqslant 4$

and $\quad -4 \leqslant u \leqslant 4$

Therefore the transformation T maps the points on a circle $|z| = 2$ in the z-plane to points in the interval $[-4, 4]$ on the real axis in the w-plane. Hence $k = 4$.

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise H, Question 16

Question:

The transformation T from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$, is given by $w = \frac{1}{z+3}$, $z \neq -3$.

Show that the line with equation $2x - 2y + 7 = 0$ is mapped by T onto a circle C. State the centre and the exact radius of C.

Solution:

T: $w = \frac{1}{z+3}$, $z \neq -3$

Line with equation $2x - 2y + 7 = 0$ in the z-plane

$$w = \frac{1}{z+3}$$

$\Rightarrow w(z + 3) = 1$

$\Rightarrow wz + 3w = 1$

$\Rightarrow wz = 1 - 3w$

$$\Rightarrow z = \frac{1 - 3w}{w}$$

$$\Rightarrow z = \frac{1 - 3(u + \text{i}v)}{u + \text{i}v}$$

$$\Rightarrow z = \frac{1 - 3u - 3\text{i}v}{u + \text{i}v}$$

$$\Rightarrow z = \frac{[(1 - 3u) - (3v)\text{i}]}{[(u + \text{i}v)]} \times \frac{(u - \text{i}v)}{(u - \text{i}v)}$$

$$\Rightarrow z = \frac{(1 - 3u)u - 3v^2 - \text{i}v(1 - 3u) - \text{i}(3uv)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{\text{i}(-v + 3uv - 3uv)}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{\text{i}(-v)}{u^2 + v^2}$$

So, $x + \text{i}y = \frac{u - 3u^2 - 3v^2}{u^2 + v^2} + \frac{\text{i}(-v)}{u^2 + v^2}$

$$\Rightarrow x = \frac{u - 3u^2 - 3v^2}{u^2 + v^2}$$

and $y = \frac{-v}{u^2 + v^2}$

As $2x - 2y + 7 = 0$, then

$$2\left(\frac{u - 3u^2 - 3v^2}{u^2 + v^2}\right) - 2\left(\frac{-v}{u^2 + v^2}\right) + 7 = 0$$

$$\Rightarrow \frac{2u - 6u^2 - 6v^2}{u^2 + v^2} + \frac{2v}{u^2 + v^2} + 7 = 0 \quad (\times\ (u^2 + v^2))$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7(u^2 + v^2) = 0$$

$$\Rightarrow 2u - 6u^2 - 6v^2 + 2v + 7u^2 + 7v^2 = 0$$

$$\Rightarrow u^2 + 2u + v^2 + 2v = 0$$

$$\Rightarrow (u + 1)^2 - 1 + (v + 1)^2 - 1 = 0$$

$$\Rightarrow (u + 1)^2 + (v + 1)^2 = 2$$

$$\Rightarrow (u + 1)^2 + (v + 1)^2 = (\sqrt{2})^2$$

Therefore the transformation T maps the line $2x - 2y + 7 = 0$ in the z-plane to a circle C with centre $(-1, -1)$, radius $\sqrt{2}$ in the w-plane.

Solutionbank FP2
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Exercise I, Question 1

Question:

Express $\frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos x - \mathrm{i} \sin x}$ in the form $\cos nx + \mathrm{i} \sin nx$ where n is an integer to be determined.

Solution:

$$\frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos x - \mathrm{i} \sin x}$$

$$= \frac{(\cos 3x + \mathrm{i} \sin 3x)^2}{\cos(-x) + \mathrm{i} \sin(-x)}$$

$$= \frac{\cos 6x + \mathrm{i} \sin 6x}{\cos(-x) + \mathrm{i} \sin(-x)}$$

$$= \cos(6x - -x) + \mathrm{i} \sin(6x - -x)$$

$$= \cos 7x + \mathrm{i} \sin 7x$$

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Exercise I, Question 2

Question:

Use de Moivre's theorem to evaluate

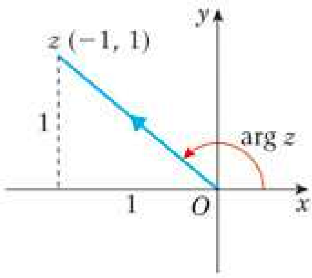
a $(-1 + i)^8$

b $\frac{1}{\left(\frac{1}{2} - \frac{1}{2}i\right)^{16}}$

Solution:

a $(-1 + i)^8$

If $z = -1 + i$, then



$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\theta = \arg z = \pi - \tan^{-1}\left(\frac{1}{1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

So, $-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore (-1 + i)^8 = \left[\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^8$

$= (\sqrt{2})^8\left(\cos\frac{24\pi}{4} + i\sin\frac{24\pi}{4}\right)$

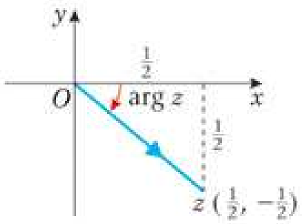
$= 16(\cos 6\pi + i\sin 6\pi)$

$= 16(1 + i(0))$

Therefore, $(-1 + i)^8 = 16$

b $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{-16}$

Let $z=\frac{1}{2}-\frac{1}{2}\mathrm{i}$, then



$$r=\sqrt{\left(\tfrac{1}{2}\right)^2+\left(-\tfrac{1}{2}\right)^2}=\sqrt{\tfrac{1}{4}+\tfrac{1}{4}}=\sqrt{\tfrac{1}{2}}=\frac{1}{\sqrt{2}}$$

$$\theta=\tan^{-1}\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)=-\frac{\pi}{4}$$

So $\frac{1}{2}-\frac{1}{2}\mathrm{i}=\frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$

$$\begin{aligned}\left(\tfrac{1}{2}-\tfrac{1}{2}\mathrm{i}\right)^{-16}&=\left[\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)\right]^{-16}\\&=\left(2^{-\frac{1}{2}}\right)^{-16}\left(\cos\left(\frac{16\pi}{4}\right)+\mathrm{i}\sin\left(\frac{16\pi}{4}\right)\right)\\&=2^8(\cos 4\pi+\mathrm{i}\sin 4\pi)\\&=256(1+\mathrm{i}(0))\\&=256\end{aligned}$$

Therefore, $\dfrac{1}{\left(\frac{1}{2}-\frac{1}{2}\mathrm{i}\right)^{16}}=256$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 3

Question:

a If $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

b Express $\left(z^2 + \frac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$.

c Hence, or otherwise, show that $\cos^3 2\theta = a\cos 6\theta + b\cos 2\theta$, where a and b are constants.

d Hence, or otherwise, show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = k\sqrt{3}$, where k is a constant.

Solution:

a $z = \cos\theta + i\sin\theta$

$$z^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

de Moivre's Theorem.

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^n$$
$$= \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta - i\sin n\theta$$

de Moivre's Theorem.

