Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

$z_1 = 2 + \text{i},\ z_2 = 3 + 4\text{i}$. Find the modulus and the tangent of the argument of each of

a $z_1 z_2^*$

b $\frac{z_1}{z_2}$

Solution:

a $z_2{}^* = 3 - 4\mathrm{i}$

$z_1 z_2{}^* = (2 + \mathrm{i})(3 - 4\mathrm{i})$

$= 6 - 8\mathrm{i} + 3\mathrm{i} - 4\mathrm{i}^2$

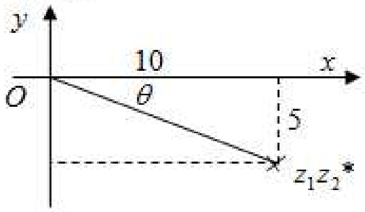
$= 10 - 5\mathrm{i}$

$|z_1 z_2{}^*|^2 = 10^2 + (-5)^2 = 125$

$|z_1 z_2{}^*| = \sqrt{125} = 5\sqrt{5}$

z^* is the symbol for the conjugate complex number of z.
If $z = a + \mathrm{i}b$, then $z^* = a - \mathrm{i}b$.

$-4\mathrm{i}^2 = -4 \times -1 = +4$



$\tan\theta = \frac{5}{10} = \frac{1}{2}$

$z_1 z_2{}^*$ is in the fourth quadrant.

$\tan \arg(z_1 z_2{}^*) = -\frac{1}{2}$

Arguments in the fourth quadrant are negative. The tangents of arguments are negative in the second and fourth quadrants.

b $\frac{z_1}{z_2} = \frac{2 + \mathrm{i}}{3 + 4\mathrm{i}} \times \frac{3 - 4\mathrm{i}}{3 - 4\mathrm{i}}$

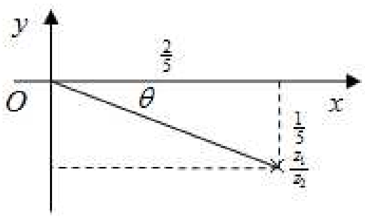
$= \frac{6 - 8\mathrm{i} + 3\mathrm{i} + 4}{25} = \frac{10 - 5\mathrm{i}}{25}$

$= \frac{2}{5} - \frac{1}{5}\mathrm{i}$

To simplify a quotient you multiply the numerator and denominator by the conjugate complex of the denominator. The conjugate complex of this denominator $3 + 4\mathrm{i}$ is $3 - 4\mathrm{i}$.

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{2}{5}\right)^2 + \left(-\frac{1}{5}\right)^2 = \frac{4}{25} + \frac{1}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$



$\tan\theta = \frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$

$\frac{z_1}{z_2}$ is in the fourth quadrant.

$\tan \arg\left(\frac{z_1}{z_2}\right) = -\frac{1}{2}$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 2

Question:

a Show that the complex number $\frac{2+3\text{i}}{5+\text{i}}$ can be expressed in the form $\lambda(1+\text{i})$, stating the value of λ.

b Hence show that $\left(\frac{2+3\text{i}}{5+\text{i}}\right)^4$ is real and determine its value.

Solution:

a
$$\frac{2+3\text{i}}{5+\text{i}}\times\frac{5-\text{i}}{5-\text{i}}=\frac{10-2\text{i}+15\text{i}+3}{26}$$

$(5+\text{i})(5-\text{i})=5^2+1^2=26$
You should practise doing such calculations mentally.

$$=\frac{13+13\text{i}}{26}=\frac{1}{2}+\frac{1}{2}\text{i}$$
$$=\frac{1}{2}(1+\text{i})$$
$$\lambda=\frac{1}{2}$$

You use the result from part (a) to simplify the working in part (b).

b
$$\left(\frac{2+3\text{i}}{5+\text{i}}\right)^4=\left[\frac{1}{2}(1+\text{i})\right]^4$$

$(1+\text{i})^4$ is expanded using the binomial expansion
$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$$=\frac{1}{16}\left(1+4\text{i}+6\text{i}^2+4\text{i}^3+\text{i}^4\right)$$

$\text{i}^3=\text{i}^2\times\text{i}=-1\times\text{i}=-\text{i}$
$\text{i}^4=\text{i}^2\times\text{i}^2=-1\times-1=1$

$$=\frac{1}{16}(1+4\text{i}-6-4\text{i}+1)$$
$$=\frac{1}{16}\times-4=-\frac{1}{4}\text{, a real number}$$

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Review Exercise
Exercise A, Question 3

Question:

$z_1 = 5 + \mathrm{i},\ z_2 = -2 + 3\mathrm{i}$

a Show that $|z_1|^2 = 2|z_2|^2$.

b Find $\arg(z_1 z_2)$.

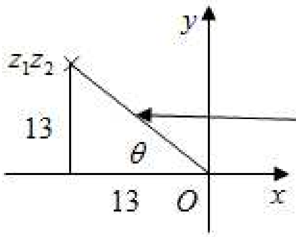
Solution:

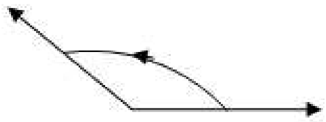
a
$$|z_1|^2 = 5^2 + 1^2 = 26$$
$$|z_2|^2 = (-2)^2 + 3^2 = 4 + 9 = 13$$
$$26 = 2 \times 13$$
Hence
$|z_1|^2 = 2|z_2|^2$, as required.

If $z = a + \mathrm{i}b$, then $|z|^2 = a^2 + b^2$

When you are asked to show or prove a result, you should conclude by saying that you have proved or shown the result. You can write the traditional q.e.d. if you like!

b
$$z_1 z_2 = (5 + \mathrm{i})(-2 + 3\mathrm{i})$$
$$= -10 + 15\mathrm{i} - 2\mathrm{i} - 3 = -13 + 13\mathrm{i}$$





The argument is the angle with the positive x-axis. Anti-clockwise is positive.

$$\tan\theta = \frac{13}{13} = 1 \Rightarrow \theta = \frac{\pi}{4}$$
$z_1 z_2$ is in the second quadrant.
$$\arg(z_1 z_2) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

As the question has not specified that you should work in radians or degrees. You could work in either and 135° would also be an acceptable answer.

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Review Exercise
Exercise A, Question 4

Question:

a Find, in the form $p + \mathrm{i}q$ where p and q are real, the complex number z which satisfies the equation $\frac{3z-1}{2-\mathrm{i}} = \frac{4}{1+2\mathrm{i}}$.

b Show on a single Argand diagram the points which represent z and z^*.

c Express z and z^* in modulus–argument form, giving the arguments to the nearest degree.

Solution:

a

$$\frac{3z-1}{2-\mathrm{i}} = \frac{4}{1+2\mathrm{i}}$$

$$3z-1 = \frac{8-4\mathrm{i}}{1+2\mathrm{i}} \times \frac{1-2\mathrm{i}}{1-2\mathrm{i}}$$

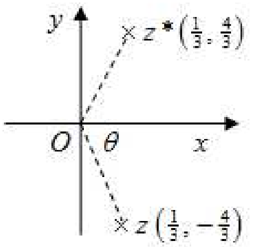
$$= \frac{8-16\mathrm{i}-4\mathrm{i}-8}{5} = \frac{-20\mathrm{i}}{5} = -4\mathrm{i}$$

$$3z = 1-4\mathrm{i}$$

$$z = \frac{1}{3} - \frac{4}{3}\mathrm{i}$$

You multiply both sides of the equation by $2-\mathrm{i}$.
Then multiply the numerator and denominator by the conjugate complex of the denominator.

b



You place the points in the Argand diagram which represent conjugate complex numbers symmetrically about the real x-axis.
Label the points so it is clear which is the original number (z) and which is the conjugate (z^*).

c

$$|z|^2 = \left(\frac{1}{3}\right)^2 + \left(-\frac{4}{3}\right)^3 = \frac{1}{9} + \frac{16}{9} = \frac{17}{9}$$

$$|z| = \frac{\sqrt{17}}{3}$$

$$\tan\theta = \frac{\frac{4}{3}}{\frac{1}{3}} = 4 \Rightarrow \theta \approx 76°$$

z is in the fourth quadrant.
$\arg z = -76°$, to the nearest degree.

The diagram you have drawn in part (b) shows that z is in the fourth quadrant. There is no need to draw it again.

$$z = \frac{\sqrt{17}}{3}\cos(-76°) + \mathrm{i}\frac{\sqrt{17}}{3}\sin(-76°)$$

$$z^* = \frac{\sqrt{17}}{3}\cos 76° + \mathrm{i}\frac{\sqrt{17}}{3}\sin 76°$$

It is always true that $|z^*| = |z|$ and $\arg z^* = -\arg z$, so you just write down the final answer without further working.

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Review Exercise
Exercise A, Question 5

Question:

$z_1 = -1 + i\sqrt{3}$, $z_2 = \sqrt{3} + i$

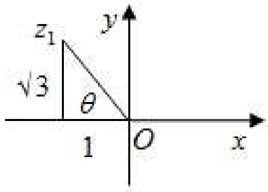
a Find **i** $\arg z_1$ **ii** $\arg z_2$.

b Express $\frac{z_1}{z_2}$ in the form $a + ib$, where a and b are real, and hence find $\arg\left(\frac{z_1}{z_2}\right)$.

c Verify that $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Solution:

a i

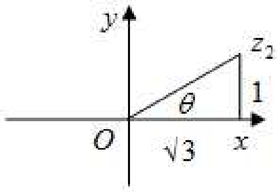


$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

z_1 is in the second quadrant

$\arg z_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

ii



$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

z_2 is in the second quadrant

$\arg z_2 = \frac{\pi}{6}$

b $\frac{z_1}{z_2} = \frac{-1+i\sqrt{3}}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{-\sqrt{3}+i+3i+\sqrt{3}}{4} = 0+i$

$(\sqrt{3}+i)(\sqrt{3}-i) = (\sqrt{3})^2 - i^2$
$= 3+1 = 4$

Although not strictly in the form $a+ib$, the answer i is acceptable.

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$

Any number on the positive imaginary axis has argument $\frac{\pi}{2}$

c $\arg z_1 - \arg z_2 = \frac{2\pi}{3} - \frac{\pi}{6}$, from part (a)

$= \frac{4\pi-\pi}{6} = \frac{3\pi}{6} = \frac{\pi}{2} = \arg\left(\frac{z_1}{z_2}\right)$

Hence the relation is satisfied by z_1 and z_2.

Verify means show that the equation is satisfied by the particular numbers in this question.
In part (a), you worked out $\arg z_1$ and $\arg z_2$.
In part (b), you worked out $\arg\left(\frac{z_1}{z_2}\right)$.
You substitute your answers into the equation in part (c) and check that it is correct.

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Review Exercise
Exercise A, Question 6

Question:

a Find the two square roots of $3 - 4\mathrm{i}$ in the form $a + \mathrm{i}b$, where a and b are real.

b Show the points representing the two square roots of $3 - 4\mathrm{i}$ in a single Argand diagram.

Solution:

a

$$z^2 = 3 - 4\mathrm{i}$$

The square root of, say, 2 is a root of the equation $z^2 = 2$. The square root of any number k, real or complex, is a root of $z^2 = k$.

Let $z = a + \mathrm{i}b$ where a and b are real.

$$(a + \mathrm{i}b)^2 = 3 - 4\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} - b^2 = 3 - 4\mathrm{i}$$

Equating real parts

$$a^2 - b^2 = 3 \quad ❶$$

Equating imaginary parts

$$2ab = -4 \quad ❷$$

Equating real and imaginary parts gives a pair of simultaneous equations one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A-level Modular Mathematics Core Mathematics 1, Chapter 3.

From ❷

$$b = -\frac{4}{2a} = -\frac{2}{a} \quad ❸$$

Substitute ❸ into ❶

$$a^2 - \left(-\frac{2}{a}\right)^2 = 3$$

$$a^2 - \frac{4}{a^2} = 3$$

$$a^4 - 3a^2 - 4 = 0$$

$$(a^2 - 4)(a^2 + 1) = 0$$

The only possible solutions of $a^2 + 1 = 0$ are complex, $a = \pm\mathrm{i}$, and as a is real you must ignore these and only consider the roots of $a^2 - 4 = 0$

$$a^2 = 4$$

$$a = 2, -2$$

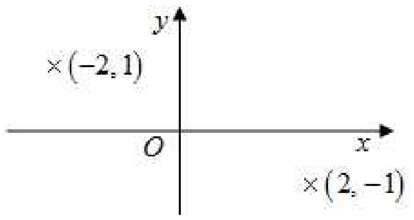
Substitute the values of a into ❸

$$a = 2 \Rightarrow b = -\frac{2}{2} = -1$$

$$a = -2 \Rightarrow b = -\frac{2}{-2} = 1$$

The square roots of $3 - 4\mathrm{i}$ are $2 - \mathrm{i}$ and $-2 + \mathrm{i}$.

b



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Review Exercise
Exercise A, Question 7

Question:

The complex number z is $-9+17\text{i}$.

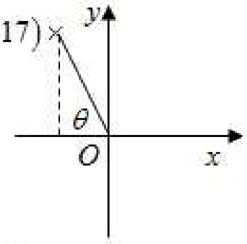
a Show z on an Argand diagram.

b Calculate arg z, giving your answer in radians to two decimal places.

c Find the complex number w for which $zw = 25+35\text{i}$, giving your answer in the form $p+\text{i}q$, where p and q are real.

Solution:

a



b $\tan\theta = \dfrac{17}{9} \Rightarrow \theta = 1.084\ \ldots$

z is in the second quadrant.

$\arg z = \pi - 1.084\ \ldots = 2.057\ \ldots$

$= 2.06$, in radians to 2 d.p.

You have to give your answer to 2 decimal places. To do this accurately you must work to at least 3 decimal places. This avoids rounding errors and errors due to premature approximation.

c $w = \dfrac{25+35\text{i}}{z} = \dfrac{25+35\text{i}}{-9+17\text{i}} = \dfrac{25+35\text{i}}{-9+17\text{i}} \times \dfrac{-9-17\text{i}}{-9-17\text{i}}$

$= \dfrac{-225-425\text{i}-315\text{i}+595}{(-9)^2+17^2}$

$= \dfrac{370-740\text{i}}{370} = 1-2\text{i}$

In this question, the arithmetic gets complicated. Use a calculator to help you with this. However, when you use a calculator, remember to show sufficient working to make your method clear.

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Review Exercise
Exercise A, Question 8

Question:

The complex numbers z and w satisfy the simultaneous equations

$2z + \mathrm{i}w = -1, \; z - w = 3 + 3\mathrm{i}.$

a Use algebra to find z, giving your answer in the form $a + \mathrm{i}b$, where a and b are real.

b Calculate arg z, giving your answer in radians to two decimal places.

Solution:

a

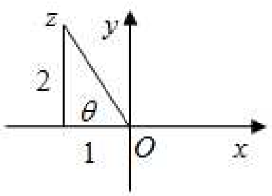
$2z + \mathrm{i}w = -1$ ❶
$z - w = 3 + 3\mathrm{i}$ ❷

❷ × i $\quad \mathrm{i}z - \mathrm{i}w = 3\mathrm{i} - 3$ ❸
❶ + ❸ $\quad (2+\mathrm{i})z = -4 + 3\mathrm{i}$

$$z = \frac{-4+3\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{-8+4\mathrm{i}+6\mathrm{i}+3}{5}$$
$$= \frac{-5+10\mathrm{i}}{5} = -1+2\mathrm{i}$$

You use the same method as you learnt for GCSE to solve simultaneous equations. To balance the coefficients of w, you multiply both sides of equation ❷ by i. Adding equations ❶ and ❸ then eliminates w.

b



$\tan\theta = \frac{2}{1} = 2 \Rightarrow \theta = 1.107\ldots$
z is in the second quadrant.
$\arg z = \pi - 1.107\ldots = 2.03$, to 2 d.p.

You must work to a least 3 decimal places to obtain an accurate answer to 2 decimal places.

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Review Exercise
Exercise A, Question 9

Question:

The complex number z satisfies the equation $\frac{z-2}{z+3\text{i}} = \lambda\text{i},\ \lambda \in \mathbb{R}$.

a Show that $z = \frac{(2-3\lambda)(1+\lambda\text{i})}{1+\lambda^2}$.

b In the case when $\lambda = 1$, find $|z|$ and arg z.

Solution:

a

$$z-2=\lambda\text{i}(z+3\text{i})$$
$$=\lambda\text{i}z-3\lambda$$

$\lambda\text{i}\times 3\text{i}=3\lambda\text{i}^2=-3\lambda$

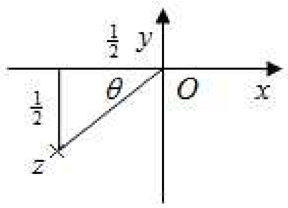
$$z(1-\lambda\text{i})=2-3\lambda$$
$$z=\frac{2-3\lambda}{1-\lambda\text{i}}\times\frac{1+\lambda\text{i}}{1+\lambda\text{i}}$$

You make z the subject of the formula and then multiply the numerator and denominator by $1+\lambda\text{i}$, which is the conjugate complex of $1-\lambda\text{i}$

$$=\frac{(2-3\lambda)(1+\lambda\text{i})}{1+\lambda^2}\text{, as required.}$$

b

$$\lambda=1 \Rightarrow z=\frac{(2-3)(1+\text{i})}{1+1}=-\frac{1}{2}-\frac{1}{2}\text{i}$$
$$|z|^2=\left(-\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$
$$|z|=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$



$$\tan\theta=\frac{\frac{1}{2}}{\frac{1}{2}}=1 \Rightarrow \theta=\frac{\pi}{4}$$

z is in the third quadrant.

$$\arg z=-\left(\pi-\frac{\pi}{4}\right)=-\frac{3\pi}{4}$$

The question does not specify radians and $\arg z=-135°$ would be an acceptable answer.

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Review Exercise
Exercise A, Question 10

Question:

The complex number z is given by $z = -2 + 2\text{i}$.

a Find the modulus and argument of z.

b Find the modulus and argument of $\frac{1}{z}$.

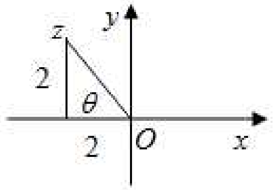
c Show on an Argand diagram the points A, B and C representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ respectively.

d State the value of $\angle ACB$.

Solution:

a $|z|^2 = (-2)^2 + 2^2 = 4 + 4 = 8$

$|z| = \sqrt{8} = 2\sqrt{2}$



$\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = \frac{\pi}{4}$

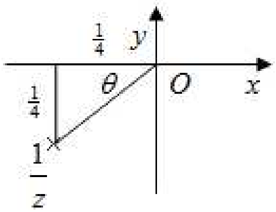
z is in the second quadrant.

$\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

b $\frac{1}{z} = \frac{1}{-2+2\mathrm{i}} \times \frac{-2-2\mathrm{i}}{-2-2\mathrm{i}} = \frac{-2-2\mathrm{i}}{8} = -\frac{1}{4} - \frac{1}{4}\mathrm{i}$

$\left|\frac{1}{z}\right|^2 = \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$

$\left|\frac{1}{z}\right| = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

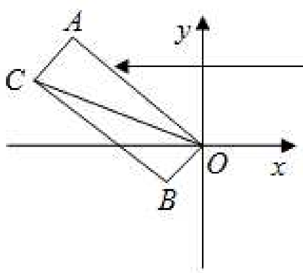


$\tan\theta = \frac{\frac{1}{4}}{\frac{1}{4}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

z is in the third quadrant.

$\arg z = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

c



The point C, representing $z + \frac{1}{z}$, must be a vertex of the parallelogram which has OA and OB as two of its sides.