$\cos(-n\theta) = \cos n\theta$
$\sin(-n\theta) = -\sin n\theta$

Therefore $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

i.e. $z^n + \frac{1}{z^n} = 2\cos n\theta$ (as required)

b $\left(z^2 + \frac{1}{z^2}\right)^3 = (z^2)^3 + {}^3C_1(z^2)^2\left(\frac{1}{z^2}\right) + {}^3C_2(z^2)\left(\frac{1}{z^2}\right)^2 + \left(\frac{1}{z^2}\right)^3$

$$= z^6 + 3z^4\left(\frac{1}{z^2}\right) + 3z^2\left(\frac{1}{z^4}\right) + \frac{1}{z^6}$$
$$= z^6 + 3z^2 + \frac{3}{z^2} + \frac{1}{z^6}$$
$$= \left(z^6 + \frac{1}{z^6}\right) = 3\left(z^2 + \frac{1}{z^2}\right)$$
$$= 2\cos 6\theta + 3(2)\cos 2\theta$$
$$= 2\cos 6\theta + 6\cos 2\theta$$

Hence, $\left(z^2 + \frac{1}{z^2}\right)^3 = 2\cos 6\theta + 6\cos 2\theta$

c $\left(z^2 + \frac{1}{z^2}\right)^3 = (2\cos 2\theta)^3 = 8\cos^3 2\theta = 2\cos 6\theta + 6\cos 2\theta$

$\therefore\ \cos^3 2\theta = \frac{2}{8}\cos 6\theta + \frac{6}{8}\cos 2\theta$

Hence, $\cos^3 2\theta = \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta$

d $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{4}\cos 6\theta + \frac{3}{4}\cos 2\theta \, d\theta$

$= \left[\frac{1}{24}\sin 6\theta + \frac{3}{8}\sin 2\theta\right]_0^{\frac{\pi}{6}}$

$= \left(\frac{1}{24}\sin \pi + \frac{3}{8}\sin\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{24}\sin 0 + \frac{3}{8}\sin 0\right)$

$= \left(\frac{1}{24}(0) + \frac{3}{8}\left(\frac{\sqrt{3}}{2}\right)\right) - (0)$

$= \frac{3}{16}\sqrt{3}$

So, $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = \frac{3}{16}\sqrt{3}$

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Exercise I, Question 4

Question:

a Use de Moivre's theorem to show that $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$.

b By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}$.

c Hence, or otherwise, write down the exact values of $\cos^2\left(\frac{3\pi}{10}\right)$, $\cos^2\left(\frac{7\pi}{10}\right)$ and $\cos^2\left(\frac{9\pi}{10}\right)$.

Solution:

a $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

de Moivre's Theorem.

$= \cos^5\theta + {}^5C_1\cos^4\theta(i\sin\theta) + {}^5C_2\cos^3\theta(i\sin\theta)^2$
$\quad + {}^5C_3\cos^2\theta(i\sin\theta)^3 + {}^5C_4\cos^3\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta$
$\quad + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

Binomial expansion.

Hence,

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin\theta$
$\quad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating the real parts gives,

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta\sin^2\theta + 5\sin^4\theta)$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 5(1-\cos^2\theta)^2)$

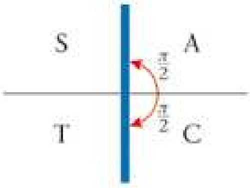
Applying $\sin^2\theta = 1 - \cos^2\theta$.

$= \cos\theta(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5(1 - 2\cos^2\theta + \cos^4\theta))$
$= \cos\theta(\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5 - 10\cos^2\theta + 5\cos^4\theta)$
$= \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$

Hence, $\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$ (as required)

b $\cos 5\theta = 0$

$\alpha = \frac{\pi}{2}$



So $5\theta = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}\right\}$

$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$

$\theta = \left\{\frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}\right\}$ for $0 < \theta \leqslant \pi$

$\cos 5\theta = 0 \Rightarrow \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Five solutions must come from: $\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

Solution ① $\cos\theta = 0$

$\alpha = \frac{\pi}{2}$

For $0 < \theta \leqslant \pi$, $\theta = \frac{\pi}{2}$ (as found earlier)

The final 4 solutions come from: $16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$$

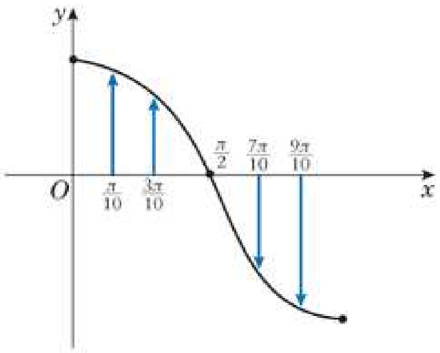
$$= \frac{20 \pm \sqrt{400 - 320}}{32}$$

$$= \frac{20 \pm \sqrt{80}}{32}$$

$$= \frac{20 \pm \sqrt{16}\sqrt{5}}{32}$$

$$= \frac{20 \pm 4\sqrt{5}}{32}$$

$$\therefore \cos^2\theta = \frac{5 \pm \sqrt{5}}{8}$$



Due to symmetry and as $\cos\left(\frac{\pi}{10}\right) > \cos\left(\frac{3\pi}{10}\right)$

$$\cos^2\left(\frac{\pi}{10}\right) = \cos^2\left(\frac{9\pi}{10}\right) > \cos^2\left(\frac{3\pi}{10}\right) = \cos^2\left(\frac{7\pi}{10}\right)$$

$$\therefore \cos^2\left(\frac{7\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

c $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$

$$\cos^2\left(\frac{7\pi}{10}\right) = \cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$$

$$\cos^2\left(\frac{9\pi}{10}\right) = \cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$$

Therefore, $\cos^2\left(\frac{3\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$, $\cos^2\left(\frac{7\pi}{10}\right) = \frac{5 - \sqrt{5}}{8}$, $\cos^2\left(\frac{9\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise I, Question 5

Question:

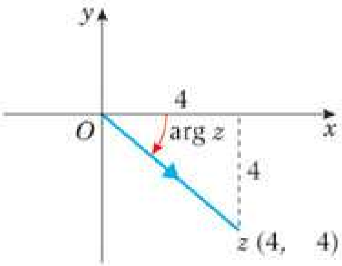
a Express $4 - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, where r and θ are exact values.

b Hence, or otherwise, solve the equation $z^5 = 4 - 4\text{i}$ leaving your answers in the form $z = R\text{e}^{\text{i}k\pi}$, where R is the modulus of z and k is a rational number such that $-1 \leqslant k \leqslant 1$.

c Show on an Argand diagram the points representing your solutions.

Solution:

a $4 - 4i$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16} = \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

argument $= \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$

$\therefore\ 4 - 4i = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

b $z^5 = 4 - 4i$

for $4 - 4i$, $r = 4\sqrt{2}$, $\theta = -\frac{\pi}{4}$

So, $z^5 = 4\sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$

$z^5 = 4\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + 2k\pi\right)}$, $k \in \mathbb{Z}$

Hence, $z = \left[4\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + 2k\pi\right)}\right]^{\frac{1}{5}}$

$= (4\sqrt{2})^{\frac{1}{5}} e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{5}\right)}$

de Moivre's Theorem.

$= \sqrt{2}\, e^{i\left(-\frac{\pi}{20} + \frac{2k\pi}{5}\right)}$

$4\sqrt{2} = 2^{\frac{5}{2}}$

So, $(4\sqrt{2})^{\frac{1}{5}} = (2^{\frac{5}{2}})^{\frac{1}{5}}$

$= 2^{\frac{1}{2}} = \sqrt{2}$

$k = 0$, $z_1 = \sqrt{2}\, e^{i\left(-\frac{\pi}{20}\right)}$

$k = 1$, $z_2 = \sqrt{2}\, e^{i\left(\frac{7\pi}{20}\right)}$

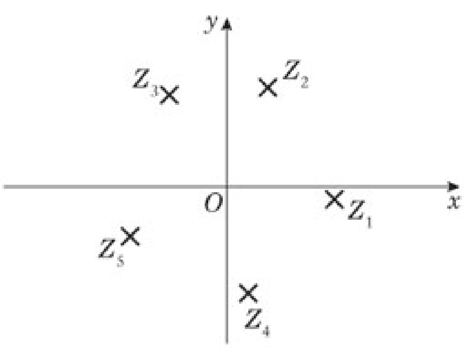
$k = 2$, $z_3 = \sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)}$

$k = -1$, $z_4 = \sqrt{2}\, e^{i\left(-\frac{9\pi}{20}\right)}$

$k = -2$, $z_5 = \sqrt{2}\, e^{i\left(-\frac{17\pi}{20}\right)}$

Therefore, $z = \sqrt{2}\, e^{-\frac{\pi i}{20}}, \sqrt{2}\, e^{\frac{7\pi i}{20}}, \sqrt{2}\, e^{\frac{3\pi i}{4}}, \sqrt{2}\, e^{-\frac{9\pi i}{20}}, \sqrt{2}\, e^{-\frac{17\pi i}{20}}$

c



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Exercise I, Question 6

Question:

a Find the Cartesian equations of

i the locus of points representing $|z-3+\mathrm{i}| = |z-1-\mathrm{i}|$,

ii the locus of points representing $|z-2| = 2\sqrt{2}$.

b Find the two values of z that satisfy both $|z-3+\mathrm{i}| = |z-1-\mathrm{i}|$ and $|z-2| = 2\sqrt{2}$.

c Hence on the same Argand diagram sketch:

i the locus of points representing $|z-3+\mathrm{i}| = |z-1-\mathrm{i}|$,

ii the locus of points representing $|z-2| = 2\sqrt{2}$.

The region R is defined by the inequalities $|z-3+\mathrm{i}| \geqslant |z-1-\mathrm{i}|$ and $|z+2| \leqslant 2\sqrt{2}$.

d On your sketch in part **c**, identify, by shading, the region R.

Solution:

a i Let $|z - 3 + i| = |z - 1 - i|$

$\Rightarrow |x + iy - 3 + i| = |x + iy - 1 - i|$

$\Rightarrow |(x - 3) + i(y + 1)| = |(x - 1) + i(y - 1)|$

$\Rightarrow |(x - 3) + i(y + 1)|^2 = |(x - 1) + i(y - 1)|^2$

$\Rightarrow (x - 3)^2 + (y + 1)^2 = (x - 1)^2 + (y - 1)^2$

$\Rightarrow x^2 - 6x + 9 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 2y + 1$

$\Rightarrow -6x + 2y + 10 = -2x - 2y + 2$

$\Rightarrow -4x + 4y + 8 = 0$

$\Rightarrow 4y = 4x - 8$

$\Rightarrow y = x - 2$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

METHOD ① **i** $|z - 3 + i| = |z - 1 - i|$

As $|z - 3 + i| = |z - 1 - i|$ is a perpendicular bisector of the line joining $A(3, -1)$ to $B(1, 1)$,

then $m_{AB} = \frac{1 - -1}{1 - 3} = \frac{2}{-2} = -1$

and perpendicular gradient $= \frac{-1}{-1} = 1$

mid-point of AB is $\left(\frac{3 + 1}{2}, \frac{-1 + 1}{2}\right)$

$= (2, 0)$

$\Rightarrow y - 0 = 1(x - 2)$

$y = x - 2$

The Cartesian equation of the locus of points representing $|z - 3 + i| = |z - 1 - i|$ is $y = x - 2$.

ii $|z - 2| = 2\sqrt{2}$

$\Rightarrow$ circle centre $(2, 0)$, radius $2\sqrt{2}$.

$\Rightarrow$ equation of circle is $(x - 2)^2 + y^2 = (2\sqrt{2})^2$

$\Rightarrow (x - 2)^2 + y^2 = 8$

The Cartesian equation of the locus of points representing $|z - 2| = 2\sqrt{2}$ is $(x - 2)^2 + y^2 = 8$.

b $|z - 3 + i| = |z - 1 + i| \Rightarrow y = x - 2$ ①

$|z - 2| = 2\sqrt{2} \Rightarrow (x - 2)^2 + y^2 = 8$ ②

①∩② $\Rightarrow (x - 2)^2 + (x - 2)^2 = 8$

$\Rightarrow 2(x - 2)^2 = 8$

$\Rightarrow (x - 2)^2 = 4$

$\Rightarrow x - 2 = \pm\sqrt{4}$

$\Rightarrow x - 2 = \pm 2$

$\Rightarrow x = 2 \pm 2$

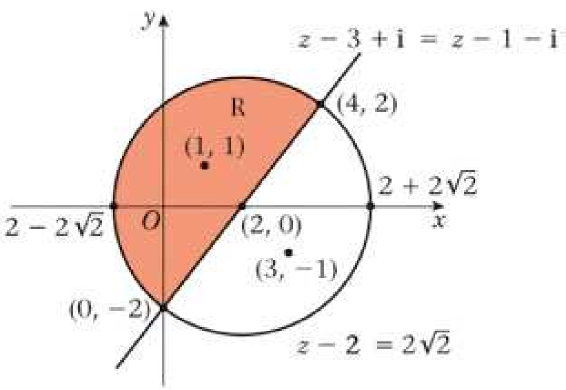
$\Rightarrow x = 0, 4$

when $x = 0$, $y = 0 - 2 = -2 \Rightarrow z = 0 - 2i$

when $x = 4$, $y = 4 - 2 = 2 \Rightarrow z = 4 + 2i$

The values of z are $-2i$ and $4 + 2i$

c



Note that $|z - 3 + i| = |z - 1 + i| \Rightarrow y = x - 2$ goes through the point $(2, 0)$ and so is a diameter of $|z - 2| = 2\sqrt{2}$.

d The region R is shaded on the Argand diagram in part **i**, which satisfies $|z - 3 + i| \geqslant |z - 1 - i|$ and $|z - 2| \leqslant 2\sqrt{2}$.