In this case, as you have already shown that OA and OB make angles of $\frac{\pi}{4}(45°)$ with the negative x-axis, the parallelogram is a rectangle.

d $\angle ACB = 90°$

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Review Exercise
Exercise A, Question 11

Question:

The complex numbers z_1 and z_2 are given by $z_1 = \sqrt{3} + \mathrm{i}$ and $z_2 = 1 - \mathrm{i}$.

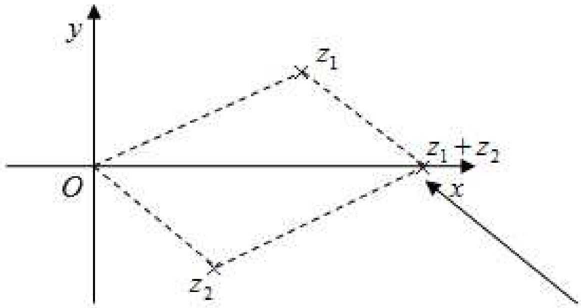
a Show, on an Argand diagram, points representing the complex numbers z_1, z_2 and $z_1 + z_2$.

b Express $\frac{1}{z_1}$ and $\frac{1}{z_2}$, each in the form $a + \mathrm{i}b$, where a and b are real numbers.

c Find the values of the real numbers A and B such that $\frac{A}{z_1} + \frac{B}{z_2} = z_1 + z_2$.

Solution:

a



The point representing z_1+z_2 must form a parallelogram with O and the points representing z_1 and z_2.
$z_1+z_2=\sqrt{3}+1$, which is real, so you must draw the point representing z_1+z_2 on the positive x-axis.

b

$$\frac{1}{z_1}=\frac{1}{\sqrt{3}+\mathrm{i}}\times\frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}-\mathrm{i}}=\frac{\sqrt{3}-\mathrm{i}}{(\sqrt{3})^2+1^2}$$

$$=\frac{\sqrt{3}-\mathrm{i}}{4}=\frac{\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}$$

$$\frac{1}{z_2}=\frac{1}{1-\mathrm{i}}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}=\frac{1+\mathrm{i}}{1^2+1^2}=\frac{1}{2}+\frac{1}{2}\mathrm{i}$$

You use your results in part (b) to simplify the working in part (c). Substitute the answers to part (b) into the printed equation in part (c)

c

$$\frac{A}{z_1}+\frac{B}{z_2}=z_1+z_2$$

$$A\left(\frac{\sqrt{3}}{4}-\frac{1}{4}\mathrm{i}\right)+B\left(\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)=\sqrt{3}+\mathrm{i}+1-\mathrm{i}=\sqrt{3}+1$$

You obtain a pair of simultaneous equations by equating the real and imaginary parts of this equation.

Equating real parts

$$\frac{\sqrt{3}}{4}A+\frac{1}{2}B=\sqrt{3}+1 \quad ❶$$

Equating imaginary parts

$$-\frac{1}{4}A+\frac{1}{2}B=0 \quad ❷$$

❶ − ❷

$$\frac{\sqrt{3}}{4}A+\frac{1}{4}A=\sqrt{3}+1$$

$$\left(\frac{\sqrt{3}+1}{4}\right)A=\sqrt{3}+1$$

$$A=4$$

Substitute in ❷

$$-1+\frac{1}{2}B=0 \Rightarrow B=2$$

$$A=4,\ B=2$$

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Review Exercise
Exercise A, Question 12

Question:

The complex numbers z and w are given by $z = \frac{A}{1 - i}$, $w = \frac{B}{1 - 3i}$, where A and B are real numbers. Given that $z + w = i$,

a find the value of A and the value of B.

b For these values of A and B, find $\tan[\arg(w - z)]$.

Solution:

a $z=\frac{A}{1-\text{i}}=\frac{A}{1-\text{i}}\times\frac{1+\text{i}}{1+\text{i}}=\frac{A}{2}(1+\text{i})$

$w=\frac{B}{1-3\text{i}}=\frac{B}{1-3\text{i}}\times\frac{1+3\text{i}}{1+3\text{i}}=\frac{B}{10}(1+3\text{i})$

The expressions for both z and w are fractions with complex denominators. You should remove these, by multiplying both the numerator and denominator by the conjugate complex of the denominator, before substituting into the equation.

$$z+w=\text{i}$$

$$\frac{A}{2}(1+\text{i})+\frac{B}{10}(1+3\text{i})=\text{i}$$

When equating the real and complex parts of both sides of the equation, think of the complex number i as $0+1\text{i}$.

Equating real parts

$$\frac{A}{2}+\frac{B}{10}=0 \quad ❶$$

Equating imaginary parts

$$\frac{A}{2}+\frac{3B}{10}=1 \quad ❷$$

❷ − ❶

$$\frac{2B}{10}=1 \Rightarrow B=5$$

Substitute into ❶

$$\frac{A}{2}+\frac{5}{10}=0 \Rightarrow \frac{A}{2}=-\frac{1}{2} \Rightarrow A=-1$$

$$A=-1, B=5$$

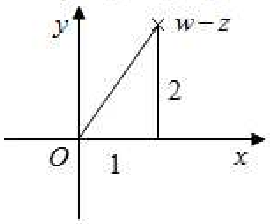
b With these values of A and B

$$z=\frac{-1}{2}(1+\text{i})=-\frac{1}{2}-\frac{1}{2}\text{i}$$

$$w=\frac{5}{10}(1+3\text{i})=\frac{1}{2}+\frac{3}{2}\text{i}$$

$$w-z=\frac{1}{2}+\frac{3}{2}\text{i}-\left(-\frac{1}{2}-\frac{1}{2}\text{i}\right)$$

$$=\frac{1}{2}+\frac{3}{2}\text{i}+\frac{1}{2}+\frac{1}{2}\text{i}=1+2\text{i}$$



$$\tan\left[\arg(w-z)\right]=\frac{2}{1}=2$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 13

Question:

a Given that $z = 2 - \text{i}$, show that $z^2 = 3 - 4\text{i}$.

b Hence, or otherwise, find the roots, z_1 and z_2, of the equation $(z + \text{i})^2 = 3 - 4\text{i}$.

c Show points representing z_1 and z_2 on a single Argand diagram.

d Deduce that $|z_1 - z_2| = 2\sqrt{5}$.

e Find the value of $\arg(z_1 + z_2)$.

Solution:

a $z^2 = (2-\mathrm{i})^2 = 4 - 4\mathrm{i} + \mathrm{i}^2$
$= 4 - 4\mathrm{i} - 1$
$= 3 - 4\mathrm{i}$, as required.

You square using the formula $(a-b)^2 = a^2 - 2ab + b^2$

b From part (a), the square roots of $3-4\mathrm{i}$ are $2-\mathrm{i}$ and $-2+\mathrm{i}$.
Taking square roots of both sides of the equation $(z+\mathrm{i})^2 = 3-4\mathrm{i}$

$z + \mathrm{i} = 2 - \mathrm{i} \Rightarrow z = 2 - 2\mathrm{i}$

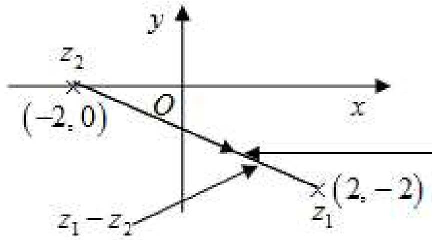
$z + \mathrm{i} = -2 + \mathrm{i} \Rightarrow z = -2$

The square root of any number k, real or complex, is a root of $z^2 = k$. Hence, part (a) shows that one square root of $3-4\mathrm{i}$ is $2-\mathrm{i}$.
If one square root of $3-4\mathrm{i}$ is $2-\mathrm{i}$, then the other is $-(2-\mathrm{i})$.

$z_1 = 2 - 2\mathrm{i}$, say, and $z_2 = -2$

z_1 and z_2 could be the other way round but that would make no difference to $|z_1 - z_2|$ or $z_1 + z_2$, the expressions you are asked about in parts (d) and (e).

c



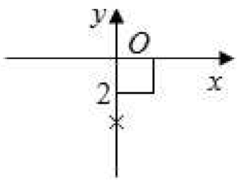
$z_1 - z_2$ can be represented on the diagram you drew in part (c) by the vector joining the point representing z_1 to the point representing z_2. The modulus of $z_1 - z_2$ is then just the length of the line joining these two points and this length can be found using coordinate geometry.

d Using the formula

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (2 - (-2))^2 + (-2 - 0)^2$$
$$= 4^2 + 2^2 = 20$$

Hence $|z_1 - z_2| = \sqrt{20} = 2\sqrt{5}$

(e) $z_1 + z_2 = 2 - 2\mathrm{i} - 2 = -2\mathrm{i}$



$\arg(z_1 + z_2) = -\frac{\pi}{2}$

The argument of any number on the negative imaginary axis is $-\frac{\pi}{2}$ or $-90°$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 14

Question:

a Find the roots of the equation $z^2 + 4z + 7 = 0$, giving your answers in the form $p + \mathrm{i}\sqrt{q}$, where p and q are integers.

b Show these roots on an Argand diagram.

c Find for each root,

i the modulus,

ii the argument, in radians, giving your answers to three significant figures.

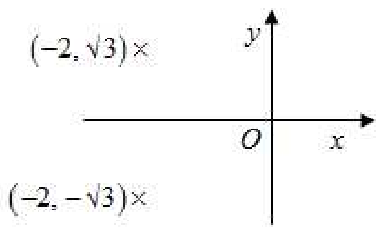
Solution:

a

$$z^2 + 4z = -7$$
$$z^2 + 4z + 4 = -7 + 4 = -3$$
$$(z+2)^2 = -3$$
$$z + 2 = \pm \mathrm{i}\sqrt{3}$$
$$z = -2 + \mathrm{i}\sqrt{3},\ -2 - \mathrm{i}\sqrt{3}$$

You may use any accurate method of solving a quadratic equation. Completing the square works well when the coefficient of z^2 is one and the coefficient of z is even.

b

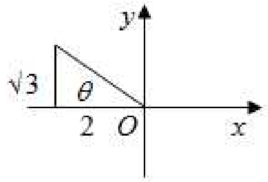


c i

$$|-2+\mathrm{i}\sqrt{3}|^2 = (-2)^2 + (\sqrt{3})^2 = 4 + 3 = 7$$
$$|-2+\mathrm{i}\sqrt{3}| = \sqrt{7}$$
$$|-2-\mathrm{i}\sqrt{3}| = \sqrt{7}$$

The moduli of conjugate complex numbers are the same so you do not have to repeat the working.

c ii



$$\tan\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 0.7137\ldots$$

$-2 + \mathrm{i}\sqrt{3}$ is in the second quadrant

$$\arg(-2+\mathrm{i}\sqrt{3}) = \pi - 0.7137\ldots$$
$$= 2.43, \text{ to 3 significant figures}$$
$$\arg(-2-\mathrm{i}\sqrt{3}) = -2.43, \text{ to 3 significant figures}$$

If z and z^* are conjugate complex numbers, then $\arg z^* = -\arg z$. Once you have worked out $\arg z$, you can just write down $\arg z^*$ without further working.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Review Exercise

Exercise A, Question 15

Question:

Given that $\lambda \varepsilon \mathbb{R}$ and that z and w are complex numbers, solve the simultaneous equations $z - \mathrm{i}w = 2$, $z - \lambda w = 1 - \lambda^2$, giving your answers in the form $a + \mathrm{i}b$, where a, $b \varepsilon \mathbb{R}$, and a and b are functions of λ.

Solution:

$z - \mathrm{i}w = 2$ .... ❶

$z - \lambda w = 1 - \lambda^2$ ... ❷

❶ − ❷

$-\mathrm{i}w + \lambda w = 2 - (1 - \lambda^2)$

$w(\lambda - \mathrm{i}) = \lambda^2 + 1$

$$w = \frac{\lambda^2 + 1}{\lambda - \mathrm{i}} \times \frac{\lambda + \mathrm{i}}{\lambda + \mathrm{i}}$$

$$= \frac{\cancel{(\lambda^2 + 1)}(\lambda + \mathrm{i})}{\cancel{(\lambda^2 + 1)}}$$

$$= \lambda + \mathrm{i}$$

Substitute in ❶

$z - \mathrm{i}(\lambda + \mathrm{i}) = 2$

$z = 2 + \mathrm{i}(\lambda + \mathrm{i}) = 2 + \mathrm{i}\lambda - 1 = 1 + \mathrm{i}\lambda$

You solve simultaneous linear equations with complex numbers in exactly the same way as you solved simultaneous equations with real numbers at GCSE. In this case, as the coefficients of z are already balanced, you subtract the equations as they stand to eliminate z.

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Review Exercise
Exercise A, Question 16

Question:

Given that $z_1 = 5 - 2\text{i}$,

a evaluate $|z_1|$, giving your answer as a surd,

b find, in radians to two decimal places, $\arg z_1$.

Given also that z_1 is a root of the equation $z^2 - 10z + c = 0$, where c is a real number,

c find the value of c.

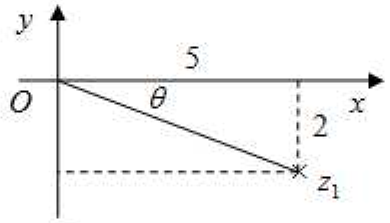
Solution:

a $|z_1|^2 = 5^2 + (-2)^2 = 25 + 4 = 29$

$|z_1| = \sqrt{29}$

If $z = a + \text{i}b$, then $|z|^2 = a^2 + b^2$

b



$\tan\theta = \dfrac{2}{5} \Rightarrow \theta = 0.3805\ \ldots$

is in the fourth quadrant

to 2 decimal places.

c If $z_1 = 5 - 2\text{i}$ is one root of a quadratic equation with real coefficients, then $z_2 = 5 + 2\text{i}$ must be the other root.

$$(z - z_1)(z - z_2) = (z - 5 + 2\text{i})(z - 5 - 2\text{i})$$
$$= (z - 5)^2 + 4$$
$$= z^2 - 10z + 25 + 4$$
$$= z^2 - 10z + 29 = 0$$

If α and β are the roots of a quadratic equation, then the equation must have the form $(z - \alpha)(z - \beta) = 0$.

Comparing this with the equation in the question

$c = 29$

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Review Exercise
Exercise A, Question 17

Question:

The complex numbers z and w are given by $z = \frac{5-10\mathrm{i}}{2-\mathrm{i}}$ and $w = \mathrm{i}z$.

a Obtain z and w in the form $p + \mathrm{i}q$, where p and q are real numbers.

b Show points representing z and w on a single Argand diagram

The origin O and the points representing z and w are the vertices of a triangle.

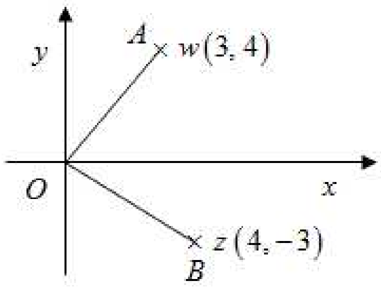
c Show that this triangle is isosceles and state the angle between the equal sides.

Solution:

a

$$z = \frac{5-10\mathrm{i}}{2-\mathrm{i}} \times \frac{2+\mathrm{i}}{2+\mathrm{i}}$$
$$= \frac{10+5\mathrm{i}-20\mathrm{i}+10}{2^2+1^2}$$
$$= \frac{20-15\mathrm{i}}{5} = 4-3\mathrm{i}$$
$$w = \mathrm{i}z = \mathrm{i}(4-3\mathrm{i}) = 4\mathrm{i}-3\mathrm{i}^2 = 3+4\mathrm{i}$$

b



c Let A be the point representing w and B be the point representing z.

$$|w|^2 = 3^2+4^2 = 25 \Rightarrow |w| = 5$$
$$|z|^2 = 4^2+(-3)^2 = 25 \Rightarrow |z| = 5$$

Hence $OA = OB = 5$ and the triangle OAB is isosceles.
The angle between the equal sides, $\angle AOB = 90°$.

As you are only asked to **state** the angle between the equal sides, you do not need to show working. If you cannot see this angle is a right angle or if working was asked for, you could argue:
the gradient of OA, $m = \frac{4}{3}$,
the gradient of OB, $m' = -\frac{3}{4}$,
$mm' = -1$, so the lines are perpendicular.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 18

Question:

$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}}$, $z_2 = \frac{\sqrt{2}}{1-\mathrm{i}}$

a Find the modulus and argument of each of the complex numbers z_1 and z_2.

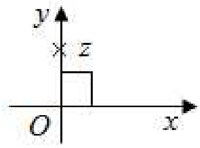
b Plot the points representing z_1, z_2 and $z_1 + z_2$ on a single Argand diagram.

c Deduce from your diagram that $\tan\left(\frac{3\pi}{8}\right) = 1 + \sqrt{2}$.

Solution:

a $z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}}$

$= \frac{1+2\mathrm{i}+\mathrm{i}^2}{1^2+1^2} = \frac{1+2\mathrm{i}-1}{2} = \frac{2\mathrm{i}}{2} = \mathrm{i}$



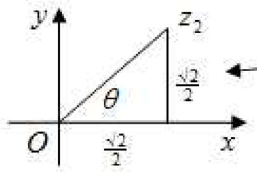
$|z_1| = 1, \ \arg z_1 = \frac{\pi}{2}$

The argument of any number on the positive imaginary axis is $\frac{\pi}{2}$ or $90°$.

$z_2 = \frac{\sqrt{2}}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{\sqrt{2}(1+\mathrm{i})}{2} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$

$|z_2|^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$

$|z_2| = 1$



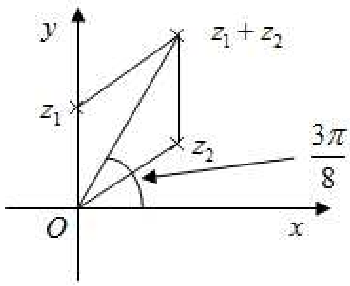
It is worth remembering that any complex number of the form $a + a\mathrm{i}$, where $a > 0$, has argument $\frac{\pi}{4}$. This working is then not necessary.

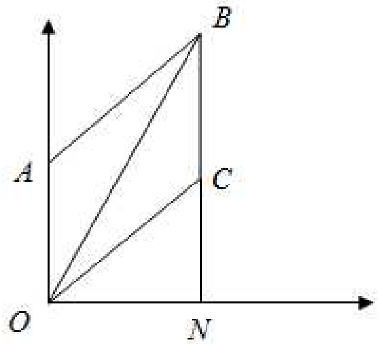
$\tan\theta = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$

z_2 is in the first quadrant

$\arg z_2 = \frac{\pi}{4}$

b





c $z_1 + z_2 = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2} + 1\right)\mathrm{i}$

$\arg(z_1 + z_2) = \frac{3\pi}{8}$

$\tan\left(\frac{3\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2} + 1}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}+2}{\sqrt{2}}$

$= \frac{\sqrt{2}}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 1 + \sqrt{2}$, as required

$\angle NOC = 45°$, the argument of z_2

$\angle COA = 90° - 45° = 45°$

$\angle COB = \frac{1}{2}\angle COA = 22\frac{1}{2}°$, the diagonal of a parallelogram bisects the angle

$\angle NOB = 45° + 22\frac{1}{2}° = 67\frac{1}{2}° = \frac{3\pi}{8}$, in radians

$\tan\left(\frac{3\pi}{8}\right) = \tan(\arg(z_1 + z_2)) = \frac{BN}{ON}$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 19

Question:

$z_1 = 1 + 2\text{i},\ z_2 = \frac{3}{5} + \frac{4}{5}\text{i}$

a Express in the form $p + q\text{i}$, where $p,\ q \in \mathbb{R}$,

i $z_1 z_2$

ii $\frac{z_1}{z_2}$.

In an Argand diagram, the origin O and the points representing $z_1 z_2$, $\frac{z_1}{z_2}$ and z_3 are the vertices of a rhombus.

b Sketch the rhombus on an Argand diagram.

c Find z_3.

d Show that $|z_3| = \frac{6\sqrt{5}}{5}$.