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Edexcel AS and A Level Modular Mathematics

Exercise I, Question 7

Question:

a Find the Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$.

b Find the value of z which satisfies both $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$.

c Hence shade in the region R on an Argand diagram which satisfies both $|z + 2| \geqslant |2z - 1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Solution:

a $|z + 2| = |2z - 1|$

$\Rightarrow |x + iy + 2| = |2(x + iy) - 1|$

$\Rightarrow |x + iy + 2| = |2x + 2iy - 1|$

$\Rightarrow |(x + 2) + iy| = |(2x - 1) + i(2y)|$

$\Rightarrow |(x + 2) + iy|^2 = |(2x - 1) + i(2y)|^2$

$\Rightarrow (x + 2)^2 + y^2 = (2x - 1)^2 + i(2y)^2$

$\Rightarrow x^2 + 4x + 4 + y^2 = 4x^2 - 4x + 1 + 4y^2$

$\Rightarrow 0 = 3x^2 - 8x + 3y^2 + 1 - 4$

$\Rightarrow 3x^2 - 8x + 3y^2 - 3 = 0$

$\Rightarrow x^2 - \frac{8}{3}x + y^2 - 1 = 0$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1 = 0$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{16}{9} + 1$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \left(\frac{5}{3}\right)^2$

This is a circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

The Cartesian equation of the locus of points representing $|z + 2| = |2z - 1|$ is

$\left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}$.

b $|z + 2| = |2z - 1| \Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}$ ①

$\arg z = \frac{\pi}{4} \Rightarrow \arg(x + iy) = \frac{\pi}{4}$

$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{y}{x} = 1$

$\Rightarrow y = x$ where $x > 0, y > 0$ ②

②^①: $\left(x - \frac{4}{3}\right)^2 + x^2 = \frac{25}{9}$

$\Rightarrow x^2 - \frac{4}{3}x - \frac{4}{3}x + \frac{16}{9} + x^2 = \frac{25}{9}$

$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{25}{9} - \frac{16}{9}$

$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{9}{9}$

$\Rightarrow 2x^2 - \frac{8}{3}x = 1 \quad (\times 3)$

$\Rightarrow 6x^2 - 8x = 3$

$\Rightarrow 6x^2 - 8x - 3 = 0$

$\Rightarrow x = \frac{8 \pm \sqrt{64 - 4(6)(-3)}}{2(6)}$

$\Rightarrow x = \frac{8 \pm \sqrt{136}}{12}$

$\Rightarrow x = \frac{8 \pm 2\sqrt{34}}{12}$

$\Rightarrow x = \frac{4 \pm \sqrt{34}}{6}$

As $x > 0$ then we reject $x = \frac{4 - \sqrt{34}}{6}$

and accept $x = \frac{4 + \sqrt{34}}{6}$

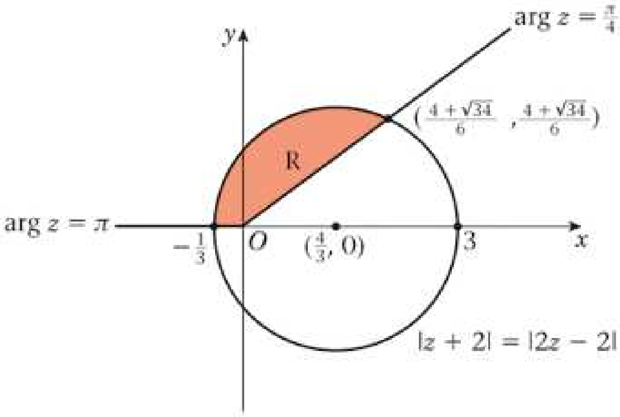
as $y = x$, then $y = \frac{4 + \sqrt{34}}{6}$

So $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)\text{i}$

The value of z satisfying $|z + 2| = |2z - 1|$ and $\arg z = \frac{\pi}{4}$

is $z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)\text{i}$ OR $z = 1.64 + 1.64\text{i}$ (2 d.p.)

c



The region R (shaded) satisfies both $|z + 2| \geqslant |2z - 1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Note that $|z + 2| \geqslant |2z - 1|$

$\Rightarrow (x + 2)^2 + y^2 \geqslant (2x - 1)^2 + (2y)^2$

$\Rightarrow 0 \geqslant 3x^2 - 8x + 3y^2 - 3$

$\Rightarrow 0 \geqslant \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$

$\Rightarrow \frac{25}{9} \geqslant \left(x - \frac{4}{3}\right)^2 + y^2$

$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leqslant \frac{25}{9}$

represents region inside and bounded by the circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

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Exercise I, Question 8

Question:

The point P represents a complex number z in an Argand diagram.
Given that $|z + 1 - i| = 1$

a find a Cartesian equation for the locus of P,

b sketch the locus of P on an Argand diagram,

c find the greatest and least values of $|z|$,

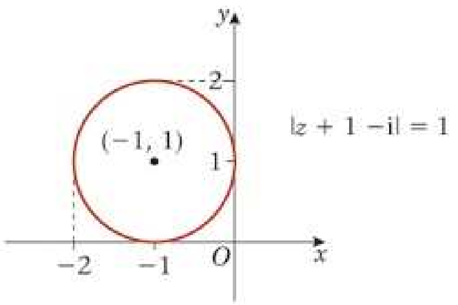
d find the greatest and least values of $|z - 1|$.

Solution:

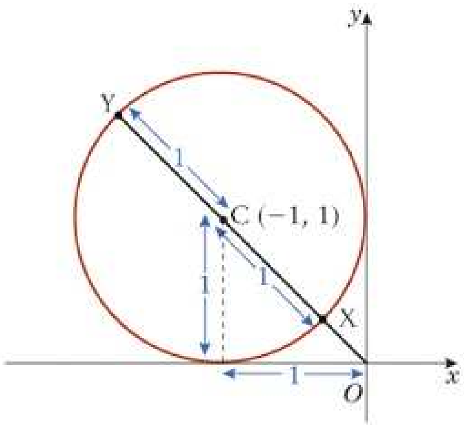
a $|z + 1 - i| = 1$ is a circle, centre $(-1, 1)$, radius 1.

The Cartesian equation for the locus of P is $(x + 1)^2 + (y - 1)^2 = 1$.

b



c



$|z|$ is the distance from (0, 0) to the locus of points.

From the Argand diagram,

$|z|_{max}$ is the distance OY

$|z|_{min}$ is the distance OX

Note that radius $= CX = CY = 1$

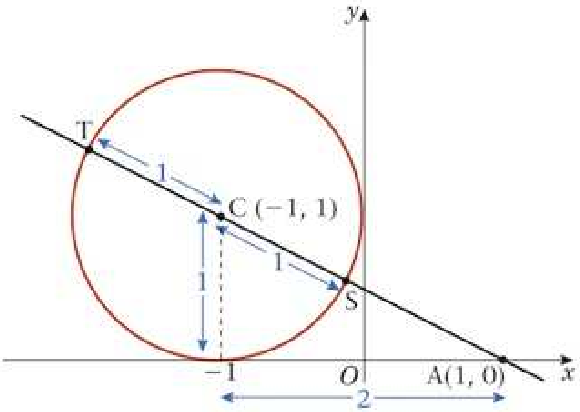
and $OC = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|z|_{max} = OC + CY = \sqrt{2} + 1$

$|z|_{min} = OC - CX = \sqrt{2} - 1$

The greatest value of $|z|$ is $\sqrt{2} + 1$ and the least value of $|z|$ is $\sqrt{2} - 1$.

d



$|z-1|$ is the distance from $A(1, 0)$ to the locus of points.