Solution:

a i $z_1 z_2 = (1+2\mathrm{i})\left(\frac{3}{5}+\frac{4}{5}\mathrm{i}\right)$

$$= \frac{3}{5}+\frac{4}{5}\mathrm{i}+\frac{6}{5}\mathrm{i}-\frac{8}{5} = -1+2\mathrm{i}$$

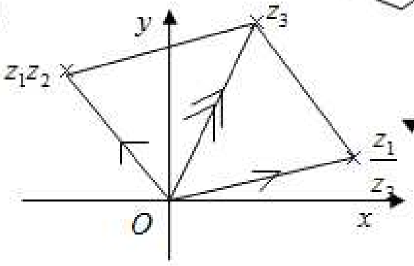
ii $\frac{z_1}{z_2} = \frac{1+2\mathrm{i}}{\frac{3}{5}+\frac{4}{5}\mathrm{i}} \times \frac{\frac{3}{5}-\frac{4}{5}\mathrm{i}}{\frac{3}{5}-\frac{4}{5}\mathrm{i}}$

$$= \frac{\frac{3}{5}-\frac{4}{5}\mathrm{i}+\frac{6}{5}\mathrm{i}+\frac{8}{5}}{1} = \frac{11}{5}+\frac{2}{5}\mathrm{i}$$

$\left(\frac{3}{5}+\frac{4}{5}\mathrm{i}\right)\left(\frac{3}{5}-\frac{4}{5}\mathrm{i}\right) = \left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = 1$

The relation between $\frac{3}{5}$, $\frac{4}{5}$ and 1 is the well-known 3, 4, 5 relation divided by 5 and, with practice, you can just write down answers like this.

b



On an Argand diagram the sum of two complex numbers can be represented by the diagonal completing the parallelogram, as shown in this diagram. (A rhombus is a special case of a parallelogram.)

c $z_3 = z_1 z_2 + \frac{z_1}{z_2} = -1+2\mathrm{i}+\frac{11}{5}+\frac{2}{5}\mathrm{i}$

$$= \frac{6}{5}+\frac{12}{5}\mathrm{i}$$

d $|z_3|^2 = \left(\frac{6}{5}\right)^2+\left(\frac{12}{5}\right)^2 = \frac{36+144}{25} = \frac{180}{25} = \frac{36\times 5}{25}$

Hence $|z_3| = \frac{6\sqrt{5}}{5}$, as required

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 20

Question:

$z_1 = -30 + 15\text{i}$.

a Find $\arg z_1$, giving your answer in radians to two decimal places.

The complex numbers z_2 and z_3 are given by $z_2 = -3 + p\text{i}$ and $z_3 = q + 3\text{i}$, where p and q are real constants and $p > q$.

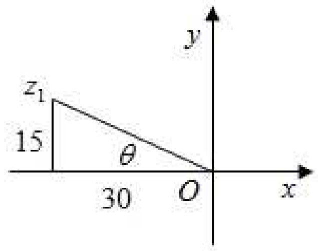
b Given that $z_2 z_3 = z_1$, find the value of p and the value of q.

c Using your values of p and q, plot the points corresponding to z_1, z_2 and z_3 on an Argand diagram.

d Verify that $2z_2 + z_3 - z_1$ is real and find its value.

Solution:

a



$\tan\theta = \frac{15}{30} = \frac{1}{2} \Rightarrow \theta \approx 0.464$

z_1 is in the second quadrant.

$\arg z_1 = \pi - \theta = 2.68$ to 2 d.p.

As you are asked to give your answer to 2 decimal places, you should work to at least 3 decimal places. This avoids rounding errors.

b

$$z_2 z_3 = z_1$$
$$(-3 + p\mathrm{i})(q + 3\mathrm{i}) = -30 + 15\mathrm{i}$$
$$-3q - 9\mathrm{i} + pq\mathrm{i} - 3p = -30 + 15\mathrm{i}$$

Equating real parts

$-3q - 3p = -30 \Rightarrow p + q = 10$ …… ❶

Equating imaginary parts

$-9 + pq = 15 \Rightarrow pq = 24$ … … … ❷

From ❷

$q = \frac{24}{p}$ … … … … ❸

Substitute ❸ into ❶

$p + \frac{24}{p} = 10$

$p^2 - 10p + 24 = (p-4)(p-6) = 0$

$p = 4, 6$

Substituting $p = 4$ into ❶ gives $q = 6$.

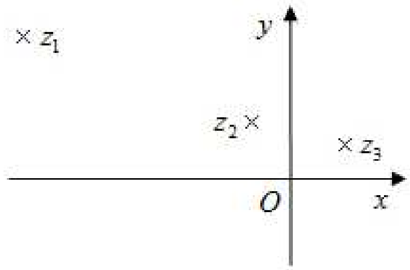
As $p > q$ is given, this solution is rejected.

Substituting $p = 6$ into ❶ gives $q = 4$.

$p = 6$, $q = 4$ is the only solution.

Equating real and imaginary parts gives a pair of simultaneous equations one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A-level Modular Mathematics Core Mathematics 1, Chapter 3.

c



d $2z_2 + z_3 - z_1 = 2(-3 + 6\mathrm{i}) + 4 + 3\mathrm{i} - (-30 + 15\mathrm{i})$

$= -6 + 12\mathrm{i} + 4 + 3\mathrm{i} + 30 - 15\mathrm{i} = 28$, a real number

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 21

Question:

Given that $z = 1 + \sqrt{3}\,\mathrm{i}$ and that $\frac{w}{z} = 2 + 2\mathrm{i}$, find

a w in the form $a + \mathrm{i}b$, where $a,\ b \in \mathbb{R}$,

b the argument of w,

c the exact value for the modulus of w.

On an Argand diagram, the point A represents z and the point B represents w.

d Draw the Argand diagram, showing the points A and B.

e Find the distance AB, giving your answer as a simplified surd.

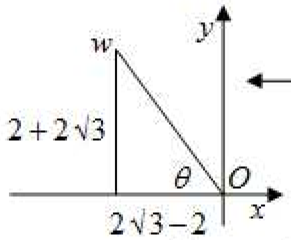
Solution:

a $w = (2+2\text{i})z = (2+2\text{i})(1+\sqrt{3}\text{i})$

$= 2 + 2\sqrt{3}\text{i} + 2\text{i} - 2\sqrt{3}$

$= (2-2\sqrt{3}) + (2+2\sqrt{3})\text{i}$

b



$2-2\sqrt{3} \approx -1.46$ so w is in the second quadrant.

The length of the side is $2\sqrt{3}-2$, not $2-2\sqrt{3}$ as lengths have to be positive.

$\tan\theta = \dfrac{2\sqrt{3}+2}{2\sqrt{3}-2} \Rightarrow \theta \approx 1.309$

w is in the second quadrant

$\arg w = \pi - \theta = 1.83$, to 3 significant figures

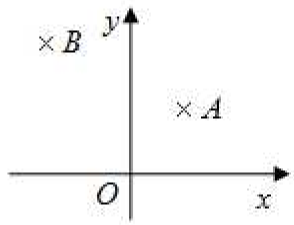
$\arg w$ is exactly $\dfrac{7\pi}{12}$. That would be an excellent answer to give, but an exact answer is not specified, so it is not essential. A calculator has been used here. Radians are not specified so degrees would also be acceptable. $\arg w = 105°$, exactly.

c $|w|^2 = (2-2\sqrt{3})^2 + (2+2\sqrt{3})^2$

$= 4 - 8\sqrt{3} + 12 + 4 + 8\sqrt{3} + 12 = 32 = 16 \times 2$

$|w| = 4\sqrt{2}$

d



A has coordinates $(1, \sqrt{3})$ and B has coordinates $(2-2\sqrt{3}, 2+2\sqrt{3})$. You use the formula $AB^2 = (x_2-x_1)^2 + (y_2-y_1)^2$ from Coordinate Geometry to calculate AB^2.

e $AB^2 = (2-2\sqrt{3}-1)^2 + (2+2\sqrt{3}-\sqrt{3})^2$

$= (1-2\sqrt{3})^2 + (2+\sqrt{3})^2$

$= 1 - 4\sqrt{3} + 12 + 4 + 4\sqrt{3} + 3$

$= 20 = 4 \times 5$

$AB = 2\sqrt{5}$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 22

Question:

The solutions of the equation $z^2 + 6z + 25 = 0$ are z_1 and z_2, where $0 < \arg z_1 < \pi$ and $-\pi < \arg z_2 < 0$.

a Express z_1 and z_2 in the form $a + \mathrm{i}b$, where a and b are integers.

b Show that $z_1^2 = -7 - 24\mathrm{i}$.

c Find $|z_1^2|$.

d Find $\arg\left(z_1^2\right)$.

e Show, on an Argand diagram, the points which represent the complex numbers $z_1,\ z_2$ and z_1^2.

Solution:

a $z^2+6z \quad =-25$

$z^2+6z+9=-25+9$

$(z+3)^2=-16$

$z=-3\pm 4\mathrm{i}$

$z_1=-3+4\mathrm{i}$, $z_2=-3-4\mathrm{i}$

$0<\arg z_1<\pi$ implies that z_1 is in the first or second quadrant. $-3+4\mathrm{i}$ is in the second quadrant, so this is z_1. The other solution is in the fourth quadrant, so that is z_2. You need to know which solution is z_1 before going on to parts (b), (c) and (d).

b $z_1^2=(-3+4\mathrm{i})^2=9-24\mathrm{i}-16$

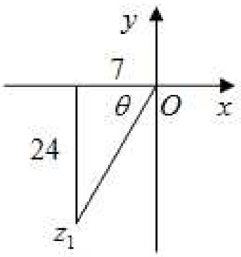
$\quad =-7-24\mathrm{i}$, as required

c $\left|z_1^2\right|^2=(-7)^2+(-24)^2=625$

$\left|z_1^2\right|=\sqrt{625}=25$

If you recognise 7, 24, 25 as a set of numbers satisfying the Pythagoras relation $a^2=b^2+c^2$, you can just write this answer down.

d



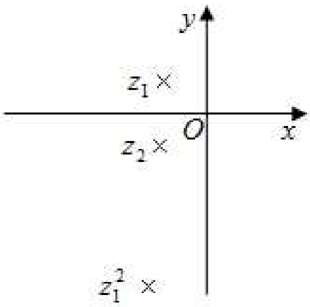
$\tan\theta=\frac{24}{7} \Rightarrow \theta\approx 1.287$

z_1 is in the fourth quadrant

$\arg z_1=-(\pi-\theta)=-1.85$, to 3 significant figures

The inequalities $0<\arg z_1<\pi$ and $-\pi<\arg z_2<0$ show that, in this question, the arguments are in radians.
Where no accuracy is specified in the question, it is reasonable to give your answer to 3 significant figures.

e



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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 23

Question:

$z = \sqrt{3} - \mathrm{i}$. z^* is the complex conjugate of z.

a Show that $\frac{z}{z^*} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$.

b Find the value of $|\frac{z}{z^*}|$.

c Verify, for $z = \sqrt{3} - \mathrm{i}$, that $\arg \frac{z}{z^*} = \arg z - \arg z^*$.

d Display z, z^* and $\frac{z}{z^*}$ on a single Argand diagram.

e Find a quadratic equation with roots z and z^* in the form $ax^2 + bx + c = 0$, where a, b and c are real constants to be found.

Solution:

a $z^* = \sqrt{3} + \mathrm{i}$

$$\frac{z}{z^*} = \frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}+\mathrm{i}} \times \frac{\sqrt{3}-\mathrm{i}}{\sqrt{3}-\mathrm{i}} = \frac{(\sqrt{3}-\mathrm{i})^2}{(\sqrt{3})^2+1}$$

$$= \frac{(\sqrt{3})^2 - 2\sqrt{3}\mathrm{i} + \mathrm{i}^2}{3+1} = \frac{3-2\sqrt{3}\mathrm{i}-1}{4}$$

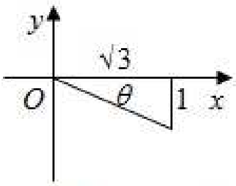
$$= \frac{2-2\sqrt{3}\mathrm{i}}{4} = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}, \text{ as required}$$

You multiply the numerator and the denominator by the conjugate complex of the denominator. The conjugate complex of $\sqrt{3}+\mathrm{i}$ is $\sqrt{3}-\mathrm{i}$, so the numerator becomes $(\sqrt{3}-\mathrm{i})^2$, which you can square using the formula $(a-b)^2 = a^2 - 2ab + b^2$.

b $$\left|\frac{z}{z^*}\right|^2 = \left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\left|\frac{z}{z^*}\right| = 1$$

c



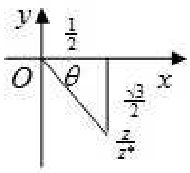
$$\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

z is in the fourth quadrant

$$\arg z = -\frac{\pi}{6}$$

$$\arg z^* = \frac{\pi}{6}$$

You use $\arg z^* = -\arg z$ and $-\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}$.



$$\tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$\frac{z}{z^*}$ is in the fourth quadrant

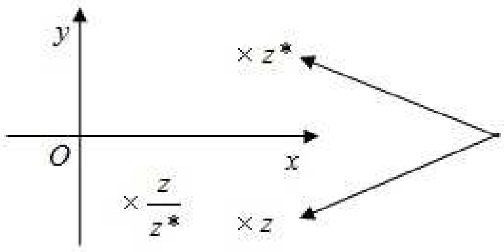
$$\arg\frac{z}{z^*} = -\frac{\pi}{3}$$

$$\arg z - \arg z^* = -\frac{\pi}{6} - \frac{\pi}{6} = -\frac{\pi}{3}$$

$$= \arg\frac{z}{z^*}, \text{ verifying the result}$$

Verify means show that the equation is satisfied by the particular numbers in this question.
You should show that the equation is satisfied exactly and not use a calculator giving approximate results.

d



In the Argand diagram, you must place points representing conjugate complex numbers symmetrically about the real x-axis.

e

$$(x-z_1)(x-z_2)=(x-\sqrt{3}+\text{i})(x-\sqrt{3}+\text{i})$$
$$=(x-\sqrt{3})^2+1$$
$$=x^2-2\sqrt{3}x+3+1$$
$$=x^2-2\sqrt{3}x+4$$

The equation is $x^2-2\sqrt{3}x+4=0$.

If $x=\alpha$ and $x=\beta$ are the solutions of a quadratic equation, then the equation, after it has been factorised, must be $(x-\alpha)(x-\beta)=0$

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Review Exercise
Exercise A, Question 24

Question:

$z = \frac{1+7i}{4+3i}$.

a Find the modulus and argument of z.

b Write down the modulus and argument of z^*.

In an Argand diagram, the points A and B represent $1+7i$ and $4+3i$ respectively and O is the origin. The quadrilateral $OABC$ is a parallelogram.

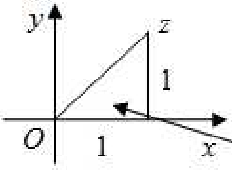
c Find the complex number represented by the point C.

d Calculate the area of the parallelogram.

Solution:

a $z = \dfrac{1+7\mathrm{i}}{4+3\mathrm{i}} \times \dfrac{4-3\mathrm{i}}{4-3\mathrm{i}} = \dfrac{4-3\mathrm{i}+28\mathrm{i}+21}{4^2+3^2}$

$= \dfrac{25+25\mathrm{i}}{25} = 1+\mathrm{i}$



$|z| = \sqrt{(1^2+1^2)} = \sqrt{2}$

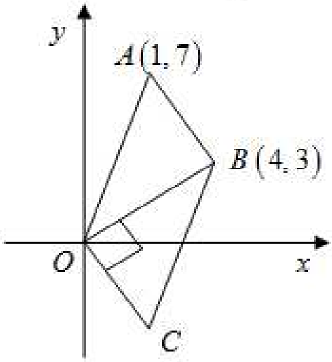
$\arg z = \dfrac{\pi}{4}$

You can see from the diagram that the argument is $45° = \dfrac{\pi}{4}$ and you need give no further working.

b $|z^*| = \sqrt{2}$, $\arg z^* = -\dfrac{\pi}{4}$

z^* is the symbol for the conjugate complex of z and you use the relations $|z^*| = |z|$ and $\arg z^* = -\arg z$ to write down the answers.

c



You are not asked to draw an Argand diagram in this question but you will certainly need to sketch one to sort out parts (c) and (d).

Let the complex number represented by the point C be w.
$OABC$ is a parallelogram. Therefore
$\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB}$
$1+7\mathrm{i}+w = 4+3\mathrm{i}$
$w = 3-4\mathrm{i}$

You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OB of the parallelogram represents the addition of the two adjacent sides, OA and OC, of the parallelogram.

d $OB^2 = 4^2+3^2 = 25 \Rightarrow OB = 5$

$OC^2 = (-3)^2 + 4^2 = 25 \Rightarrow OC = 5$

The gradient of OB is given by $m = \dfrac{3}{4}$

The gradient of OC is given by $m' = -\dfrac{4}{3}$

$mm' = -1$ and, hence, OB is perpendicular to OC.
The area of the right-angled triangle OBC is given by

$\text{area} = \dfrac{1}{2}\,\text{base} \times \text{height} = \dfrac{1}{2} \times 5 \times 5 = 12\dfrac{1}{2}$

The area of the parallelogram is $2 \times 12\dfrac{1}{2} = 25$.

The diagonal of the parallelogram divides the parallelogram into two congruent triangles.

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Review Exercise
Exercise A, Question 25

Question:

Given that $\frac{z+2\text{i}}{z-\lambda\text{i}}=\text{i}$, where λ is a positive, real constant,

a show that $z=\left(\frac{\lambda}{2}+1\right)+\text{i}\left(\frac{\lambda}{2}-1\right)$.

Given also that $\tan(\arg z)=\frac{1}{2}$, calculate

b the value of λ,

c the value of $|z|^2$.

Solution:

a

$$\frac{z+2\text{i}}{z-\lambda\text{i}}=\text{i}$$

$$z+2\text{i}=\text{i}(z-\lambda\text{i})=\text{i}z+\lambda$$

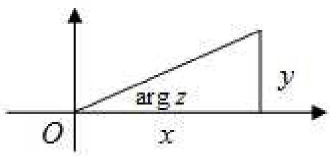
$$z(1-\text{i})=\lambda-2\text{i}$$

You start this question by "making z the subject of the formula"; a method you learnt for GCSE.

$$z=\frac{\lambda-2\text{i}}{1-\text{i}}\times\frac{1+\text{i}}{1+\text{i}}=\frac{\lambda+\lambda\text{i}-2\text{i}+2}{2}$$

$$=\left(\frac{\lambda+2}{2}\right)+\text{i}\left(\frac{\lambda-2}{2}\right)$$

$$=\left(\frac{\lambda}{2}+1\right)+\text{i}\left(\frac{\lambda}{2}-1\right)\text{, as required}$$



If $z=x+\text{i}y$, then $\tan(\arg z)=\frac{y}{x}$

b

$$\tan(\arg z)=\frac{\frac{\lambda}{2}-1}{\frac{\lambda}{2}+1}=\frac{\lambda-2}{\lambda+2}=\frac{1}{2}$$

$$2\lambda-4=\lambda+2\quad\Rightarrow\lambda=6$$

Multiplying all terms in both the numerator and denominator by 2.

c Substitute $\lambda=6$ into the result of part (a).

$$z=\left(\frac{6}{2}+1\right)+\text{i}\left(\frac{6}{2}-1\right)=4+2\text{i}$$

$$|z|^2=4^2+2^2=20$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 26

Question:

The complex numbers $z_1 = 2 + 2\text{i}$ and $z_2 = 1 + 3\text{i}$ are represented on an Argand diagram by the points P and Q respectively.

a Display z_1 and z_2 on the same Argand diagram.

b Find the exact values of $|z_1|$, $|z_2|$ and the length of PQ.

Hence show that

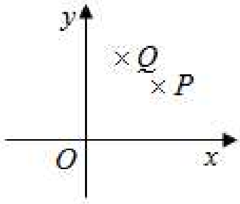
c ΔOPQ, where O is the origin, is right-angled.

Given that $OPQR$ is a rectangle in the Argand diagram,

d find the complex number z_3 represented by the point R.

Solution:

a



b $|z_1|^2 = 2^2 + 2^2 = 8 = 4 \times 2 \Rightarrow |z_1| = 2\sqrt{2}$

$|z_2|^2 = 1^2 + 3^2 = 10 \Rightarrow |z_2| = \sqrt{10}$

P has coordinates $(2, 2)$ and Q $(1, 3)$

$PQ^2 = (1-2)^2 + (3-2)^2 = (-1)^2 + 1^2 = 2$

$PQ = \sqrt{2}$

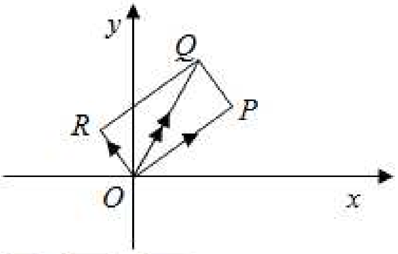
You use the formula $PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ from Coordinate Geometry to calculate PQ^2.

c From (b), $OP = 2\sqrt{2}$ and $OQ = \sqrt{10}$.

$$OP^2 + PQ^2 = (2\sqrt{2})^2 + (\sqrt{2})^2$$
$$= 8 + 2 = 10$$
$$= OQ^2$$

By the converse of Pythagoras' Theorem, $\triangle OPQ$ is right-angled.

d



$\overline{OP} + \overline{OR} = \overline{OQ}$

$2 + 2\text{i} + z_3 = 1 + 3\text{i}$

$z_3 = -1 + \text{i}$

You use the representation of the addition of complex numbers in an Argand diagram. The diagonal OQ of the parallelogram represents the addition of the two adjacent sides, OP and OR, of the parallelogram. (A rectangle is a special case of a parallelogram.)

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Review Exercise
Exercise A, Question 27

Question:

The complex number z is given by $z = (1 + 3\text{i})(p + q\text{i})$, where p and q are real and $p > 0$.

Given that $\arg z = \frac{\pi}{4}$,

a show that $p + 2q = 0$.

Given also that $|z| = 10\sqrt{2}$,

b find the value of p and the value of q.

c Write down the value of $\arg z^*$.

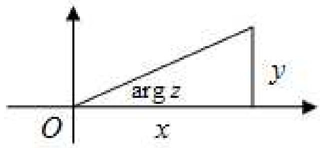
Solution:

a $z = (1+3\text{i})(p+q\text{i})$
$= p + q\text{i} + 3p\text{i} - 3q$
$= (p-3q) + (3p+q)\text{i}$

$\arg z = \dfrac{\pi}{4}$, given

$\tan(\arg z) = \tan\dfrac{\pi}{4}$

$\dfrac{3p+q}{p-3q} = 1$



If $z = x + \text{i}\,y$, then $\tan(\arg z) = \dfrac{y}{x}$.

$3p + q = p - 3q \Rightarrow 2p + 4q = 0$
$\Rightarrow p + 2q = 0$, as required

b $|z|^2 = (p-3q)^2 + (3p+q)^2 = (10\sqrt{2})^2$
$p^2 - 6pq + 9q^2 + 9p^2 + 6pq + q^2 = 200$
$10p^2 + 10q^2 = 200$
$p^2 + q^2 = 20$ … … ❶
From the result of part (a)
$p = -2q$ … … … ❷
Substitute ❷ into ❶
$4q^2 + q^2 = 20 \Rightarrow 5q^2 = 20 \Rightarrow q^2 = 4$
$q = \pm 2$
$q = 2$ substituted into ❷ gives $p = -4$. As $p > 0$ is given in the question, this solution is rejected and $q = -2$ is the only answer.
$p = 4$, $q = -2$

❶ and ❷ are a pair of simultaneous equations, one of which is quadratic and the other linear. The method of solving these is given in Edexcel AS and A Level Modular Mathematics Core Mathematics 1, Chapter 3.

c $\arg z^* = -\dfrac{\pi}{4}$

You use $\arg z^* = -\arg z$ to write down this answer. You were given that $\arg z = \dfrac{\pi}{4}$.

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 28

Question:

The complex numbers z_1 and z_2 are given by $z_1 = 5 + \text{i}$, $z_2 = 2 - 3\text{i}$.

a Show points representing z_1 and z_2 on an Argand diagram.

b Find the modulus of $z_1 - z_2$.

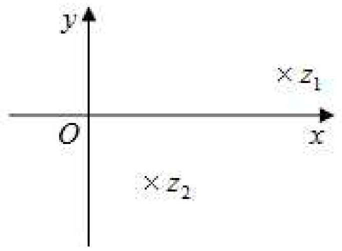
c Find the complex number $\frac{z_1}{z_2}$ in the form $a + \text{i}b$, where a and b are rational numbers.

d Hence find the argument of $\frac{z_1}{z_2}$, giving your answer in radians to three significant figures.

e Determine the values of the real constants p and q such that $\frac{p + \text{i}q + 3z_1}{p - \text{i}q + 3z_2} = 2\text{i}$.

Solution:

a



b $z_1 - z_2 = 5 + \mathrm{i} - (2 - 3\mathrm{i}) = 3 + 4\mathrm{i}$

$|z_1 - z_2|^2 = 3^2 + 4^2 = 25$

$|z_1 - z_2| = 5$

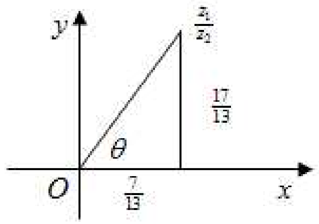
If you recognise the 3,4,5 "triangle", you can write the answer 5 down without further working.

c $\dfrac{z_1}{z_2} = \dfrac{5+\mathrm{i}}{2-3\mathrm{i}} \times \dfrac{2+3\mathrm{i}}{2+3\mathrm{i}} = \dfrac{10+15\mathrm{i}+2\mathrm{i}-3}{2^2+3^2}$

$= \dfrac{7+17\mathrm{i}}{13} = \dfrac{7}{13} + \dfrac{17}{13}\mathrm{i}$

The question asks you to put your answer in the form $a + \mathrm{i}b$, where a and b are rational numbers. Rational numbers are exact fractions and so $\frac{7}{13}$ and $\frac{17}{13}$ satisfy the conditions of the question. Approximate decimals would not be acceptable

d



$\tan\theta = \dfrac{\frac{17}{13}}{\frac{7}{13}} = \dfrac{17}{7} \Rightarrow \theta \approx 1.180$

$\dfrac{z_1}{z_2}$ is in the first quadrant

$\arg \dfrac{z_1}{z_2} = 1.18$, to 3 significant figures

e $p + \mathrm{i}q + 3z_1 = 2\mathrm{i}(p - \mathrm{i}q + 3z_2)$

$p + \mathrm{i}q + 15 + 3\mathrm{i} = 2p\mathrm{i} + 2q + 6\mathrm{i}(2 - 3\mathrm{i})$

$= 2p\mathrm{i} + 2q + 12\mathrm{i} + 18$

Equating real parts

$p + 15 = 2q + 18 \Rightarrow p - 2q = 3 \ldots$ ❶

Equating imaginary parts

$q + 3 = 2p + 12 \Rightarrow -2p + q = 9 \ldots$ ❷

You find two simultaneous equations by equating the real and imaginary parts of the equation.

❶ $\times 2$ $\quad 2p - 4q = 6 \ldots$ ❸

❷ + ❸ $\quad -3q = 15 \Rightarrow q = -5$

Substitute into ❶

$p + 10 = 3 \Rightarrow p = -7$

$p = -7, q = -5$

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Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 29

Question:

$z = a + \mathrm{i}b$, where a and b are real and non-zero.

a Find z^2 and $\frac{1}{z}$ in terms of a and b, giving each answer in the form $x + \mathrm{i}y$, where x and y are real.

b Show that $|z^2| = a^2 + b^2$.

c Find $\tan(\arg z^2)$ and $\tan\left(\arg \frac{1}{z}\right)$, in terms of a and b.

On an Argand diagram the point P represents z^2 and the point Q represents $\frac{1}{z}$ and O the origin.

d Using your answer to **c**, or otherwise, show that if P, O and Q are collinear, then $3a^2 = b^2$.

Solution:

a $z^2 = (a + \mathrm{i}b)^2 = a^2 + 2ab\mathrm{i} - b^2$

$$= (a^2 - b^2) + 2ab\mathrm{i}$$

$$\frac{1}{z} = \frac{1}{a+\mathrm{i}b} \times \frac{a-\mathrm{i}b}{a-\mathrm{i}b} = \frac{a-\mathrm{i}b}{a^2+b^2}$$

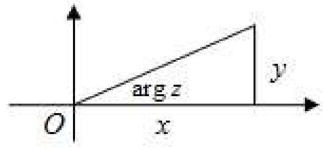
$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\mathrm{i}$$

b $|z^2|^2 = (a^2 - b^2)^2 + (2ab)^2$

$$= a^4 - 2a^2b^2 + b^4 + 4a^2b^2$$

$$= a^4 + 2a^2b^2 + b^4 = (a^2 + b^2)^2$$

Hence $|z^2| = a^2 + b^2$, as required.



c $\tan(\arg z^2) = \dfrac{2ab}{a^2 - b^2}$

$$\tan\left(\arg \frac{1}{z}\right) = \frac{-\dfrac{b}{a^2+b^2}}{\dfrac{a}{a^2+b^2}} = -\frac{b}{a}$$

If $z = x + \mathrm{i}\,y$, then $\tan(\arg z) = \dfrac{y}{x}$. You the use the answers in part (a).

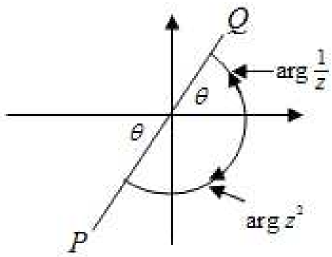
d If P, O and Q are in a straight line then $\tan(\arg z^2)$ and $\tan\left(\arg \dfrac{1}{z}\right)$ must be equal.

$$\frac{2ab}{a^2 - b^2} = -\frac{b}{a}$$

$$2a^2\cancel{b} = -\cancel{b}(a^2 - b^2)$$

$$2a^2 = -a^2 + b^2$$

$$3a^2 = b^2, \text{ as required}$$



If P and Q are in the same quadrant, this is obvious, but when they are in opposite quadrants this is not so clear. A possible case is shown above.

$$\tan(\arg z^2) = \tan(-(\pi - \theta)) = \tan(\theta - \pi)$$

$$= \tan\theta = \tan\left(\arg \frac{1}{z}\right)$$

$\tan(\theta - \pi) = \tan\theta$ because the function tan has period π. (This is in the C2 specification) You would not be expected to explain this in an examination.

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Review Exercise
Exercise A, Question 30

Question:

Starting with $x = 1.5$, apply the Newton–Raphson procedure once to $f(x) = x^3 - 3$ to obtain a better approximation to the cube root of 3, giving your answer to three decimal places.

Solution:

$$f(x) = x^3 - 3$$

The cube root of x is the solution of $x^3 - 3 = 0$. 1.5 is the first approximation and you have to find a second approximation using the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

$$f'(x) = 3x^2$$

$$f(1.5) = 1.5^3 - 3 = 0.375$$

$$f'(1.5) = 3 \times 1.5^2 = 6.75$$

$$x = 1.5 - \frac{f(1.5)}{f'(1.5)}$$

It is a good idea to write down the Newton-Raphson formula and adapt it to your solution. The examiner will then know what you are doing.

$$= 1.5 - \frac{0.375}{6.75}$$

$= 1.444$, to 3 decimal places

Correct the answer to the number of decimal places you are asked for.

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Review Exercise
Exercise A, Question 31

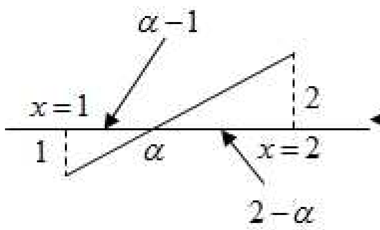
Question:

$f(x) = 2^x + x - 4$. The equation $f(x) = 0$ has a root α in the interval [1, 2]. Use linear interpolation on the values at the end points of this interval to find an approximation to α.

Solution:

$$f(x) = 2^x + x - 4$$
$$f(1) = 2^1 + 1 - 4 = -1$$
$$f(2) = 2^2 + 2 - 4 = 2$$

The first stage of a linear interpolation is to evaluate the function at both ends of the interval.



A diagram helps you to see what is going on and, as you are going to use similar triangles, to see which sides in one triangle correspond to which sides in the other triangle.

By similar triangles

$$\frac{\alpha - 1}{1} = \frac{2 - \alpha}{2}$$

Solve the equation to find α.

$$2\alpha - 2 = 2 - \alpha$$
$$3\alpha = 4$$
$$\alpha = 1\frac{1}{3}$$

This is an exact answer. There is no need to correct to a given number of decimal places as you have not been asked to do this.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 32

Question:

Given that the equation $x^3 - x - 1 = 0$ has a root near 1.3, apply the Newton–Raphson procedure once to $f(x) = x^3 - x - 1$ to obtain a better approximation to this root, giving your answer to three decimal places.

Solution:

$$\text{Let } f(x) = x^3 - x - 1$$
$$f'(x) = 3x^2 - 1$$

$$f(1.3) = -0.103$$
$$f'(1.3) = 4.07$$

$$x = 1.3 - \frac{f(1.3)}{f'(1.3)}$$

$$= 1.3 + \frac{0.103}{4.07}$$
$$= 1.325307\ \ldots$$
$$\approx 1.325$$

Remember to correct your answer to the number of decimal places asked for in the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 33

Question:

$f(x) = x^3 - 12x + 7.$

a Use differentiation to find $f'(x)$.

The equation $f(x) = 0$ has a root α in the interval $\frac{1}{2} < x < 1$.

b Taking $x = \frac{1}{2}$ as a first approximation to α, use the Newton–Raphson procedure twice to obtain two further approximations to α. Give your final answer to four decimal places.

Solution:

a $f'(x) = 3x^2 - 12$

b $x_1 = 0.5$

$f(0.5) = 0.5^3 - 12 \times 0.5 + 7 = 1.125$

$f'(0.5) = 3 \times 0.5^2 - 12 = -11.25$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{1.125}{-11.25} = 0.5 + 0.1$$

$$= 0.6$$

$0.5 \left(\text{or } \frac{1}{2} \right)$ is the first approximation, which you are given. You have to find two more approximations. It is useful to call the first approximation x_1, the second x_2, the third x_3, etc. This helps you keep track of where you are in a long calculation.

The signs need care here. Missing that "minus a minus is a plus" is a major source of error, even at A level!

$f(0.6) = 0.6^3 - 12 \times 0.6 + 7 = 0.016$

$f'(0.6) = 3 \times 0.6^2 - 12 = -10.92$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6 - \frac{0.016}{-10.92} = 0.6 + 0.001465 \ \ldots$$

$= 0.6015$, to 4 decimal places

Remember to correct your answer to the number of decimal places asked for in the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 34

Question:

The equation $\sin x = \frac{1}{2}x$ has a root in the interval [1.8, 2]. Use linear interpolation once on the interval [1.8, 2] to find an estimate of the root, giving your answer to two decimal places.

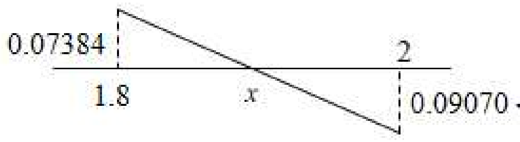
Solution:

$$f(x) = \sin x - \frac{1}{2}x$$

In this question you have not been given $f(x)$ and have to choose a function yourself. To choose $f(x) = \sin x - \frac{1}{2}x$ is sensible as $f(x) = 0$ is obviously equivalent to the equation $\sin x = \frac{1}{2}x$, which you are asked to solve.

$$f(1.8) = 0.07384 \ldots$$
$$f(2) = -0.09070 \ldots$$

Remember to work in radians. The final answer has to be given to 2 decimal places and you must work to sufficient accuracy to achieve this. 5 decimal places will certainly be enough to achieve this but there is no harm in giving more decimal places.



It is a common error to use -0.09070 instead of 0.09070 here. $f(2)$ is negative but the number is the length of a side in the diagram and lengths have to be positive.

By similar triangles

$$\frac{x-1.8}{0.07384} = \frac{2-x}{0.09070}$$

$$0.09070x - 0.16326 = 0.14768 - 0.07384x$$

$$0.16454x = 0.31094$$

The numbers are quite difficult here. Use your calculator to help you "cross multiply" and collect together correctly.

$$x = \frac{0.31094}{0.16454} \approx 1.89, \text{ to 2 decimal places.}$$

Your answer must be corrected to 2 decimal places – the accuracy specified in the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 35

Question:

$f(x) = x^4 + 3x^3 - 4x - 5$. The equation $f(x) = 0$ has a root between $x = 1.2$ and $x = 1.6$. Starting with the interval [1.2, 1.6], use interval bisection three times to obtain an interval of width 0.05 which contains this root.

Solution:

The mid-point of the interval $[1.2, 1.6]$ is

$$\frac{1.2+1.6}{2} = 1.4$$

You start interval bisection by dividing the interval into two equal parts by finding the mid-point of an interval.

$$f(1.2) = -2.5424 < 0$$

$$f(1.4) = 1.4736 > 0$$

$$(f(1.6) = 7.4416)$$

It is not always necessary to calculate the values at both ends and the mid-point. In this case you already have a sign change between $x = 1.2$ and $x = 1.4$ and, so it is not necessary to calculate the value of $f(1.6)$.

There is a sign change between $x = 1.2$ and $x = 1.6$.
Hence, the root lies in the interval $(1.2, 1.4)$.

The mid-point of the interval $[1.2, 1.4]$ is

$$\frac{1.2+1.4}{2} = 1.3$$

$$f(1.3) = -0.7529 < 0$$

$f(1.4) = 1.4736 > 0$, from above.

You calculated $f(1.4)$ earlier and there is no need to calculate it again.

There is a sign change between $x = 1.3$ and $x = 1.4$.
Hence, the root lies in the interval $(1.3, 1.4)$.

The mid-point of the interval $[1.3, 1.4]$ is

$$\frac{1.3+1.4}{2} = 1.35$$

$f(1.35) = 0.30263 > 0$, from above.

$$f(1.3) = -0.7529 < 0$$

There is a sign change between $x = 1.3$ and $x = 1.35$.
Hence, the root lies in the interval $(1.3, 1.35)$.

$1.35 - 1.3 = 0.05$ and so this interval satisfies the requirements of the question.

Quartic equations can be solved exactly. You may have access to a computer package or advanced calculator which can do this. $x = 1.336\ 20$ is accurate to 5 decimal places, which confirms the result of your calculation.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 36

Question:

$f(x) = 3\tan\left(\frac{x}{2}\right) - x - 1,\ -\pi < x < \pi.$

Given that $f(x) = 0$ has a root between 1 and 2, use linear interpolation once on the interval [1, 2] to find an approximation to this root. Give your answer to two decimal places.

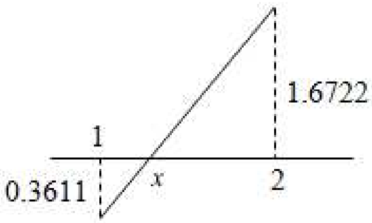
Solution:

$$f(x) = 3\tan\left(\frac{x}{2}\right) - x - 1$$

Unless it is clearly stated otherwise, all questions on this topic require you to work in radians. Make sure your calculator is in the correct mode.

$$f(1) = -0.3611$$
$$f(2) = 1.6722$$

The final answer must be to 2 decimal places. To achieve this you must work to at least one more decimal place and it's safer to work to more. 4 decimal places will certainly be enough here.