From the Argand diagram,

$|z-1|_{max}$ is the distance AS

$|z-1|_{min}$ is the distance AT

Note that radius $= CS = CT = 1$

and $AC = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|z-1|_{max} = AC + CT = \sqrt{5} + 1$

$|z-1|_{min} = AC - CS = \sqrt{5} - 1$

The greatest value of $|z-1|$ is $\sqrt{5} + 1$ and the least value of $|z-1|$ is $\sqrt{5} - 1$.

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Exercise I, Question 9

Question:

Given that $\arg\left(\frac{z-4-2i}{z-6i}\right)=\frac{\pi}{2}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

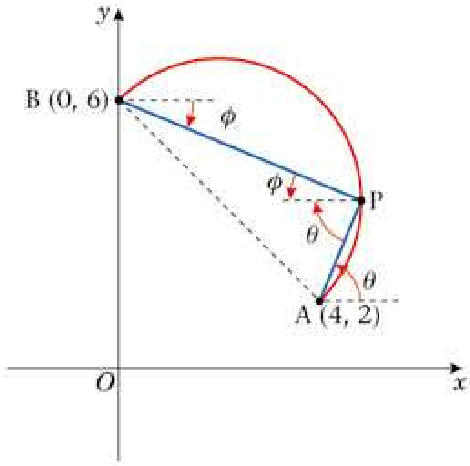
b deduce the exact value of $|z-2-4i|$.

Solution:

a $\arg\left(\frac{z-4-2i}{z-6i}\right)=\frac{\pi}{2}$

$\Rightarrow \arg(z-4-2i)-\arg(z-6i)=\frac{\pi}{2}$

$\Rightarrow \theta-\phi=\frac{\pi}{2}$, where $\arg(z-4-2i)=\theta$ and $\arg(z-6i)=\phi$.



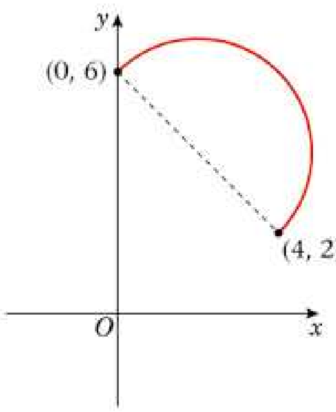
Using geometry,

$\Rightarrow A\hat{P}B=-\phi+\theta$

$\Rightarrow A\hat{P}B=\theta-\phi$

$\Rightarrow A\hat{P}B=\frac{\pi}{2}$

The locus of z is the arc of a circle (in this case, a semi-circle) cut off at (4, 2) and (0, 6) as shown below.



$\arg\left(\frac{z-4-2i}{z-6i}\right)=\frac{\pi}{2}$

b $|z - 2 - 4i|$ is the distance from the point (2, 4) to the locus of points P.

Note, as the locus is a semi-circle, its centre is $\left(\frac{4+0}{2}, \frac{2+6}{2}\right) = (2, 4)$.

Therefore $|z - 2 - 4i|$ is the distance from the centre of the semi-circle to points on the locus of points P.

Hence $|z - 2 - 4i|$ = radius of semi-circle

$$= \sqrt{(0-2)^2 + (6-4)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

The exact value of $|z - 2 - 4i|$ is $2\sqrt{2}$

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Exercise I, Question 10

Question:

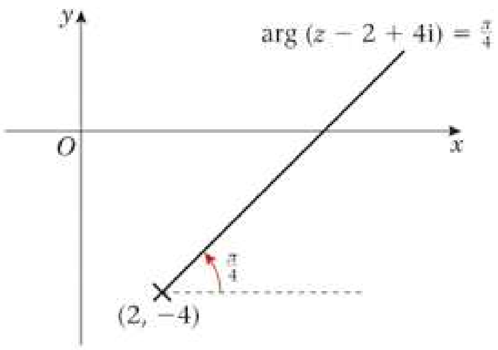
Given that $\arg(z - 2 + 4\mathrm{i}) = \frac{\pi}{4}$,

a sketch the locus of $P(x, y)$ which represents z on an Argand diagram,

b find the minimum value of $|z|$ for points on this locus.

Solution:

a $\arg(z-2+4i)=\frac{\pi}{4}$ is a half-line from $(2,-4)$ as shown



b $\arg(z-2+4i)=\frac{\pi}{4} \Rightarrow \arg(x+iy-2+4i)=\frac{\pi}{4}$

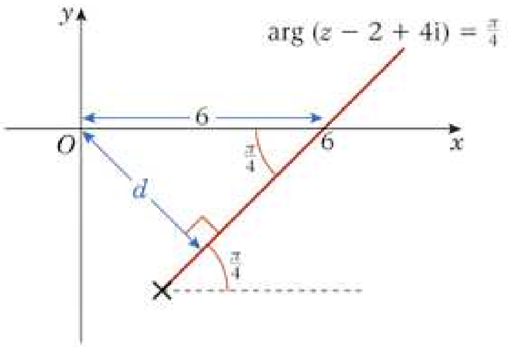
$$\Rightarrow \arg((x-2)+i(y+4))=\frac{\pi}{4}$$

$$\Rightarrow \frac{y+4}{x-2}=\tan\frac{\pi}{4}=1$$

$$\Rightarrow y+4=x-2$$

$$\Rightarrow y=x-6,\ x>0,\ y>0$$

Half-line cuts x-axis at $0=x-6 \Rightarrow x=6$.



$|z|$ is the distance from (0, 0) to the locus of points.

$|z|_{\min}=d \Rightarrow \frac{d}{6}=\sin\left(\frac{\pi}{4}\right) \Rightarrow d=6\sin\left(\frac{\pi}{4}\right)=6\left(\frac{1}{\sqrt{2}}\right)=\frac{6\sqrt{2}}{2}=3\sqrt{2}$.

Therefore the minimum value of $|z|$ is $3\sqrt{2}$.

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Exercise I, Question 11

Question:

The transformation T from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$, is given by $w = \frac{1}{z}$, $z \neq 0$.

a Show that the image, under T, of the line with equation $x = \frac{1}{2}$ in the z-plane is a circle C in the w-plane. Find the equation of C.

b Hence, or otherwise, shade and label on an Argand diagram the region R of the w-plane which is the image of $x \geqslant \frac{1}{2}$ under T.