By similar triangles

$$\frac{x-1}{0.3611} = \frac{2-x}{1.6722}$$

$$1.6722x - 1.6722 = 0.7222 - 0.3611x$$

$$2.0333x = 2.39442$$

$$x = \frac{2.39442}{2.0333} \approx 1.1776$$

Correct your approximation to x to 2 decimal places.

$x \approx 1.18$, to 2 decimal places

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 37

Question:

$f(x) = 3^x - x - 6.$

a Show that $f(x) = 0$ has a root α between $x = 1$ and $x = 2$.

b Starting with the interval [1, 2], use interval bisection three times to find an interval of width 0.125 which contains α.

Solution:

a

$$f(x) = 3^x - x - 6$$
$$f(1) = 3 - 1 - 6 = -4 < 0$$
$$f(2) = 9 - 2 - 6 = 1 > 0$$

There is a sign change between $x = 1$ and $x = 2$.
Hence the function $f(x)$ has a root α between $x = 1$ and $x = 2$.

When you are asked to "show that", or "prove that" a result is true, you should give a conclusion to your argument. It is always safe to base the wording of your conclusion on the wording of the question, as has been done here.

b

$$\frac{1+2}{2} = 1.5$$

At each stage of an interval bisection question, you begin by dividing the interval into two equal parts by finding its mid-point.

$f(1.5) = -2.3038 \ldots < 0$

$f(2) = 1 > 0$, from above.

You calculated $f(1)$ and $f(2)$ in part (a) of the question and there is no need to calculate them again in part (b).

There is a sign change between $x = 1.5$ and $x = 2$.
Hence $\alpha \in (1.5, 2)$.

$$\frac{1.5+2}{2} = 1.75$$

$f(1.75) = -0.9114 < 0$

$f(2) = 1 > 0$, from above.

There is a sign change between $x = 1.75$ and $x = 2$.
Hence $\alpha \in (1.75, 2)$.

$$\frac{1.75+2}{2} = 1.875$$

$f(1.875) = -0.0298 < 0$

$f(2) = 1 > 0$, from above.

There is a sign change between $x = 1.875$ and $x = 2$.
Hence $\alpha \in (1.875, 2)$.

$2 - 1.875 = 0.125$ and so this interval satisfies the conditions of the question.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 38

Question:

Given that x is measured in radians and $\text{f}(x) = \sin x - 0.4x$,

a find the values of f(2) and f(2.5) and deduce that the equation $\text{f}(x) = 0$ has a root α in the interval [2, 2.5],

b use linear interpolation once on the interval [2, 2.5] to estimate the value of α, giving your answer to two decimal places.

Solution:

a

$$\text{f}(x) = \sin x - 0.4x$$

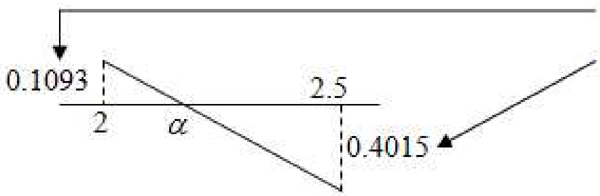
$$\text{f}(2) = 0.10929\ \ldots > 0$$

$$\text{f}(2.5) = -0.40152\ \ldots < 0$$

There is a sign change between $x = 2$ and $x = 2.5$.

Hence the equation $\text{f}(x) = 0$ has a root α in the interval $[2, 2.5]$.

b



You have calculated the values of $\text{f}(x)$ at the end points of the interval in part (a) and these values can be used in part (b).

By similar triangles

$$\frac{\alpha - 2}{0.1093} = \frac{2.5 - \alpha}{0.4015}$$

The answer needs to be given to 2 decimal places; that will be 3 significant figures. It will be sufficient to work to 4 significant figures here. There would be no harm in using more significant figures but if you only worked to 3 significant figures the last figure might be inaccurate.

$$0.4015\alpha - 0.8030 = 0.2733 - 0.1093\alpha$$

$$0.5108\alpha = 1.0765$$

$$\alpha \approx 2.11, \text{ to 2 decimal places.}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 39

Question:

$f(x) = \tan x + 1 - 4x, \ -\frac{\pi}{2} < x < \frac{\pi}{2}.$

a Show that $f(x) = 0$ has a root α in the interval [1.42, 1.44].

b Use linear interpolation once on the interval [1.42, 1.44] to find an estimate of α, giving your answer to three decimal places.

Solution:

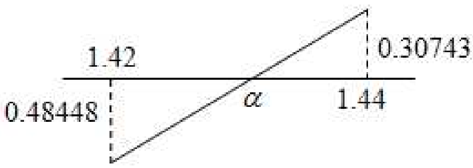
a

$$f(x) = \tan x + 1 - 4x^2$$
$$f(1.42) \approx -0.48448 < 0$$
$$f(1.44) \approx 0.30743 > 0$$

To show a change of sign, you only need to calculate the values of the function to one significant figure. However later in the question you are asked to give your answer to 3 decimal places (which will be 4 significant figures). It is sensible to work out and write down at least 5 significant figures here. You do not want to carry out or write out the calculations twice. It often pays to read quickly through a question before you start it.

There is a sign change between $x = 1.42$ and $x = 1.44$.
Hence the equation $f(x) = 0$ has a root α in the interval $[1.42, 1.44]$.

b



By similar triangles
$$\frac{\alpha - 1.42}{0.48448} = \frac{1.44 - \alpha}{0.30743}$$

$$(0.30743 + 0.48448)\alpha = 1.44 \times 0.48448 + 1.42 \times 0.30743$$

$$0.7919\alpha = 1.1342018$$
$$\alpha \approx 1.432, \text{ to 3 decimal places.}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 40

Question:

$f(x) = \cos\sqrt{x} - x$

a Show that $f(x) = 0$ has a root α in the interval [0.5, 1].

b Use linear interpolation on the interval [0.5, 1] to obtain an approximation to α. Give your answer to two decimal places.

c By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to **b** is accurate to two decimal places.

Solution:

a

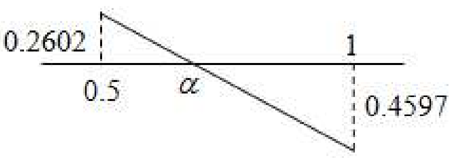
$$f(x) = \cos\sqrt{x} - x$$
$$f(0.5) = 0.2602 \ > 0$$
$$f(1) = -0.4597 \ < 0$$

In this topic, angles are measured in radians, unless otherwise stated.

There is a sign change between $x = 0.5$ and $x = 1$.

Hence the equation $f(x) = 0$ has a root α in the interval $[0.5, 1]$.

b



By similar triangles

$$\frac{\alpha - 0.5}{0.2602} = \frac{1-\alpha}{0.4597}$$

$$0.4597\alpha - 0.2299 = 0.2602 - 0.2602\alpha$$
$$0.7199\alpha = 0.4901$$
$$\alpha \approx 0.68\text{, to 2 decimal places}$$

c

$$f(0.675) = 0.006\,06\ \ldots > 0$$
$$f(0.685) = -0.008\,38\ \ldots < 0$$

There is a change of sign and, hence, $\alpha \in (0.675, 0.685)$.

Hence $\alpha = 0.68$ is accurate to 2 decimal places.

If 0.68 is accurate to 2 decimal places then α must lie in the interval $0.675 \leqslant \alpha < 0.685$. Any number in this interval rounded to two decimal places is 0.68. You evaluate $f(x)$ at the end points of this interval and, if there is a change of sign, you know that α lies in the interval and you can deduce that 0.68 is **accurate** to 2 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 41

Question:

$f(x) = 2^x - x^2 - 1$

The equation $f(x) = 0$ has a root α between $x = 4.256$ and $x = 4.26$.

a Starting with the interval [4.256, 4.26] use interval bisection three times to find an interval of width 5×10^{-4} which contains α.

b Write down the value of α, correct to three decimal places.

Solution:

a $$\frac{4.256+4.26}{2} = 4.258$$

$f(4.256) = -0.0069 \ldots < 0$

$f(4.258) = 0.0025 \ldots > 0$

As you already have a change of sign, there is no need to calculate $f(4.26)$.

There is a sign change between $x = 4.256$ and $x = 4.258$.
Hence $\alpha \in [4.256, 4.258]$.

$$\frac{4.256+4.258}{2} = 4.257$$

$f(4.257) = -0.0021 \ldots < 0$

$f(4.258) = 0.0025 \ldots > 0$, from above

There is a sign change between $x = 4.257$ and $x = 4.258$.
Hence $\alpha \in [4.257, 4.258]$.

$$\frac{4.257+4.258}{2} = 4.2575$$

$f(4.257) = -0.0021 \ldots < 0$, from above

$f(4.2575) = 0.00018 \ldots > 0$

There is a sign change between $x = 4.257$ and $x = 4.2575$.
Hence $\alpha \in [4.257, 4.2575]$.

$4.2575 - 4.257 = 0.0005$, which is the same as 5×10^{-4}, and so the interval $[4.257, 4.2575]$ satisfies the conditions in the question. The open interval $(4.257, 4.2575)$ would also be correct.

b As $\alpha \in [4.257, 4.2575]$, then $\alpha = 4.257$ is accurate to 3 decimal places.

Any number in the interval $[4.257, 4.2575]$ rounded to 3 decimal places would be 4.257. Accurately $\alpha = 4.257\,4619 \ldots$ which is 4.257, to 3 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 42

Question:

$f(x) = 2x^2 + \frac{1}{x} - 3$

The equation $f(x) = 0$ has a root α in the interval $0.3 < x < 0.5$.

a Use linear interpolation once on the interval $0.3 < x < 0.5$ to find an approximation to α. Give your answer to three decimal places.

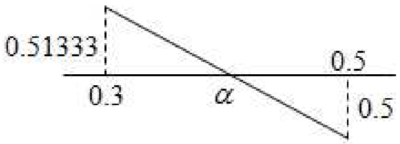
b Find $f'(x)$.

c Taking 0.4 as an approximation to α, use the Newton–Raphson procedure once to find another approximation to α.

Solution:

a

$$f(x) = 2x^2 + \frac{1}{x} - 3$$
$$f(0.3) = 0.513\,33 \ \ldots > 0$$
$$f(0.5) = -0.5 < 0$$



By similar triangles

$$\frac{\alpha - 0.3}{0.51333} = \frac{0.5 - \alpha}{0.5}$$

$$(0.5 + 0.51333)\alpha = 0.5 \times 0.51333 + 0.3 \times 0.5$$
$$1.01333\alpha = 0.4066$$
$\alpha \approx 0.401$, to 3 decimal places.

b $f'(x) = 4x - \frac{1}{x^2}$

$\frac{1}{x} = x^{-1}$ and the rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ gives $\frac{\mathrm{d}}{\mathrm{d}x}(x^{-1}) = -1 \times x^{-2} = -\frac{1}{x^2}$

c

$$f(0.4) = -0.18$$
$$f(0.4) = -4.65$$
$$\alpha = 0.4 - \frac{f(0.4)}{f'(0.4)}$$
$$= 0.4 - \frac{0.18}{4.65}$$
$$\alpha \approx 0.361$$

No accuracy has been specified in the question. Giving the answer to 2 or 3 significant figures is reasonable.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 43

Question:

$f(x) = 0.25x - 2 + 4\ \sin\sqrt{x}$.

a Show that the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$.

b Starting with the interval [0.24, 0.28], use interval bisection three times to find an interval of width 0.005 which contains α

Solution:

a

$f(x) = 0.25x - 2 + 4\sin\sqrt{x}$

Remember to carry out the calculations in radian mode.

$f(0.24) \approx -0.06 < 0$

$f(0.28) \approx 0.09 > 0$

In a question where you only have to consider sign changes, you need only work to one significant figure. The solution shown here gives the minimum of working. You can, of course, show more decimal places if you wish.

There is a sign change between $x = 0.24$ and $x = 0.28$.

Hence the equation $f(x) = 0$ has a root α between $x = 0.24$ and $x = 0.28$.

b

$$\frac{0.24 + 0.28}{2} = 0.26$$

$f(0.26) \approx 0.02 > 0$

$f(0.24) \approx -0.06 < 0$, from above

There is a sign change between $x = 0.24$ and $x = 0.26$.

Hence $\alpha \in [0.24, 0.26]$.

$$\frac{0.24 + 0.26}{2} = 0.25$$

$f(0.25) \approx -0.02 < 0$

$f(0.26) \approx 0.02 > 0$, from above

There is a sign change between $x = 0.25$ and $x = 0.26$.

Hence $\alpha \in [0.25, 0.26]$.

$$\frac{0.25 + 0.26}{2} = 0.255$$

$f(0.255) \approx -0.001 < 0$

$f(0.26) \approx 0.02 > 0$, from above

There is a sign change between $x = 0.255$ and $x = 0.26$.

Hence $\alpha \in [0.255, 0.26]$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 44

Question:

$f(x) = x^3 + 8x - 19$.

a Show that the equation $f(x) = 0$ has only one real root.

b Show that the real root of $f(x) = 0$ lies between 1 and 2.

c Obtain an approximation to the real root of $f(x) = 0$ by performing two applications of the Newton–Raphson procedure to $f(x)$, using $x = 2$ as the first approximation. Give your answer to three decimal places.

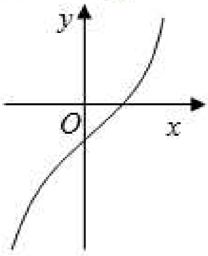
d By considering the change of sign of $f(x)$ over an appropriate interval, show that your answer to **c** is accurate to three decimal places.

Solution:

a $$f'(x) = 3x^2 + 8$$

As, for all x, $x^2 \geqslant 0$, $f'(x) \geqslant 8 > 0$ for all x.

As the derivative of $f(x)$ is always positive, $f(x)$ is always increasing.



Drawing a sketch diagram helps you to see what is going on. If the function is always increasing, after crossing the x-axis it can never turn round and cross the axis again.

As $f(x)$ is always increasing it can only cross the x-axis once, as shown in the sketch and, hence, the equation $f(x) = 0$ has only one real root.

b

$$f(1) = -10 < 0$$
$$f(2) = 5 > 0$$

There is a sign change between $x = 1$ and $x = 2$.

Hence the real root of $f(x) = 0$ lies between $x = 1$ and $x = 2$.

You should give a conclusion to this part of the question. You can word the conclusion by modelling it upon the wording in the question.

c

$$x_1 = 2$$
$$f(2) = 20$$
$$f'(2) = 5$$
$$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{5}{20} = 1.75$$

This is the Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with the values that apply in this question.

$$f(1.75) = 0.359375$$
$$f'(1.75) = 17.1875$$
$$x_3 = 1.75 - \frac{f(1.75)}{f'(1.75)} = 1.75 - \frac{0.359387}{17.1975}$$
$$\approx 1.729, \text{ to 3 decimal places}$$

d

$$f(1.7285) \approx -0.0077 < 0$$
$$f(1.7295) \approx 0.0092 > 0$$

If 1.729 is accurate to 3 decimal places then α must lie in the interval $1.7285 \leqslant \alpha < 1.7295$. Any number in this interval rounded to 3 decimal places is 1.729. You evaluate $f(x)$ at the end points of this interval and, if there is a change of sign, you know that the root lies in the interval your answer is correct to 3 decimal places.

There is a change of sign between $x = 1.7285$ and $x = 1.7295$. Hence the root of the equation lies in the interval $(1.7285, 1.7295)$.

It follows that the root is 1.729 correct to 3 decimal places.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 45

Question:

$f(x) = x^3 - 3x - 1$

The equation $f(x) = 0$ has a root α in the interval $[-2, -1]$.

a Use linear interpolation on the values at the ends of the interval $[-2, -1]$ to obtain an approximation to α.

The equation $f(x) = 0$ has a root β in the interval $[-1, 0]$.

b Taking $x = -0.5$ as a first approximation to β, use the Newton–Raphson procedure once to obtain a second approximation to β.

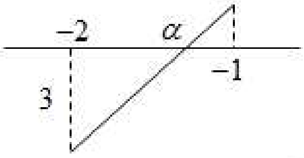
The equation $f(x) = 0$ has a root γ in the interval [1.8, 1.9].

c Starting with the interval [1.8, 1.9] use interval bisection twice to find an interval of width 0.025 which contains γ.

Solution:

a $f(-1) = (-1)^3 - 3(-1) - 1 = -1 + 3 - 1 = 1$

$f(-2) = (-2)^3 - 3(-2) - 1 = -8 + 6 - 1 = -3$



$$\frac{\alpha - (-2)}{3} = \frac{-1-\alpha}{1}$$

$$\alpha + 2 = -3 - 3\alpha$$

$$4\alpha = -5$$

$$\alpha \approx -1.25$$

Finding distances on the negative x-axis can be difficult. The distance is the positive difference between the coordinates, so you must subtract the coordinates and, as $\alpha - (-2) = \alpha + 2$, this will be positive when α is between -1 and -2.

b

$$f'(x) = 3x^2 - 3$$

$$f(-0.5) = 0.375$$

$$f'(-0.5) = -2.25$$

$$\beta = -0.5 - \frac{f(-0.5)}{f'(-0.5)} = -0.5 - \frac{0.375}{-2.25}$$

$$\beta \approx -0.33$$

This expression evaluates as exactly $-\frac{1}{3}$ but as this is an estimate of β, and not an exact value of β, it is sensible to give the answer to 2 decimal places.

c

$$\frac{1.8 + 1.9}{2} = 1.85$$

$$f(1.8) = -0.568 < 0$$

$$f(1.85) = -0.218 \ldots < 0$$

$$f(1.9) = 0.159 > 0$$

There is a sign change between $x = 1.85$ and $x = 1.9$.
Hence $\gamma \in (1.85, 1.9)$.

$$\frac{1.85 + 1.9}{2} = 1.875$$

$$f(1.875) \approx -0.0332 < 0$$

$f(1.9) = 0.159 > 0$, as above

There is a sign change between $x = 1.875$ and $x = 1.9$.
Hence $\gamma \in (1.875, 1.9)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

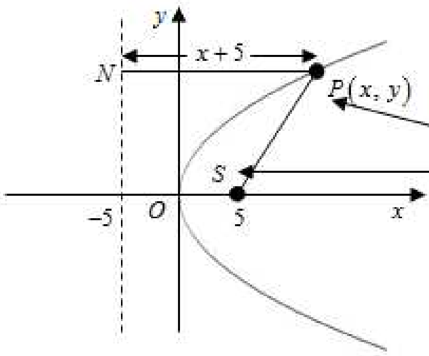
Review Exercise
Exercise A, Question 46

Question:

A point P with coordinates (x, y) moves so that its distance from the point $(5, 0)$ is equal to its distance from the line with equation $x = -5$.

Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of a.

Solution:



A diagram helps you see and understand what is going on. You should label important points. If letters are not given in the question, you can make your own up. Putting them on a diagram makes your method clear to the examiner.

By the definition of a parabola

$$SP = PN$$
$$SP^2 = PN^2$$

Using the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$, it is often easier to work with the distances squared rather than with distances.

$S(5, 0)$, $P(x, y)$

$$SP^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x-5)^2 + y^2$$

$$PN = x + 5$$

Along PN, the distance from P to the y-axis is x, and it is a further distance 5 from the y-axis to N.

$$SP^2 = PN^2$$
$$(x-5)^2 + y^2 = (x+5)^2$$
$$\cancel{x^2} - 10x + \cancel{25} + y^2 = \cancel{x^2} + 10x + \cancel{25}$$
$$y^2 = 20x$$

Multiply out the brackets using $(a+b)^2 = a^2 + 2ab + b^2$. Then "cancel" the equal terms on both sides of the equation.