Solution:

$T: w = \frac{1}{z}$

a line $x = \frac{1}{2}$ in the z-plane

$w = \frac{1}{z}$

$\Rightarrow wz = 1$

$\Rightarrow z = \frac{1}{w}$

$\Rightarrow z = \frac{1}{u + \text{i}v}$

$\Rightarrow z = \frac{1}{(u + \text{i}v)} \times \frac{(u - \text{i}v)}{(u - \text{i}v)}$

$\Rightarrow z = \frac{u - \text{i}v}{u^2 + v^2}$

$\Rightarrow z = \frac{u}{u^2 + v^2} + \text{i}\left(\frac{-v}{u^2 + v^2}\right)$

So, $x + \text{i}y = \frac{u}{u^2 + v^2} + \text{i}\left(\frac{-v}{u^2 + v^2}\right)$

$\Rightarrow x = \frac{u}{u^2 + v^2}$ and $y = \frac{-v}{u^2 + v^2}$

As $x = \frac{1}{2}$, then $\frac{1}{2} = \frac{u}{u^2 + v^2}$

$\Rightarrow u^2 + v^2 = 2u$

$\Rightarrow u^2 - 2u + v^2 = 0$

$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$

$\Rightarrow (u - 1)^2 + v^2 = 1$

Therefore the transformation T maps the line $x = \frac{1}{2}$ in the z-plane to a circle C, with centre $(1, 0)$, radius 1. The equation of C is $(u - 1)^2 + v^2 = 1$.

b $x \geqslant \frac{1}{2}$ $\frac{u}{u^2 + v^2} \geqslant \frac{1}{2}$

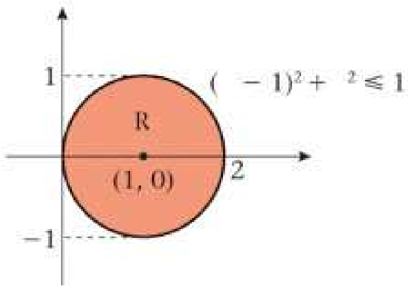
$\Rightarrow 2u \geqslant u^2 + v^2$

$\Rightarrow 0 \geqslant u^2 + v^2 - 2u$

$\Rightarrow 0 \geqslant (u - 1)^2 + v^2 - 1$

$\Rightarrow 1 \geqslant (u - 1)^2 + v^2$

$\Rightarrow (u - 1)^2 + v^2 \leqslant 1$



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Exercise I, Question 12

Question:

The point P represents the complex number z on an Argand diagram.
Given that $|z + 4i| = 2$,

a sketch the locus of P on an Argand diagram.

b Hence find the maximum value of $|z|$.

T_1, T_2, T_3 and T_4 represent transformations from the z-plane to the w-plane. Describe the locus of the image of P under the transformations

c T_1: $w = 2z$,

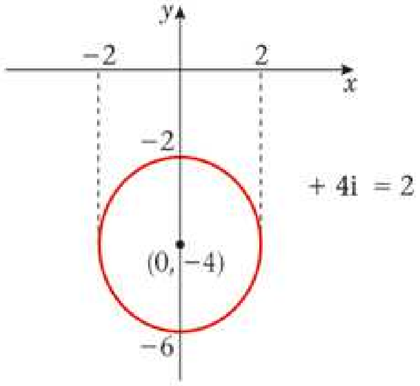
d T_2: $w = iz$,

e T_3: $w = -iz$,

f T_4: $w = z^*$

Solution:

a $|z + 4i| = 2$ is represented by a circle centre $(0, -4)$, radius 2.



b $|z|$ represents the distance from (0, 0) to points on the locus of P. Hence $|z|_{max}$ is the distance OY.

$|z|_{max} = OY = 6$.

c T_1: $w = 2z$

METHOD ① z lies on circle with equation $|z + 4i| = 2$

$$\Rightarrow w = 2z$$
$$\Rightarrow \frac{w}{2} = z$$
$$\Rightarrow \frac{w}{2} + 4i = z + 4i$$
$$\Rightarrow \frac{w + 8i}{2} = z + 4i$$
$$\Rightarrow \left|\frac{w + 8i}{2}\right| = |z + 4i|$$
$$\Rightarrow \frac{|w + 8i|}{|2|} = |z + 4i|$$
$$\Rightarrow \frac{|w + 8i|}{2} = 2$$
$$\Rightarrow |w + 8i| = 4$$

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

METHOD ② z lies on circle centre $(0, -4)$, radius 2

↓ enlargement scale factor 2, centre 0.

$w = 2z$ lies on a circle centre $(0, -8)$, radius 4.

So the locus of the image of P under T_1 is a circle centre $(0, -8)$, radius 4, with equation $u^2 + (v + 8)^2 = 16$.

d T_2: $w = \text{i}z$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = \text{i}z$

$\Rightarrow \frac{w}{\text{i}} = z$

$\Rightarrow \frac{w}{\text{i}}\left(\frac{\text{i}}{\text{i}}\right) = z$

$\Rightarrow \frac{w\text{i}}{(-1)} = z$

$\Rightarrow -w\text{i} = z$

$\Rightarrow z = -w\text{i}$

Hence $|z + 4\text{i}| = 2 \Rightarrow |-w\text{i} + 4\text{i}| = 2$

$\Rightarrow |(-\text{i})(w - 4)| = 2$

$\Rightarrow |(-\text{i})|\,|w - 4| = 2$

$\Rightarrow |w - 4| = 2$

So the locus of the image of P under T_2 is a circle centre (4, 0), radius 2, with equation $(u - 4)^2 + v^2 = 4$.

e T_3: $w = -\text{i}z$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = -\text{i}z$

$\Rightarrow \text{i}w = \text{i}(-\text{i}z)$

$\Rightarrow \text{i}w = z$

$\Rightarrow z = \text{i}w$

Hence $|z + 4\text{i}| = 2 \Rightarrow |\text{i}w + 4\text{i}| = 2$

$\Rightarrow |\text{i}(w + 4)| = 2$

$\Rightarrow |\text{i}||w + 4| = 2$

$\Rightarrow |w + 4| = 2$ — $|\text{i}| = 1$

So the locus of the image of P under T_3 is a circle centre $(-4, 0)$, radius 2, with equation $(u + 4)^2 + v^2 = 4$.

f T_4: $w = z^*$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = z^* \Rightarrow u + \text{i}v = x - \text{i}y$ — $z = x + \text{i}y \Rightarrow z^* = x - \text{i}y$

So $u = x, v = -y$ and $x = u$ and $y = -v$

$|z + 4\text{i}| = 2 \Rightarrow |x + \text{i}y + 4\text{i}| = 2$

$\Rightarrow |x + \text{i}(y + 4)| = 2$

$\Rightarrow |u + \text{i}(-v + 4)| = 2$

$\Rightarrow |u + \text{i}(4 - v)| = 2$

$\Rightarrow |u + \text{i}(4 - v)|^2 = 2^2$

$\Rightarrow u^2 + (4 - v)^2 = 4$

$\Rightarrow u^2 + (v - 4)^2 = 4$

So the locus of the image of P under T_4 is a circle centre (0, 4), radius 2, with equation $u^2 + (v - 4)^2 = 4$.