Comparing with $y^2 = 4ax$, this is the required form with $a = 5$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 47

Question:

A parabola C has equation $y^2 = 16x$. The point S is the focus of the parabola.

a Write down the coordinates of S.

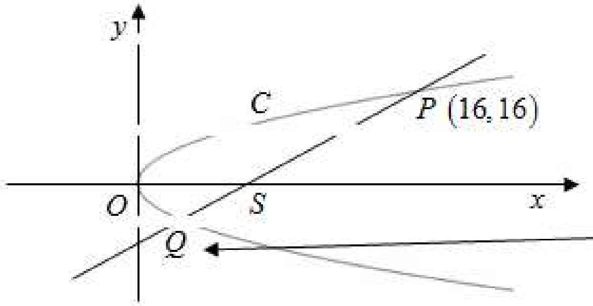
The point P with coordinates (16, 16) lies on C.

b Find an equation of the line SP, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line SP intersects C at the point Q, where P and Q are distinct points.

c Find the coordinates of Q.

Solution:



You should mark on your diagram any points given in the question. Here mark S, P and Q. Diagrams often help you check your working. Here, for example, it is obvious from the diagram that Q must have a negative y-coordinate. If you got $y = 4$ (a mistake it is easy to make), you would know you were wrong and look for an error in your working.

a $S(4, 0)$

The focus of the parabola with equation $y^2 = 4ax$ has coordinates $(a, 0)$. Here $a = 4$. The question asks you to write down the answer, so you do not have to show working.

b Using $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ with $(x_1, y_1) = (4, 0)$ and $(x_2, y_2) = (16, 16)$, an equation of SP is

$$\frac{y-0}{16-0} = \frac{x-4}{16-4}$$

$$\frac{y}{16} = \frac{x-4}{12}$$

$$\cancel{12}^{3} y = \cancel{16}^{4}(x-4)$$

$$3y = 4x - 16$$

$$4x - 3y - 16 = 0$$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

c From (b)

$$x = \frac{3y+16}{4}$$

Substitute for x in $y^2 = 16x$

$$y^2 = \cancel{16}^{4}\left(\frac{3y+16}{\cancel{4}}\right) = 12y + 64$$

$$y^2 - 12y - 64 = 0.$$

$$(y-16)(y+4) = 0$$

$y = 16$ corresponds to the point P.

To find Q you solve the simultaneous equations $4x - 3y - 16 = 0$ and $y^2 = 16x$. The method of using substitution, when one equation is linear and the other is quadratic, is given in Chapter 3 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1.

For Q, $y = -4$

$$x = \frac{3 \times -4 + 16}{4} = \frac{4}{4} = 1$$

The coordinates of Q are $(1, -4)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 48

Question:

The curve C has equations $x = 3t^2$, $y = 6t$.

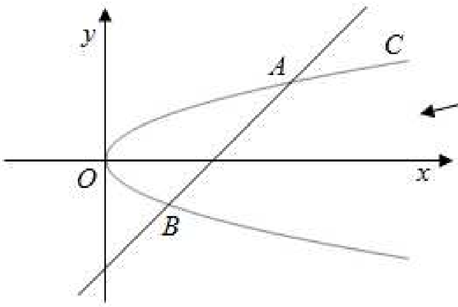
a Sketch the graph of the curve C.

The curve C intersects the line with equation $y = x - 72$ at the points A and B.

b Find the length AB, giving your answer as a surd in its simplest form.

Solution:

a



You have to recognise that $x = 3t^2$, $y = 6t$ is a parabola and draw it passing through the origin with the correct orientation.

b For the intersections, substitute $x = 3t^2$, $y = 6t$ into $y = x - 72$

$$6t = 3t^2 - 72$$
$$3t^2 - 6t - 72 = 0$$
$$(\div 3) \quad t^2 - 2t - 24 = 0$$
$$(t-6)(t+4) = 0$$
$$t = 6, -4$$

$(3t^2, 6t)$ is a general point on the parabola. The points A and B must be of this form and, if they also lie on the line with equation $y = x - 72$, the points on the parabola must also satisfy the equation of the line.

For A, say, $t = 6$

$$x = 3t^2 = 108,\ y = 6t = 36$$

For B, say, $t = -4$

$$x = 3t^2 = 3 \times (-4)^2 = 48,\ y = 6t = -24$$

The question does not tell you which point is A and which point is B but, as you are only asked to find the distance between them, it does not matter which is which and you can make your own choice.

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$AB^2 = (108 - 48)^2 + (36 - (-24))^2$$
$$= 60^2 + 60^2 = 2 \times 60^2$$
$$AB = \sqrt{(2 \times 60^2)} = 60\sqrt{2}$$

You are asked to give your answer as a surd in its simplest form. 84.85 is not acceptable as it is not a surd and $\sqrt{7200}$ is not the simplest form. A surd in its simplest form contains the square root of the smallest possible single number.

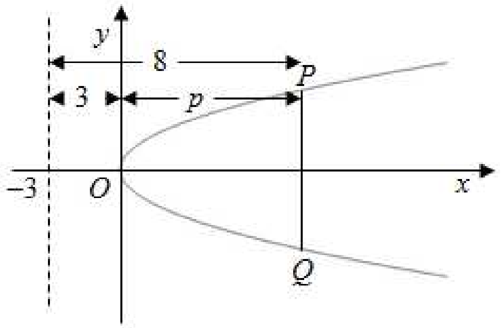
Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 49

Question:

A parabola C has equation $y^2 = 12x$. The points P and Q both lie on the parabola and are both at a distance 8 from the directrix of the parabola. Find the length PQ, giving your answer in surd form.

Solution:



The directrix of $y^2 = 4ax$ is $x = -a$. Comparison of $y^2 = 4ax$ with $y^2 = 12x$ shows that, in this question, $a = 3$.

The equation of the directrix is $x = -3$.
If the x-coordinate of P is p,

$$p + 3 = 8 \Rightarrow p = 5$$

The y-coordinate of P is given by

$$y^2 = 12x = 60 \Rightarrow y = \sqrt{60}$$

By symmetry, the coordinates of Q are $(5, -\sqrt{60})$

$$PQ = 2\sqrt{60} = 4\sqrt{15}$$

P is vertically above Q and the distance from P to Q is twice the y–coordinate of P.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 50

Question:

The point $P(2, 8)$ lies on the parabola C with equation $y^2 = 4ax$. Find

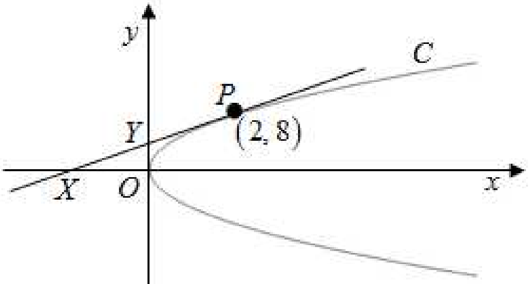
a the value of a,

b an equation of the tangent to C at P.

The tangent to C at P cuts the x-axis at the point X and the y-axis at the point Y.

c Find the exact area of the triangle OXY.

Solution:



a Substitute $(2, 8)$ into $y^2 = 4ax$.

$64 = 4a \times 2 = 8a \Rightarrow a = \frac{64}{8} = 8$

b $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

When $a = 8$ and $x = 2$

$\frac{dy}{dx} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

Using $y - y_1 = m(x - x_1)$, the tangent to C at P is

$y - 8 = 2(x - 2) = 2x - 4$

$y = 2x + 4$

The method for obtaining the equation of a straight line when you know its gradient and one point which it passes through is given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1.

c At X, $y = 0 \Rightarrow 0 = 2x + 4 \Rightarrow x = -2$

So $OX = 2$

At Y, $x = 0 \Rightarrow y = 2 \times 0 + 4 \Rightarrow y = 4$

So $OY = 4$

Area $\triangle OXY = \frac{1}{2} OX \times OY = \frac{1}{2} \times 2 \times 4 = 4$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 51

Question:

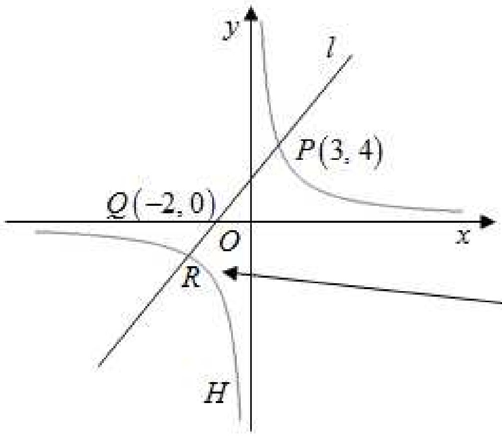
The point P with coordinates (3, 4) lies on the rectangular hyperbola H with equation $xy = 12$. The point Q has coordinates $(-2,\ 0)$. The points P and Q lie on the line l.

a Find an equation of l, giving your answer in the form $y = mx + c$, where m and c are real constants.

The line l cuts H at the point R, where P and R are distinct points.

b Find the coordinates of R.

Solution:



You can see from the diagram that both the x- and y-coordinates of R are negative. This will help you check your work in part (b).

a Using $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ with

$P(x_1, y_1)=(3, 4)$ and $Q(x_2, y_2)=(-2, 0)$.

$$\frac{y-4}{0-4}=\frac{x-3}{-2-3}$$

$$\frac{y-4}{-4}=\frac{x-3}{-5}$$

$$5(y-4)=4(x-3)$$

$$5y-20=4x-12$$

$$5y=4x+8$$

$$y=\frac{4}{5}x+\frac{8}{5} \quad \ldots\ldots \ast$$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

b Substitute $\ast$ into the equation of H.

$$xy=12$$

$$x\left(\frac{4}{5}x+\frac{8}{5}\right)=12$$

$$\frac{4}{5}x^2+\frac{8}{5}x=12 \Rightarrow 4x^2+8x=60$$

$$x^2+2x-15=0$$

$$(x+5)(x-3)=0$$

$$x=-5, 3$$

$x=3$ corresponds to the point P.

For R, $x=-5$

$$y=\frac{12}{x}=\frac{12}{-5}=-2.4$$

The coordinates of R are $(-5, -2.4)$

Part (b) involves solving a pair of simultaneous equations, one of which is linear and one of which is quadratic, as xy is a quadratic term. Substitution is used to solve such pairs of equations.

There are two points on both the line l and the parabola H. You are given the coordinates of one $(3, 4)$ in the question. You are looking for the other point, so you must reject the value of x (3) which belongs to P and take the other value.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 52

Question:

The point $P(12, 3)$ lies on the rectangular hyperbola H with equation $xy = 36$.

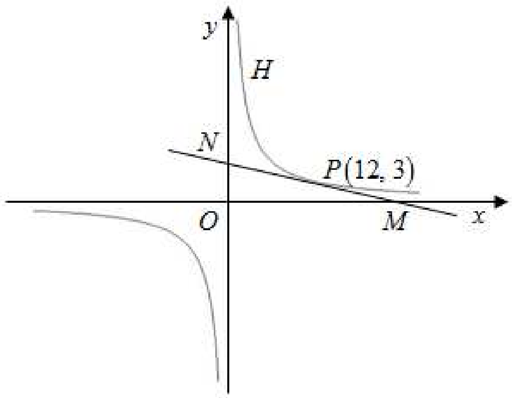
a Find an equation of the tangent to H at P.

The tangent to H at P cuts the x-axis at the point M and the y-axis at the point N.

b Find the length MN, giving your answer as a simplified surd.

Solution:

a



$$y = \frac{36}{x} = 36x^{-1}$$

$$\frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2}$$

At P, $x = 12$

$$\left(\frac{dy}{dx}\right)_{12} = -\frac{36}{12^2} = -\frac{1}{4}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,

$$\frac{d}{dx}(x^{-1}) = -1 \times x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Using $y - y_1 = m(x - x_1)$, the tangent to H at P is

$$y - 3 = -\frac{1}{4}(x - 12)$$

$$4y - 12 = -x + 12$$

$$x + 4y = 24$$

No particular form of the equation of the tangent has been specified and any form would be accepted. $x + 4y = 24$ has been chosen here as, reading ahead, you will have to substitute $x = 0$ and $y = 0$ into the equation to find the corresponding y and x. It is very easy to do this with this equation. Reading ahead can often save time.

b At M, $y = 0 \Rightarrow x = 24 \Rightarrow OM = 24$

At M, $x = 0 \Rightarrow y = 6 \Rightarrow ON = 6$

$$MN^2 = OM^2 + ON^2$$

$$= 24^2 + 6^2 = 612 = 36 \times 7$$

$$MN = 6\sqrt{7}$$

A surd in its simplest form has the square root of the smallest possible single number.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 53

Question:

The point $P(5, 4)$ lies on the rectangular hyperbola H with equation $xy = 20$. The line l is the normal to H at P.

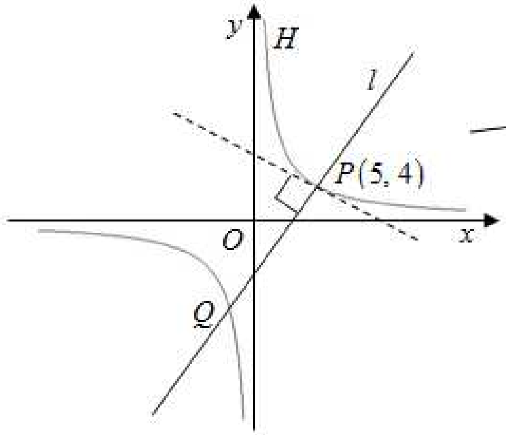
a Find an equation of l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line l meets H again at the point Q.

b Find the coordinates of Q.

Solution:

a



The normal to H at P is perpendicular to the tangent at P. The tangent has here been drawn as a dotted line. To work out perpendicular gradients you will need the formula $mm' = -1$.

$$y = \frac{20}{x} = 20x^{-1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -20x^{-2} = -\frac{20}{x^2}$$

At P, $x = 5$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_5 = -\frac{20}{5^2} = -\frac{4}{5}$$

You have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent by differentiating.

For the gradient of the normal, using $mm' = -1$,

$$\left(-\frac{4}{5}\right)m' = -1 \Rightarrow m' = \frac{5}{4}$$

Using $y - y_1 = m(x - x_1)$, the normal to H at P is $y - 4 = \frac{5}{4}(x-5)$

$$4(y-4) = 5(x-5)$$

$$4y - 16 = 5x - 25$$

$$5x - 4y - 9 = 0$$

You were asked to give your answer in the form $ax + by + c = 0$, where a, b and c are integers. The answer $-5x + 4y + 9 = 0$ would also have been acceptable.

b Rearranging the answer to part (a)

$$x = \frac{4y+9}{5}$$

Substitute this expression for x into $xy = 20$

$$\left(\frac{4y+9}{5}\right)y = 20$$

$$(4y+9)y = 100$$

$$4y^2 + 9y - 100 = 0$$

$$(y-4)(4y+25) = 0$$

$$y = 4, \ -\frac{25}{4}$$

Expressions like $4y^2 + 9y - 100$ are not easy to factorise but, as P lies on both l and H, you know that the y-coordinate of P, $y = 4$, must be one answer to the equation. So $(y-4)$ has to be one factor and the other can just be written down using $y \times 4y = 4y^2$ and $-4 \times +25 = -100$.

$y = 4$ corresponds to the point P.

For Q, $y = -6.25 \Rightarrow x(-6.25) = 20 \Rightarrow x = -\frac{20}{6.25} = -3.2$

The coordinates of Q are $(-3.2, -6.25)$.

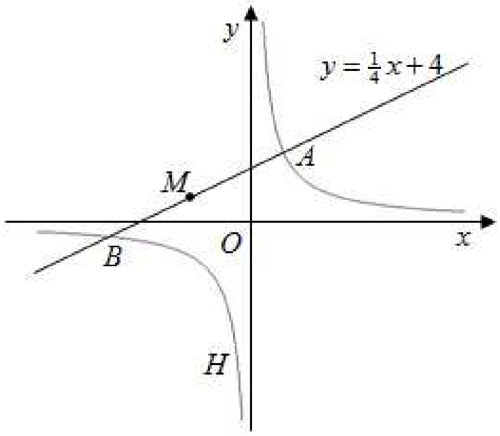
Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 54

Question:

The curve H with equation $x = 8t$, $y = \frac{8}{t}$ intersects the line with equation $y = \frac{1}{4}x + 4$ at the points A and B. The mid-point of AB is M. Find the coordinates of M.

Solution:



$\left(8t, \frac{8}{t}\right)$ is a general point on the rectangular hyperbola. The points A and B must be of this form and, if they also lie on the line with equation $y=\frac{1}{4}x+4$, the points on the parabola must also satisfy the equation of the line.

Substitute $x=8t,\ y=\frac{8}{t}$ into $y=\frac{1}{4}x+4$

$$\frac{8}{t}=\frac{1}{4}\times 8t+4$$

$$\frac{8}{t}=2t+4$$

Multiplying by t and rearranging

$$2t^2+4t-8=0$$

$(\div 2)$ $$t^2+2t-4=(t+4)(t-2)=0$$

$$t=2,\ -4$$

For A, say, $t=2 \Rightarrow x=8t=8\times 2=16$

and $y=\frac{8}{t}=\frac{8}{2}=4$

The coordinates of A are $(16, 4)$

For B, say, $t=-4 \Rightarrow x=8t=8\times -4=-32$

and $y=\frac{8}{t}=\frac{8}{-4}=-2$

The coordinates of B are $(-32, -2)$

The question does not tell you which point is A and which point is B but, as the mid-point is not affected by the choice, it does not matter which is which and you can make your own choice.

The x-coordinate of the mid-point of AB is given by

$$x_M=\frac{16-32}{2}=-8$$

The y-coordinate of the mid-point of AB is given by

$$y_M=\frac{4-2}{2}=1$$

The coordinates of the mid-point M of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $(x_M, y_M)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

The coordinates of M are $(-8, 1)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 55

Question:

The point $P(24t^2,\ 48t)$ lies on the parabola with equation $y^2 = 96x$. The point P also lies on the rectangular hyperbola with equation $xy = 144$.

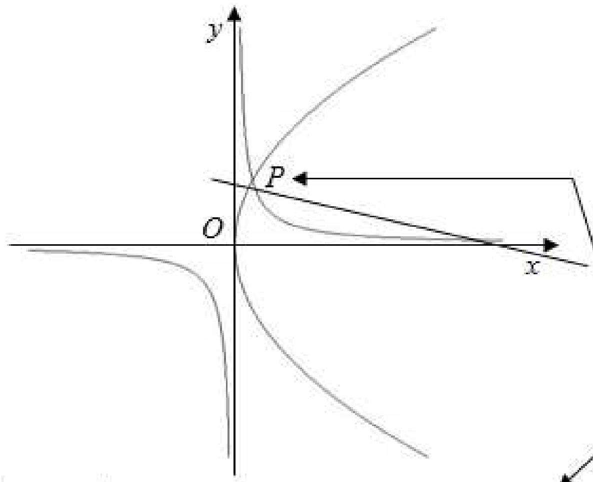
a Find the value of t and, hence, the coordinates of P.

b Find an equation of the tangent to the parabola at P, giving your answer in the form $y = mx + c$, where m and c are real constants.

c Find an equation of the tangent to the rectangular hyperbola at P, giving your answer in the form $y = mx + c$, where m and c are real constants.

Solution:

a



The point with coordinates $(at^2, 2at)$ always lies on the parabola with equation $y^2 = 4ax$, in this case $a = 24$, so P is on the parabola for all t. There will however only be one value of t for which P also lies on the rectangular hyperbola and you find it by substituting $(24t^2, 48t)$ into $xy = 144$.

$(24t^2, 48t)$ must satisfy the equation $xy = 144$

$$24t^2 \times 48t = 144$$

$$t^3 = \frac{144}{24 \times 48} = \frac{1}{8} \Rightarrow t = \frac{1}{2}$$

For P, $x = 24t^2 = 24 \times \left(\frac{1}{2}\right)^2 = 6$

$$y = 48t = 48 \times \frac{1}{2} = 24$$

The coordinates of P are $(6, 24)$.

$96 = 16 \times 6 = 4^2 \times 6$.
So $\sqrt{96} = \sqrt{(4^2 \times 6)} = 4\sqrt{6}$

b

$$y^2 = 96x \Rightarrow y = (96)^{\frac{1}{2}} x^{\frac{1}{2}} = 4\sqrt{6}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 4\sqrt{6}x^{-\frac{1}{2}} = \frac{2\sqrt{6}}{x^{\frac{1}{2}}}$$

At $x = 6$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{\sqrt{6}} = 2$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the parabola at P is

$$y - 24 = 2(x - 6) = 2x - 12$$

$$y = 2x + 12$$

c

$$y = \frac{144}{x} = 144x^{-1}$$

$$\frac{dy}{dx} = -144x^{-2} = -\frac{144}{x^2}$$

At $x = 6$, $\frac{dy}{dx} = -\frac{144}{6^2} = -4$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the hyperbola at P is

$$y - 24 = -4(x - 6) = -4x + 24$$

$$y = -4x + 48$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 56

Question:

The points $P(9, 8)$ and $Q(6, 12)$ lie on the rectangular hyperbola H with equation $xy = 72$.

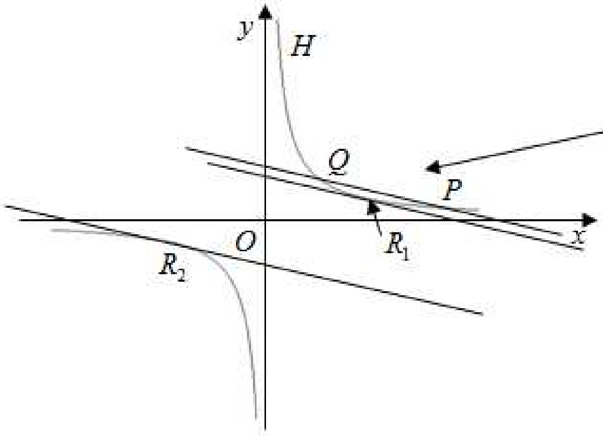
a Show that an equation of the chord PQ of H is $4x + 3y = 60$.

The point R lies on H. The tangent to H at R is parallel to the chord PQ.

b Find the exact coordinates of the two possible positions of R.

Solution:

a



The diagram shows that there are two positions of R, labelled R_1 and R_2 in the diagram, where the tangents to H are parallel to PQ.

Using $\dfrac{y-y_1}{y_2-y_1}=\dfrac{x-x_1}{x_2-x_1}$ with

$P(x_1, y_1)=(9, 8)$ and $Q(x_2, y_2)=(6, 12)$,

an equation of the chord PQ is

$$\frac{y-8}{12-8}=\frac{x-9}{6-9}$$

$$\frac{y-8}{4}=\frac{x-9}{-3}$$

$$-3(y-8)=4(x-9)$$

$$-3y+24=4x-36$$

$4x+3y=60$, as required.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Rearranging the answer to part (a)

$$3y=-4x+60 \Rightarrow y=-\frac{4}{3}x+20$$

The gradient of the chord is $-\dfrac{4}{3}$.

If the tangents are parallel to AB, the gradients of the tangents must also be $-\dfrac{4}{3}$.

The lines $y=mx+c$ and $y=m'x+c'$ are parallel if $m=m'$. For AB, $m=-\dfrac{4}{3}$. For a tangent, $m'=\dfrac{dy}{dx}$. The key step is, therefore, solving $\dfrac{dy}{dx}=-\dfrac{4}{3}$.

$$y=72x^{-1} \Rightarrow \frac{dy}{dx}=-72x^{-2}=-\frac{72}{x^2}$$

$$-\frac{72}{x^2}=-\frac{4}{3}$$

$$72\times 3=4x^2 \Rightarrow x^2=\frac{72\times 3}{4}=54$$

$$x=\pm\sqrt{54}=\pm 3\sqrt{6}$$

At R_1, $x=3\sqrt{6}$, $y=\dfrac{72}{3\sqrt{6}}=\dfrac{12\times 6}{3\sqrt{6}}=4\sqrt{6}$

At R_2, $x=-3\sqrt{6}$, $y=-\dfrac{72}{3\sqrt{6}}=-\dfrac{12\times 6}{3\sqrt{6}}=-4\sqrt{6}$

The coordinates of the two possible positions of R are $(3\sqrt{6}, 4\sqrt{6})$ and $(-3\sqrt{6}, -4\sqrt{6})$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 57

Question:

A rectangular hyperbola H has cartesian equation $xy = 9$. The point $\left(3t,\ \frac{3}{t}\right)$ is a general point on H.

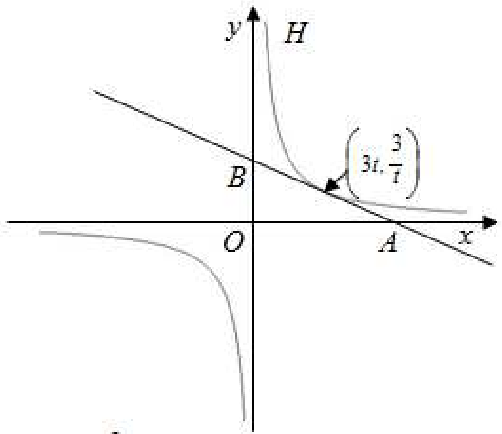
a Show that an equation of the tangent to H at $\left(3t,\ \frac{3}{t}\right)$ is $x + t^2y = 6t$.

The tangent to H at $\left(3t,\ \frac{3}{t}\right)$ cuts the x-axis at A and the y-axis at B. The point O is the origin of the coordinate system.

b Show that, as t varies, the area of the triangle OAB is constant.

Solution:

a



$y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

When $x = 3t, \frac{dy}{dx} = -\frac{9}{(3t)^2} = -\frac{1}{t^2}$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}(9x^{-1}) = -1 \times 9x^{-1-1} = -9x^{-2} = -\frac{9}{x^2}$.

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$,
the tangent to H

$$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$$

$(\times t^2)$ $\quad t^2y - 3t = -x + 3t$

$x + t^2y = 6t$, as required.

You can use $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$ in the formula $y - y_1 = m(x - x_1)$ in exactly the same way as you use coordinates with numerical values like, say, $(6, 4)$.

b For A, substitute, $y = 0$ into $x + t^2y = 6t$.

$x = 6t \Rightarrow OA = 6t$

For B, substitute, $x = 0$ into $x + t^2y = 6t$.

$t^2y = 6t \Rightarrow y = \frac{6}{t} \Rightarrow OB = \frac{6}{t}$

Area $\triangle OAB = \frac{1}{2} OA \times OB = \frac{1}{2} \times 6t \times \frac{6}{t} = 18$

This area, 18, is a constant independent of t.

This result means that no matter which point you take on this rectangular hyperbola the area of the triangle OAB is always the same, 18.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 58

Question:

The point $P\left(ct, \frac{c}{t}\right)$ lies on the hyperbola with equation $xy = c^2$, where c is a positive constant.

a Show that an equation of the normal to the hyperbola at P is

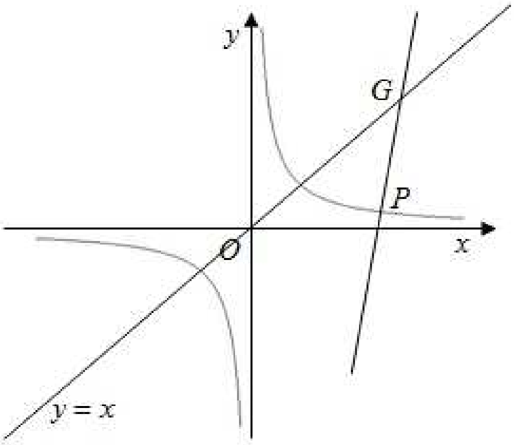
$t^3x - ty - c(t^4 - 1) = 0.$

The normal to the hyperbola at P meets the line $y = x$ at G. Given that $t \neq \pm 1$,

b show that $PG^2 = c^2\left(t^2 + \frac{1}{t^2}\right)$.

Solution:

a



$$y = \frac{c^2}{x} = c^2 x^{-1}$$

$$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$$

At P, $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{c^2 t^2} = -\frac{1}{t^2}$$

For the gradient of the normal, using $mm' = -1$,

$$\left(-\frac{1}{t^2}\right) m' = -1 \Rightarrow m' = t^2$$

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$. So you have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent using differentiation.

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$,

an equation of the normal to the hyperbola at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

$$y - \frac{c}{t} = t^2 x - ct^3$$

$(\times t)$ $\quad yt - c = t^3 x - ct^4$

$$t^3 x - ty - ct^4 + c = 0$$

$t^3 x - ty - c(t^4 - 1) = 0$, as required

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b For G, substitute $y = x$ into the result in part (a)

$$t^3x - tx - c\left(t^4 - 1\right) = 0$$

$$\left(t^3 - t\right)x = c\left(t^4 - 1\right)$$

$$x = \frac{c\left(t^4 - 1\right)}{t^3 - t} = \frac{c\cancel{\left(t^2 - 1\right)}\left(t^2 + 1\right)}{t\cancel{\left(t^2 - 1\right)}} = \frac{ct^2 + c}{t} = ct + \frac{c}{t}$$

You could not "cancel" the $\left(t^2 - 1\right)$ terms if $t = \pm 1$, as then $\left(t^2 - 1\right)$ would be 0, but these cases are explicitly ruled out in the question.

The coordinates of G are $\left(ct + \frac{c}{t}, ct + \frac{c}{t}\right)$

$$PG^2 = \left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2$$

$$= \left(ct + \frac{c}{t} - ct\right)^2 + \left(ct + \frac{c}{t} - \frac{c}{t}\right)^2$$

$$= \frac{c^2}{t^2} + c^2t^2 = c^2\left(t^2 + \frac{1}{t^2}\right), \text{ as required.}$$

Using $d^2 = \left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2$ with $\left(x_1, y_1\right) = \left(ct + \frac{c}{t}, ct + \frac{c}{t}\right)$ and $\left(x_2, y_2\right) = \left(ct, \frac{c}{t}\right)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 59

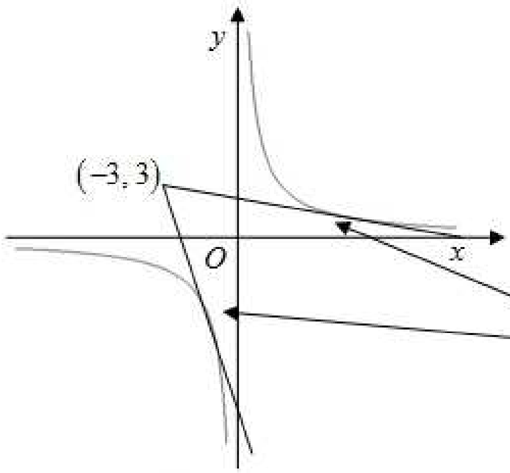
Question:

a Show that an equation of the tangent to the rectangular hyperbola with equation $xy = c^2$ at the point $\left(ct,\ \frac{c}{t}\right)$ is $t^2y + x = 2ct$.

Tangents are drawn from the point $(-3,\ 3)$ to the rectangular hyperbola with equation $xy = 16$.

b Find the coordinates of the points of contact of these tangents with the hyperbola.

Solution:



The diagram shows that, in part (b), the tangents have two points of contact with the hyperbola. One is in the first quadrant and the other in the third.

a $y = \frac{c^2}{x} = c^2x^{-1}$

$$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$,

an equation of the tangent to the hyperbola is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$y - \frac{c}{t} = -\frac{x}{t^2} + \frac{c}{t}$$

$$y + \frac{x}{t^2} = \frac{2c}{t}$$

$(\times t^2)$ $\quad t^2y + x = 2ct$, as required.

Part (a) is a general question. Part (b) is about the specific rectangular hyperbola with $c^2 = 16$. The first step in part (b) is to adapt the answer in (a) to (b) by substituting $c = 4$.

b When $c = 4$, the equation of the tangent is

$$t^2y + x = 8t$$

$(-3, 3)$ satisfies the equation

$$3t^2 - 3 = 8t$$

$(-3, 3)$ must lie on both tangents and you use this to obtain a quadratic in t.

$$3t^2 - 8t - 3 = (3t+1)(t-3) = 0$$
$$t = -\frac{1}{3}, 3$$

The points on the hyperbola are $\left(4t, \frac{4}{t}\right)$

When $t = -\frac{1}{3}$, the point is $\left(-\frac{4}{3}, \frac{4}{-\frac{1}{3}}\right) = \left(-\frac{4}{3}, -12\right)$

When $t = 3$, the point is $\left(12, \frac{4}{3}\right)$

The points of contact of the tangents with the hyperbola are $\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

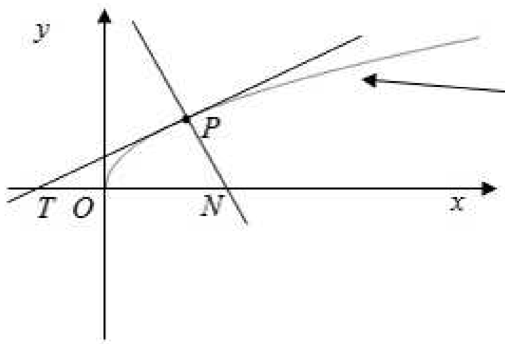
Review Exercise
Exercise A, Question 60

Question:

The point $P(at^2,\ 2at)$, where $t > 0$, lies on the parabola with equation $y^2 = 4ax$.

The tangent and normal at P cut the x-axis at the points T and N respectively. Prove that $\frac{PT}{PN} = t$.

Solution:



The $t > 0$ in the question implies that you only need to consider the part of the parabola where $y > 0$, that is the part above the x-axis.

To find an equation of the tangent PT

$y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \quad (y > 0)$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$,
an equation of the tangent to the parabola at P is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at = x - at^2$$

$$ty = x + at^2 \quad \ldots\ldots ❶$$

The tangent crosses the x-axis where $y = 0$.

To find the x-coordinate of T, substitute $y = 0$ into ❶

$$0 = x + at^2 \Rightarrow x = -at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with
$(x_1, y_1) = (at^2, 2at)$ and $(x_2, y_2) = (-at^2, 0)$

$$PT^2 = \left(at^2 - (-at^2)\right)^2 + (2at - 0)^2$$

$$= (2at^2)^2 + 4a^2t^2 = 4a^2t^4 + 4a^2t^2$$

$$= 4a^2t^2(t^2 + 1) \quad \ldots\ldots ❷$$

To find an equation of the normal PN.

Using $mm' = -1$,

$$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$$

The normal is perpendicular to the tangent. From working earlier in the question, you know that the gradient of the tangent is $\frac{1}{t}$.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$,
an equation of the normal to the parabola at P is

$$y - 2at = -t\left(x - at^2\right)$$
$$= -tx + at^3$$
$$y + tx = 2at + at^3 \quad \ldots\ldots \text{❸}$$

The normal crosses the x-axis where $y = 0$.

To find the x-coordinate of N, substitute $y = 0$ into ❸

$$tx = 2at + at^3 \Rightarrow x = 2a + at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = \left(at^2, 2at\right)$ and $(x_2, y_2) = \left(2a + at^2, 0\right)$

$$PN^2 = \left(at^2 - \left(2a + at^2\right)\right)^2 + (2at - 0)^2$$
$$= (2a)^2 + (2at)^2 = 4a^2 + 4a^2t^2$$
$$= 4a^2\left(1 + t^2\right) \quad \ldots\ldots \text{❹}$$

From ❷ and ❹

$$\frac{PT^2}{PN^2} = \frac{4a^2t^2\left(t^2+1\right)}{4a^2\left(t^2+1\right)} = t^2$$

Hence

$$\frac{PT}{PN} = t \text{, as required.}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 61

Question:

The point P lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to the parabola $P(ap^2,\ 2ap),\ p > 0$, is $py = x + ap^2$.

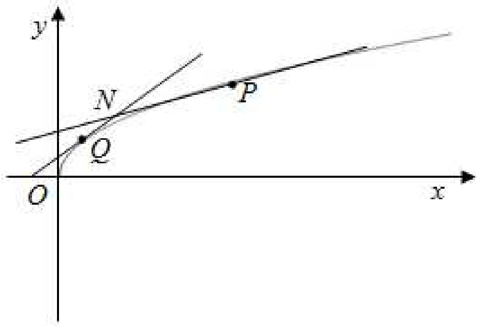
The tangents at the points $P(ap^2,\ 2ap)$ and $Q(aq^2,\ 2aq)(p \neq q,\ p > 0,\ q > 0)$ meet at the point N.

b Find the coordinates of N.

Given further that N lies on the line with equation $y = 4a$,

c find p in terms of q.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} \quad (y > 0)$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

At $x = ap^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}p$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}p} = \frac{1}{p}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (ap^2, 2ap)$,
an equation of the tangent to the parabola at P is

$$y - 2ap = \frac{1}{p}(x - ap^2)$$

$$py - 2ap^2 = x - ap^2$$

$$py = x + ap^2, \text{ as required.}$$

b An equation of the tangent to the parabola at Q is

$$qy = x + aq^2 \quad \ldots\ldots \text{❶}$$

$$py = x + ap^2 \quad \ldots\ldots \text{❷}$$

The equation of the tangent at Q is the same as the equation of the tangent at P with the ps replaced by qs. You do not have to work out the equation again.

❷ − ❶ $\quad py - qy = ap^2 - aq^2$

$$(p - q)y = a(p^2 - q^2)$$

$$y = \frac{a(p^2 - q^2)}{p - q} = \frac{a\cancel{(p-q)}(p+q)}{\cancel{p-q}} = a(p+q)$$

To find x and y, in terms of p and q, you solve equations ❶ and ❷ as a pair of simultaneous linear equations.

Substitute into ❶

$$qa(p+q) = x + aq^2$$

$$x = qa(p+q) - aq^2 = apq + aq^2 - aq^2 = apq$$

The coordinates of N are $(apq, a(p+q))$.

c If N lies on $y = 4a$,

$$\cancel{a}(p+q) = 4\cancel{a}$$

$$p = 4 - q$$

If N lies on the line with equation $y = 4a$, then the y-coordinate of N, which is $a(p+q)$, must be $4a$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 62

Question:

The point $P(at^2,\ 2at),\ t \neq 0$ lies on the parabola with equation $y^2 = 4ax$, where a is a positive constant.

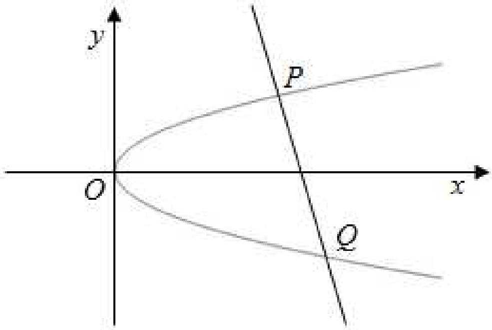
a Show that an equation of the normal to the parabola at P is

$y + xt = 2at + at^3$.

The normal to the parabola at P meets the parabola again at Q.

b Find, in terms of t, the coordinates of Q.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$$

Using $mm' = -1$,

$$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$$

The normal is perpendicular to the tangent, so you must first find the gradient of the tangent. Then you use $mm' = -1$ to find the gradient of the normal.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$,

an equation of the normal to the parabola at P is

$$y - 2at = -t(x - at^2)$$

$$= -tx + at^3$$

$y + tx = 2at + at^3$, as required.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Let the coordinates of Q be $(aq^2, 2aq)$

The point Q lies on the normal at P, so

You substitute $x = aq^2$ and $y = 2aq$ into the answer to part (a).

$$2aq + taq^2 = 2at + at^3$$

$$2aq - 2at + atq^2 - at^3 = 0$$

$$2a(q - t) + at(q^2 - t^2) = 0$$

$$2a(q - t) + at(q - t)(q + t) = 0$$

$$a(q - t)(2 + t(q + t)) = 0$$

$$2 + tq + t^2 = 0$$

$$q = -\frac{t^2 + 2}{t}$$

There are two possibilities here: $q - t = 0$ and $2 + t(q + t) = 0$. As P and Q are different points, $q \neq t$, so you need only consider the second possibility. You use this to find q in terms of t.