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Exercise I, Question 13

Question:

The transformation T from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$, is given by $w = \frac{z+2}{z+\text{i}}$, $z \neq -\text{i}$.

a Show that the image, under T, of the imaginary axis in the z-plane is a line l in the w-plane. Find the equation of l.

b Show that the image, under T, of the line $y = x$ in the z-plane is a circle C in the w-plane. Find the centre of C and show that the radius of C is $\frac{1}{2}\sqrt{10}$.

Solution:

T: $w = \dfrac{z+2}{z+\mathrm{i}}, z \neq -\mathrm{i}$

a the imaginary axis in z-plane $\Rightarrow x = 0$

$$w = \frac{z+2}{z+\mathrm{i}}$$

$\Rightarrow w(z + \mathrm{i}) = z + 2$

$\Rightarrow wz + \mathrm{i}w = z + 2$

$\Rightarrow wz - z = 2 - \mathrm{i}w$

$\Rightarrow z(w - 1) = 2 - \mathrm{i}w$

$$\Rightarrow z = \frac{2 - \mathrm{i}w}{w - \mathrm{i}}$$

$$\Rightarrow z = \frac{2 - \mathrm{i}(u + \mathrm{i}v)}{u + \mathrm{i}v - 1}$$

$$\Rightarrow z = \frac{2 - \mathrm{i}u + v}{(u-1) + \mathrm{i}v}$$

$$\Rightarrow z = \left[\frac{(2+v) - \mathrm{i}u}{(u-1) + \mathrm{i}v}\right] \times \left[\frac{(u-1) - \mathrm{i}v}{(u-1) - \mathrm{i}v}\right]$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv - \mathrm{i}v(2+v) - \mathrm{i}u(u-1)}{(u-1)^2 + v^2}$$

$$\Rightarrow z = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - \mathrm{i}\left(\frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2}\right)$$

So $x + \mathrm{i}y = \dfrac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - \mathrm{i}\left(\dfrac{v(2+v) + u(u-1)}{(u-1)^2 + v^2}\right)$

$\Rightarrow x = \dfrac{(2+v)(u-1) - uv}{(u-1)^2 + v^2}$ and $y = \dfrac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2}$

As $x = 0$, then

$$\frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} = 0$$

$\Rightarrow (2 + v)(u - 1) - uv = 0$

$\Rightarrow 2u - 2 + vu - v - uv = 0$

$\Rightarrow 2u - 2 - v = 0$

$\Rightarrow v = 2u - 2$

The transformation T maps the imaginary axis in the z-plane to the line l with equation $v = 2u - 2$ in the w-plane.

b As $y = x$, then

$$\frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2} = \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2}$$

$\Rightarrow -v(2+v) - u(u-1) = (2+v)(u-1) - uv$

$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 + vu - v - uv$

$\Rightarrow -2v - v^2 - u^2 + u = 2u - 2 - v$

$\Rightarrow 0 = u^2 + v^2 + u + v - 2$

$\Rightarrow \left(u + \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = 0$

$\Rightarrow \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{2}$

$\Rightarrow \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{10}}{2}\right)^2$

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2} = \frac{1}{2}\sqrt{10}$$

The transformation T maps the line $y = x$ in the z-plane to the circle C with centre $\left(\frac{-1}{2}, \frac{-1}{2}\right)$, radius $\frac{1}{2}\sqrt{10}$ in the w-plane.

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Exercise I, Question 14

Question:

The transformation T from the z-plane, where $z = x + \mathrm{i}y$ to the w-plane where $w = u + \mathrm{i}v$, is given by $w = \frac{4 - z}{z + \mathrm{i}}$, $z \neq -\mathrm{i}$.

The circle $|z| = 1$ is mapped by T onto a line l. Show that l can be written in the form $au + bv + c = 0$, where a, b and c are integers to be determined.

Solution:

T: $\quad w = \frac{4 - z}{z + \mathrm{i}} \quad z \neq -\mathrm{i}$

circle with equation $|z| = 1$ in the z-plane.

$$w = \frac{4 - z}{z + \mathrm{i}}$$

$\Rightarrow w(z + \mathrm{i}) = 4 - z$

$\Rightarrow wz + \mathrm{i}w = 4 - z$

$\Rightarrow wz + z = 4 - \mathrm{i}w$

$\Rightarrow z(w + 1) = 4 - \mathrm{i}w$

$\Rightarrow z = \frac{4 - \mathrm{i}w}{w + 1}$

$\Rightarrow |z| = \left|\frac{4 - \mathrm{i}w}{w + 1}\right|$

$\Rightarrow |z| = \frac{|4 - \mathrm{i}w|}{|w + 1|}$

Applying $|z| = 1$ gives $1 = \frac{|4 - \mathrm{i}w|}{|w + 1|}$

$\Rightarrow |w + 1| = |4 - \mathrm{i}w|$

$\Rightarrow |w + 1| = |-\mathrm{i}(w + 4\mathrm{i})|$

$\Rightarrow |w + 1| = |-\mathrm{i}|\,|w + 4\mathrm{i}|$

$\Rightarrow |w + 1| = |w + 4\mathrm{i}|$

$\Rightarrow |u + \mathrm{i}v + 1| = |u + \mathrm{i}v + 4\mathrm{i}|$

$\Rightarrow |(u + 1) + \mathrm{i}v| = |u + \mathrm{i}(v + 4)|$

$\Rightarrow |(u + 1) + \mathrm{i}v|^2 = |u + \mathrm{i}(v + 4)|^2$

$\Rightarrow (u + 1)^2 + v^2 = u^2 + (v + 4)^2$

$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$

$\Rightarrow 2u + 1 = 8v + 16$

$\Rightarrow 2u - 8v - 15 = 0$

The circle $|z| = 1$ is mapped by T onto the line l: $2u - 8v - 15 = 0$ (i.e. $a = 2$, $b = -8$, $c = -15$).

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Exercise I, Question 15

Question:

The transformation T from the z-plane, where $z = x + iy$, to the w-plane where $w = u + iv$, is given by $w = \frac{3iz + 6}{1 - z}$, $z \neq 1$.

Show that the circle $|z| = 2$ is mapped by T onto a circle C. State the centre of C and show that the radius of C can be expressed in the form $k\sqrt{5}$ where k is an integer to be determined.