The coordinates of Q are $\left(a\left(\frac{t^2+2}{t}\right)^2, -2a\left(\frac{t^2+2}{t}\right)\right)$

Replace the q in $(aq^2, 2aq)$ by $-\frac{t^2+2}{t}$. You need not attempt to simplify this further.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 63

Question:

a Show that the normal to the rectangular hyperbola $xy = c^2$, at the point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, has equation $y = t^2x + \frac{c}{t} - ct^3$.

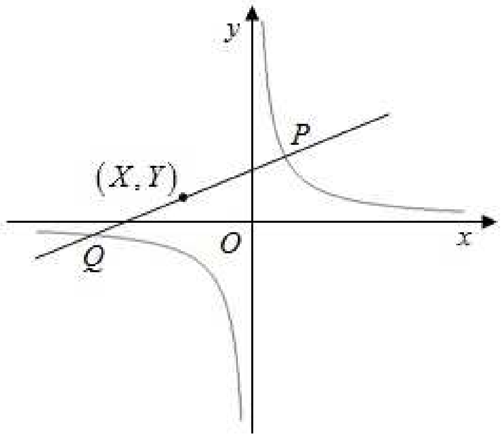
The normal to the hyperbola at P meets the hyperbola again at the point Q.

b Find, in terms of t, the coordinates of the point Q.

Given that the mid-point of PQ is (X, Y) and that $t \neq \pm 1$,

c show that $\frac{X}{Y} = -\frac{1}{t^2}$.

Solution:



a $y = \dfrac{c^2}{x} = c^2x^{-1}$

$\dfrac{dy}{dx} = -c^2x^{-2} = -\dfrac{c^2}{x^2}$

At P, $x = ct$

$\dfrac{dy}{dx} = -\dfrac{c^2}{c^2t^2} = -\dfrac{1}{t^2}$

For the gradient of the normal, using $mm' = -1$,

$\left(-\dfrac{1}{t^2}\right)m' = -1 \Rightarrow m' = t^2$

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$. So you have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent by differentiating.

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \dfrac{c}{t}\right)$,

an equation of the normal to the hyperbola at P is

$y - \dfrac{c}{t} = t^2(x - ct)$

$= t^2x - ct^3$

$y = t^2x + \dfrac{c}{t} - ct^3$, as required. … ❶

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question. In this case, the form of the printed equation suggests a method for the next part of the question.

b $xy = c^2 \Rightarrow y = \dfrac{c^2}{x}$ … … ❷

Writing the equation of the rectangular hyperbola, in the form $y = \ldots$, enables you to eliminate y quickly between ❶ and ❷.

For Q, from ❶ and ❷

$t^2x + \dfrac{c}{t} - ct^3 = \dfrac{c^2}{x}$

$\times t$ and collect terms as a quadratic in x

$t^3x^2 + (c - ct^4)x - c^2t = 0$

$(x - ct)(t^3x + c) = 0$

$x = ct$ corresponds to P

The x-coordinate of P, ct, must be a solution of the equation. So $(x - ct)$ must be a factor of the quadratic and, so, you can write down the second factor using $x \times t^3x = t^3x^2$ and $-ct \times c = -c^2t$.

For Q, $x=-\frac{c}{t^3}$

Substitute the x-coordinate into ❷

$$y=\frac{c^2}{x}=\frac{c^2}{-\frac{c}{t^3}}=-ct^3$$

The coordinates of Q are $\left(-\frac{c}{t^3}, -ct^3\right)$

c

$$X=\frac{ct+\left(-\frac{c}{t^3}\right)}{2}=\frac{ct^4-c}{2t^3}=\frac{c\left(t^4-1\right)}{2t^3}$$

$$Y=\frac{\frac{c}{t}+\left(-ct^3\right)}{2}=\frac{ct-ct^4}{2t}=\frac{c\left(1-t^4\right)}{2t}$$

$$\frac{X}{Y}=\frac{\frac{c\left(t^4-1\right)}{2t^3}}{\frac{c\left(1-t^4\right)}{2t}}=\frac{c\left(t^4-1\right)}{2t^3}\times\frac{2t}{c\left(1-t^4\right)}$$

$=-\frac{1}{t^2}$, as required.

The coordinates of the mid-point of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $(X, Y)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Multiplying all terms on the top and bottom of the fraction by t^3.

Multiplying all terms on the top and bottom of the fraction by t.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 64

Question:

The rectangular hyperbola C has equation $xy = c^2$, where c is a positive constant.

a Show that the tangent to C at the point $P\left(cp,\ \frac{c}{p}\right)$ has equation $p^2y = -x + 2cp$.

The point Q has coordinates $Q\left(cq,\ \frac{c}{q}\right),\ q \neq p$.

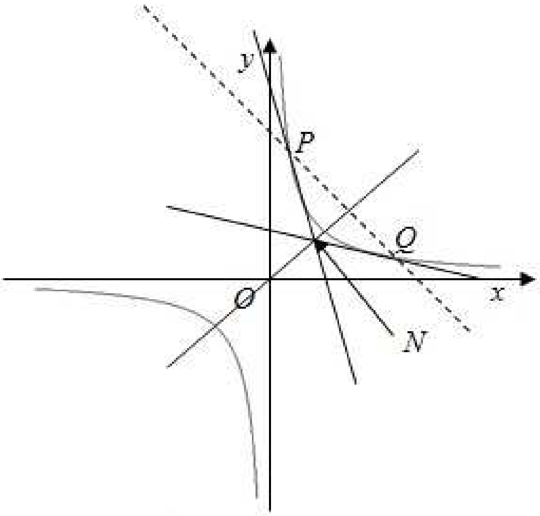
The tangents to C at P and Q meet at N. Given that $p + q \neq 0$,

b show that the y-coordinate of N is $\frac{2c}{p+q}$.

The line joining N to the origin O is perpendicular to the chord PQ.

c Find the numerical value of p^2q^2.

Solution:



a

$$y = \frac{c^2}{x} = c^2 x^{-1}$$

$$\frac{dy}{dx} = -c^2 x^{-2} = -\frac{c^2}{x^2}$$

At $x = cp$

$$\frac{dy}{dx} = -\frac{c^2}{c^2 p^2} = -\frac{1}{p^2}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$,

an equation of the tangent to the hyperbola is

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$y - \frac{c}{p} = -\frac{x}{p^2} + \frac{c}{p}$$

$$y = -\frac{x}{p^2} + \frac{2c}{p}$$

$(\times p^2)$ $\quad p^2 y = -x + 2cp$, as required. … ❶

The equation of the tangent at Q is the same as the equation of the tangent at P with the ps replaced by qs. You do not have to work out the equation twice.

b The tangent at Q is

$q^2 y = -x + 2cq$ … … ❷

To find the y-coordinate of N subtract ❷ from ❶

$$(p^2 - q^2)y = 2c(p - q)$$

$$y = \frac{2c(p-q)}{p^2 - q^2} = \frac{2c\cancel{(p-q)}}{\cancel{(p-q)}(p+q)} = \frac{2c}{p+q}$$, as required.

To find y, you eliminate x from equations ❶ and ❷. These equations are a pair of simultaneous linear equations and the method of solving them is essentially the same as you learnt for GCSE.

c To find the x-coordinate of N substitute the result of part (b) into ➊

$$\frac{2cp^2}{p+q} = -x + 2cp$$

$$x = 2cp - \frac{2cp^2}{p+q} = \frac{2cp(p+q) - 2cp^2}{p+q} = \frac{2cpq}{p+q}$$

The gradient of PQ, m say, is given by

$$m = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\frac{\cancel{c}\cancel{(q-p)}^{-1}}{pq}}{\cancel{c}\cancel{(p-q)}} = -\frac{1}{pq}$$

The gradient m is found using $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ and $(x_2, y_2) = \left(cq, \frac{c}{q}\right)$

The gradient of ON, m' say, is given by

$$m' = \frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}} = \frac{1}{pq}$$

Given that ON is perpendicular to PQ

$$mm' = -1$$

$$-\frac{1}{pq} \times \frac{1}{pq} = -1 \Rightarrow p^2q^2 = 1$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 65

Question:

The point P lies on the rectangular hyperbola $xy = c^2$, where c is a positive constant.

a Show that an equation of the tangent to the hyperbola at the point $P\left(cp,\ \frac{c}{p}\right),\ p > 0$, is $yp^2 + x = 2cp$.

This tangent at P cuts the x-axis at the point S.

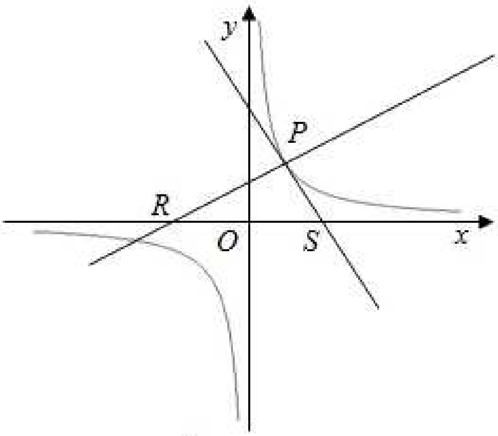
b Write down the coordinates of S.

c Find an expression, in terms of p, for the length of PS.

The normal at P cuts the x-axis at the point R. Given that the area of ΔRPS is $41c^2$,

d find, in terms of c, the coordinates of the point P.

Solution:



a $y = \dfrac{c^2}{x} = c^2 x^{-1}$

$\dfrac{dy}{dx} = -c^2 x^{-2} = -\dfrac{c^2}{x^2}$

At $x = cp$

$\dfrac{dy}{dx} = -\dfrac{c^2}{c^2 p^2} = -\dfrac{1}{p^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(cp, \dfrac{c}{p}\right)$,

an equation of the tangent to the hyperbola is

$y - \dfrac{c}{p} = -\dfrac{1}{p^2}(x - cp)$

$y - \dfrac{c}{p} = -\dfrac{x}{p^2} + \dfrac{c}{p}$

$y + \dfrac{x}{p^2} = \dfrac{2c}{p}$

$(\times p^2)$ $p^2 y + x = 2cp$, as required. … ❶

The tangent crosses the x-axis at $y = 0$. You can put $y = 0$ into ❶ in your head and just write down the coordinates of S. No working is needed.

b $(2cp, 0)$

c Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = \left(cp, \dfrac{c}{p}\right)$ and $(x_2, y_2) = (2cp, 0)$

$PS^2 = (cp - 2cp)^2 + \left(\dfrac{c}{p} - 0\right)^2 = c^2 p^2 + \dfrac{c^2}{p^2}$

$= c^2\left(p^2 + \dfrac{1}{p^2}\right) = c^2\left(\dfrac{p^4 + 1}{p^2}\right)$

$PS = \dfrac{c}{p}(1 + p^4)^{\frac{1}{2}}$

There are many possible forms for this answer. Any equivalent form would gain full marks.

d To find the equation of the normal at P.

The working in part (a) shows the gradient of the tangent is $-\frac{1}{p^2}$.

Let the gradient of the normal be m'.

Using $mm' = -1$,

$$-\frac{1}{p^2} \times m' = -1 \Rightarrow m' = p^2$$

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$,

an equation of the normal to the hyperbola at P is

$$y - \frac{c}{p} = p^2(x - cp)$$

$$= p^2x - cp^3$$

$$p^2x = y - \frac{c}{p} + cp^3$$

To find the x-coordinate of R, substitute $y = 0$

$$p^2x = -\frac{c}{p} + cp^3 \Rightarrow x = cp - \frac{c}{p^3}$$

To find an expression for the area of the triangle you can obtain the length of the side RS and use that as the base of the triangle in the formula for the area of the triangle. First you need to obtain an equation of the normal and use it to find the coordinates of R.

$$RS = 2cp - \left(cp - \frac{c}{p^3}\right) = cp + \frac{c}{p^3} = c\left(\frac{p^4+1}{p^3}\right)$$

$$\text{Area } \triangle RPS = \frac{1}{2}RS \times \text{height}$$

$$41c^2 = \frac{1}{2} \times c\left(\frac{p^4+1}{p^3}\right) \times \frac{c}{p}$$

$$= \frac{c^2}{2p^4}(p^4+1)$$

If RS is taken as the base of the triangle, the height of the triangle is the y-coordinate of P.

$$82p^4 = p^4 + 1 \Rightarrow p^4 = \frac{1}{81} \Rightarrow p = \frac{1}{3}$$

As $p > 0$ is given in the question, you need not consider the alternative solution $p = -\frac{1}{3}$.

The coordinates of P are $\left(cp, \frac{c}{p}\right) = \left(\frac{c}{3}, 3c\right)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 66

Question:

The curve C has equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the normal to C at the point $P(ap^2,\ 2ap),\ (p \neq 0)$ is $y + px = 2ap + ap^3$.

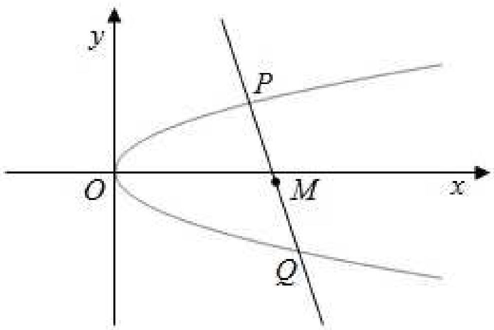
The normal at P meets C again at the point $Q(aq^2,\ 2aq)$.

b Find q in terms of p.

Given that the mid-point of PQ has coordinates $\left(\frac{125}{18}a\ ,\ -3a\right)$,

c use your answer to **b**, or otherwise, to find the value of p.

Solution:



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}p$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}p} = \frac{1}{p}$$

Using $mm' = -1$,

$$\frac{1}{p} \times m' = -1 \Rightarrow m' = -p$$

The normal is perpendicular to the tangent, so you must first find the gradient of the tangent. Then you use $mm' = -1$ to find the gradient of the normal.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (ap^2, 2ap)$,

an equation of the normal to the parabola at P is

$$y - 2ap = -p(x - ap^2)$$

$$= -px + ap^3$$

$y + xp = 2ap + ap^3$, as required.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b Let the coordinates of Q be $(aq^2, 2aq)$

The point Q lies on the normal at P, so

$$2aq + paq^2 = 2ap + ap^3$$

$$2aq - 2ap + apq^2 - ap^3 = 0$$

$$2a(q - p) + ap(q^2 - p^2) = 0$$

$$2a\cancel{(q - p)} + ap\cancel{(q - p)}(q + p) = 0$$

$$2 + p(q + p) = 0$$

$$2 + pq + p^2 = 0$$

$$pq = -p^2 - 2 \Rightarrow q = -p - \frac{2}{p}$$

As P and Q are different points, $p \neq q$ and it follows that $q - p \neq 0$. You can, therefore, divide throughout this line by $(q - p)$.

Any equivalent of this expression is acceptable, e.g. $q = -\frac{p^2 + 2}{p}$.

c The y-coordinate of the mid-point of PQ is given by

$$\frac{y_P + y_Q}{2} = \frac{2ap + 2aq}{2} = \frac{\cancel{2}a(p+q)}{\cancel{2}} = a(p+q)$$

You only need one equation to find p and so you do not need to consider both coordinates of the mid-point. Either would do, but it is sensible to choose the coordinate with the easier numbers. In this case, that is the y-coordinate.

The answer to part (b) is

$$q = -p - \frac{2}{p}$$

Therefore $p + q = -\frac{2}{p}$

The y-coordinate of the mid-point is

$$a(p+q) = a \times -\frac{2}{p} = -3a \text{, given.}$$

$$-\frac{2a}{p} = -3a \Rightarrow p = \frac{2}{3}$$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 67

Question:

The parabola C has equation $y^2 = 32x$.

a Write down the coordinates of the focus S of C.

b Write down the equation of the directrix of C.

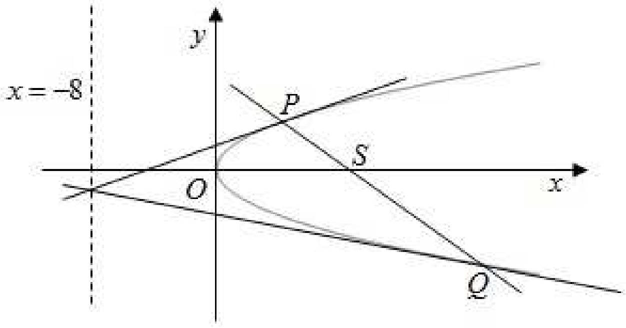
The points P (2, 8) and Q (32, −32) lie on C.

c Show that the line joining P and Q goes through S.

The tangent to C at P and the tangent to C at Q intersect at the point D.

d Show that D lies on the directrix of C.

Solution:



a $(8, 0)$

b $x = -8$

If $y^2 = 4ax$, the focus has coordinates $(a, 0)$ and the directrix has equation $x = -a$. Comparison of $y^2 = 4ax$ with $y^2 = 32x$, shows that, in this case, $a = 8$.

c Using $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$ with $(x_1, y_1) = (2, 8)$ and $(x_2, y_2) = (32, -32)$, an equation of PQ is

$$\frac{y-8}{-32-8} = \frac{x-2}{32-2}$$

$$\frac{y-8}{-4\cancel{0}} = \frac{x-2}{3\cancel{0}}$$

$$3y - 24 = -4x + 8$$

$$3y + 4x = 32$$

Substitute $y = 0$

$$0 + 4x = 32 \Rightarrow x = 8$$

The coordinates of S $(8, 0)$ satisfy the equation of PQ.
Hence S lies on the line joining P and Q.

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A-Level Modular Mathematics, Core Mathematics 1. You can use any correct method for finding the line.

d $y^2 = 32x \Rightarrow y = \pm 4\sqrt{2}x^{\frac{1}{2}}$

$\sqrt{32} = \sqrt{(16\times 2)} = \sqrt{16}\times\sqrt{2} = 4\sqrt{2}$

P is on the upper half of the parabola where $y = +4\sqrt{2}x^{\frac{1}{2}}$

On the upper half of the parabola, in the first quadrant, the y-coordinates of P are positive.

$$\frac{dy}{dx} = \frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = \frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 2$, $\frac{dy}{dx} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

Using $y - y_1 = m(x - x_1)$, the tangent to C at P is

$$y - 8 = 2(x - 2) = 2x - 4$$

$$y = 2x + 4 \quad \ldots\ldots ❶$$

On the lower half of the parabola, in the fourth quadrant, the y-coordinates of P are negative.

Q is on the lower half of the parabola where $y = -4\sqrt{2}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = -\frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = -\frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 32$, $\frac{dy}{dx} = -\frac{2\sqrt{2}}{\sqrt{32}} = -\frac{2\sqrt{2}}{4\sqrt{2}} = -\frac{1}{2}$

Using $y - y_1 = m(x - x_1)$, the tangent to C at Q is

$$y + 32 = -\frac{1}{2}(x - 32) = -\frac{1}{2}x + 16$$

$$y = -\frac{1}{2}x - 16 \quad \ldots\ldots ❷$$

To find the x-coordinate of the intersection of the tangents, from ❶ and ❷

$$2x + 4 = -\frac{1}{2}x - 16$$

$$\frac{5}{2}x = -20 \Rightarrow x = -20\times\frac{2}{5} = -8$$

The equation of the directrix is $x = -8$ and, hence, the intersection of the tangents lies on the directrix.