Solution:

T: $w = \frac{3iz + 6}{1 - z}$; $z \neq 1$

circle with equation $|z| = 2$

$$w = \frac{3iz + 6}{1 - z}$$

$\Rightarrow w(1 - z) = 3iz + 6$

$\Rightarrow w - wz = 3iz + 6$

$\Rightarrow w - 6 = 3iz + wz$

$\Rightarrow w - 6 = z(3i + w)$

$\Rightarrow \frac{w - 6}{w + 3i} = z$

$\Rightarrow \left|\frac{w - 6}{w + 3i}\right| = |z|$

$\Rightarrow \frac{|w - 6|}{|w + 3i|} = |z|$

Applying $|z| = 2 \Rightarrow \frac{|w - 6|}{|w + 3i|} = 2$

$\Rightarrow |w - 6| = 2|w + 3i|$

$\Rightarrow |u + iv - 6| = 2|u + iv + 3i|$

$\Rightarrow |(u - 6) + iv| = 2|u + i(v + 3)|$

$\Rightarrow |(u - 6) + iv|^2 = 2^2|u + i(v + 3)|^2$

$\Rightarrow (u - 6)^2 + v^2 = 4[u^2 + (v + 3)^2]$

$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$

$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$

$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$

$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$

$\Rightarrow 0 = (u + 2)^2 - 4 + (v + 4)^2 - 16$

$\Rightarrow 20 = (u + 2)^2 + (v + 4)^2$

$\Rightarrow (u + 2)^2 + (v + 4)^2 = (2\sqrt{5})^2$

$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

Therefore the circle with equation $|z| = 2$ is mapped onto a circle C, centre $(-2, -4)$, radius $2\sqrt{5}$. So $k = 2$.

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Exercise I, Question 16

Question:

A transformation from the z-plane to the w-plane is defined by $w = \frac{az + b}{z + c}$,
where $a, b, c \in \mathbb{R}$.

Given that $w = 1$ when $z = 0$ and that $w = 3 - 2\text{i}$ when $z = 2 + 3\text{i}$,

a find the values of a, b and c,

b find the exact values of the two points in the complex plane which remain invariant under the transformation.

Solution:

a $w = \frac{az + b}{z + c} \quad a, b, c \in \mathbb{R}.$

$w = 1$ when $z = 0$ ①

$w = 3 - 2i$ when $z = 2 + 3i$ ②

① $\Rightarrow 1 = \frac{a(0) + b}{0 + c} \Rightarrow 1 = \frac{b}{c} \Rightarrow c = 6$ ③

③ $\Rightarrow w = \frac{az + b}{z + b}$

② $\Rightarrow 3 - 2i = \frac{a(2 + 3i) + b}{2 + 3i + b}$

$3 - 2i = \frac{(2a + b) + 3ai}{(2 + b) + 3i}$

$(3 - 2i)[(2 + b) + 3i] = 2a + b + 3ai$

$6 + 3b + 9i - 4i - 2bi + 6 = 2a + b + 3ai$

$(12 + 3b) + (5 - 2b) = (2a + b) + 3ai$

Equate real parts: $12 + 3b = 2a + b$

$\Rightarrow \quad 12 = 2a - 2b$ ④

Equate imaginary parts: $5 - 2b = 3a$

$\Rightarrow \quad 5 = 3a + 2b$ ⑤

④ + ⑤: $\quad 17 = 5a$

$\Rightarrow \quad \frac{17}{5} = a$

⑤ $\Rightarrow \quad 5 = \frac{51}{5} + 2b$

$\Rightarrow \quad \frac{-26}{5} = 2b$

$\Rightarrow \quad \frac{-13}{5} = b$

As $b = c$ then $c = \frac{-13}{5}$

The values are $a = \frac{17}{5}$, $b = \frac{-13}{5}$, $c = \frac{-13}{5}$

b $w = \dfrac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}}$ $\quad\begin{matrix}(\times 5)\\(\times 5)\end{matrix}$

$w = \dfrac{17z - 13}{5z - 13}$

invariant points $\Rightarrow z = \dfrac{17z - 13}{5z - 13}$

$z(5z - 13) = 17z - 13$

$5z^2 - 13z = 17z - 13$

$5z^2 - 30z + 13 = 0$

$z = \dfrac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$

$z = \dfrac{30 \pm \sqrt{900} - 260}{10}$

$z = \dfrac{30 \pm \sqrt{640}}{10}$

$z = \dfrac{30 \pm \sqrt{64}\sqrt{10}}{10}$

$z = \dfrac{30 \pm 8\sqrt{10}}{10} = 3 \pm \frac{4}{5}\sqrt{10}$

The exact values of the two points which remain invariant are

$z = 3 + \frac{4}{5}\sqrt{10}$ and $z = 3 - \frac{4}{5}\sqrt{10}$.

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Exercise I, Question 17

Question:

The transformation T from the z-plane, where $z = x + \text{i}y$, to the w-plane where $w = u + \text{i}v$, is given by

$$w = \frac{z + \text{i}}{z}, z \neq 0.$$

a The transformation T maps the points on the line with equation $y = x$ in the z-plane other than $(0, 0)$, to points on the l in the w-plane. Find an equation of l.

b Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has equation $u^2 + v^2 - u + v = 0$.

c On the same Argand diagram, sketch l and C.

Solution:

T: $w = \frac{z + i}{z}$, $z \neq 0$.

a the line $y = x$ in the z-plane other than $(0, 0)$

$$w = \frac{z + i}{z}$$

$$\Rightarrow wz = z + i$$

$$\Rightarrow wz - z = i$$

$$\Rightarrow z(w - 1) = i$$

$$\Rightarrow z = \frac{i}{w - 1}$$

$$\Rightarrow z = \frac{i}{(u + iv) - 1} = \frac{i}{(u - 1) + iv}$$

$$\Rightarrow z = \left[\frac{i}{(u - 1) + iv}\right]\left[\frac{(u - 1) - iv}{(u - 1) - iv}\right]$$

$$\Rightarrow z = \frac{i(u - 1) + v}{(u - 1)^2 + v^2}$$

$$\Rightarrow z = \frac{v}{(u - 1)^2 + v^2} + i\frac{(u - 1)}{(u - 1)^2 + v^2}$$

So $x + iy = \frac{v}{(u - 1)^2 + v^2} + i\frac{(u - 1)}{(u - 1)^2 + v^2}$

$\Rightarrow x = \frac{v}{(u - 1)^2 + v^2}$ and $y = \frac{u - 1}{(u - 1)^2 + v^2}$

Applying $y = x$, gives $\frac{u - 1}{(u - 1)^2 + v^2} = \frac{v}{(u - 1)^2 + v^2}$

$$\Rightarrow u - 1 = v$$

$$\Rightarrow v = u - 1$$

Therefore the line l has equation $v = u - 1$.

b the line with equation $x + y + 1 = 0$ in the z-plane

$x + y + 1 = 0 \Rightarrow \frac{v}{(u - 1)^2 + v^2} + \frac{u - 1}{(u - 1)^2 + v^2} + 1 = 0$ $[\times (u - 1)^2 + v^2]$

$$\Rightarrow v + (u - 1) + (u - 1)^2 + v^2 = 0$$

$$\Rightarrow v + u - 1 + u^2 - 2u + 1 + v^2 = 0$$

$$\Rightarrow u^2 + v^2 - u + v = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

The image of $x + y + 1 = 0$ under T is a circle C, centre $\left(\frac{1}{2}, \frac{-1}{2}\right)$, radius $\frac{\sqrt{2}}{2}$ with equation $u^2 + v^2 - u + v = 0$, as required.

c