Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	3	2	3
A plays 2	−2	1	3
A plays 3	4	2	1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution for this game and determine the saddle point.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	3	2	3	2	←
A plays 2	−2	1	3	−2	
A plays 3	4	2	1	1	
Column max	4	2	3		
		↑			

A should play 1 (row maximin = 2)
B should play 2 (column minimax = 2)

b row maximin = 2 = column minimax
$\therefore$ game is stable

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Exercise A, Question 2

Question:

Robert and Steve play a zero-sum game. This game is represented by the following pay-off matrix for Robert.

	Steve plays 1	Steve plays 2	Steve plays 3	Steve plays 4
Robert plays 1	−2	−1	−3	1
Robert plays 2	2	3	1	−2
Robert plays 3	1	1	−1	3

a Determine the play safe strategy for each player.
b Verify that there is no stable solution for this game.

Solution:

a

	S plays 1	S plays 2	S plays 3	S plays 4	Row min	
R plays 1	−2	−1	−3	1	−3	
R plays 2	2	3	1	−2	−2	
R plays 3	1	1	−1	3	−1	←
Column max	2	3	1	3		
			↑			

R should play 3 (row maximin $= -1$)
S should play 3 (column minimax $= 1$)

b row maximin ≠ column minimax

$$-1 \neq 1$$

so game is not stable

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Exercise A, Question 3

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	−3	−2	2
A plays 2	−1	−1	3
A plays 3	4	−3	1
A plays 4	3	−1	−1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution for this game and determine the saddle points.
c State the value of the game to player A.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	−3	−2	2	−3	
A plays 2	−1	−1	3	−1	←
A plays 3	4	−3	1	−3	
A plays 4	3	−1	−1	−1	←
Column max	4	−1	3		
		↑			

A should play 2 or 4 (row maximin −1)
B should play 2 (column minimax −1)

b Since row maximin = column minimax

$$-1 = -1$$

game is stable

Saddle points are (A2, B2) and (A4, B2).

c Value of the game is −1 to A (if A players 2 or 4 and B plays 2 the value of the game is −1).

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Exercise A, Question 4

Question:

Claire and David play a two person zero-sum game, which is represented by the following pay-off matrix for Claire.

	D plays 1	D plays 2	D plays 3	D plays 4
C plays 1	7	2	−3	5
C plays 2	4	−1	1	3
C plays 3	−2	5	2	−1
C plays 4	3	−3	−4	2

a Determine the play safe strategy for each player.
b Verify that there is no stable solution for this game.
c State the value of the game for Claire if both players play safe.
d State the value of the game for David if both players play safe.
e Determine the pay-off matrix for David.

Solution:

a

	D plays 1	D plays 2	D plays 3	D plays 4	Row min	
C plays 1	7	2	−3	5	−3	
C plays 2	4	−1	1	3	−1	←
C plays 3	−2	5	2	−1	−2	
C plays 4	3	−3	−4	2	−4	
Column max	7	5	2	5		
			↑			

C plays 2 (row maximin $= -1$)
D plays 3 (column minimax $= 2$)

b $-1 \neq 2$
row maximin ≠ column minimax
so no stable solution

c If C plays 2 and D plays 3, the value of the game is 1 to Claire

d *either* since the value of the game is 1 to Claire and it is a zero-sum game, the value of the game must be −1 to David
or
If C plays 2 and D plays 3 Claire wins 1, so David wins −1

e

	C plays 1	C plays 2	C plays 3	C plays 4
D plays 1	−7	−4	2	−3
D plays 2	−2	1	−5	3
D plays 3	3	−1	−2	4
D plays 4	5	−3	1	−2

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Exercise A, Question 5

Question:

Hilary and Denis play a two person zero-sum game, which is represented by the following pay-off matrix for Hilary.

	D plays 1	D plays 2	D plays 3	D plays 4	D plays 5
H plays 1	2	1	0	0	2
H plays 2	4	0	0	0	2
H plays 3	1	4	−1	−1	3
H plays 4	1	1	−1	−2	0
H plays 5	0	−2	−3	−3	−1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution for this game and state the saddle points.
c State the value of the game for Hilary if both players play safe.
d State the value of the game for Denis if both players play safe.
e Determine the pay-off matrix for Denis.

Solution:

	D plays 1	D plays 2	D plays 3	D plays 4	D plays 5	Row min
H plays 1	2	1	0	0	2	0 ←
H plays 2	4	0	0	0	2	0 ←
H plays 3	1	4	−1	−1	3	−1
H plays 4	1	1	−1	−2	0	−2
H plays 5	0	−2	−3	−3	−1	−3
Column max	4	4	0	0	3	
			↑	↑		

a H plays 1 or 2
D plays 3 or 4

b row maximin = column minimax

$$0 = 0$$

so game stable

saddle points (H1, D3) (H2, D3) (H1, D4) (H2, D4)

c The value of the game to Hilary = 0

d The value of the game to Denis = 0

e

	H plays 1	H plays 2	H plays 3	H plays 4	H plays 5
D plays 1	−2	−4	−1	−1	0
D plays 2	−1	0	−4	−1	2
D plays 3	0	0	1	1	3
D plays 4	0	0	1	2	3
D plays 5	−2	−2	−3	0	1

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Exercise B, Question 1

Question:

	Freya plays 1	Freya plays 2
Ellie plays 1	1	−5
Ellie plays 2	−1	6
Ellie plays 3	3	−3

Ellie and Freya play a zero-sum game, represented by the pay-off matrix for Ellie shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Row 3 dominates row 1 $(3 > 1, -3 > -5)$ so game can be reduced to

Ellie would always choose to play row 3 over row 1

	Freya plays 1	Freya plays 2
Ellie plays 2	−1	6
Ellie plays 3	3	−3

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Exercise B, Question 2

Question:

	Harry plays 1	Harry plays 2	Harry plays 3
Doug plays 1	−5	2	−1
Doug plays 2	2	−3	−6

Doug and Harry play a zero-sum game, represented by the pay-off matrix for Doug shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Column 3 dominates 2 $(-1 < 2 \quad -6 < -3)$

	Harry plays 1	Harry plays 3
Doug plays 1	−5	−1
Doug plays 2	2	−6

Harry would always choose to play 3 over 1

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Exercise B, Question 3

Question:

	Nick plays 1	Nick plays 2	Nick plays 3
Chris plays 1	1	2	3
Chris plays 2	−1	−3	1
Chris plays 3	2	−1	5

Chris and Nick play a zero-sum game, represented by the pay-off matrix for Chris shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Row 1 dominates row 2 $(1 > -1, 2 > -3, 3 > 1)$

Chris would always choose to play 1 over 2

Column 1 (or column 2) dominates column 3

$(1 < 3, -1 < 1, 2 < 5$ or $2 < 3, -3 < 1, -1 < 5$

	Nick plays 1	Nick plays 2
Chris plays 1	1	2
Chris plays 3	2	−1

Nick would always choose 1 (or 2) over 3

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Exercise B, Question 4

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	2	−4
A plays 2	−1	3

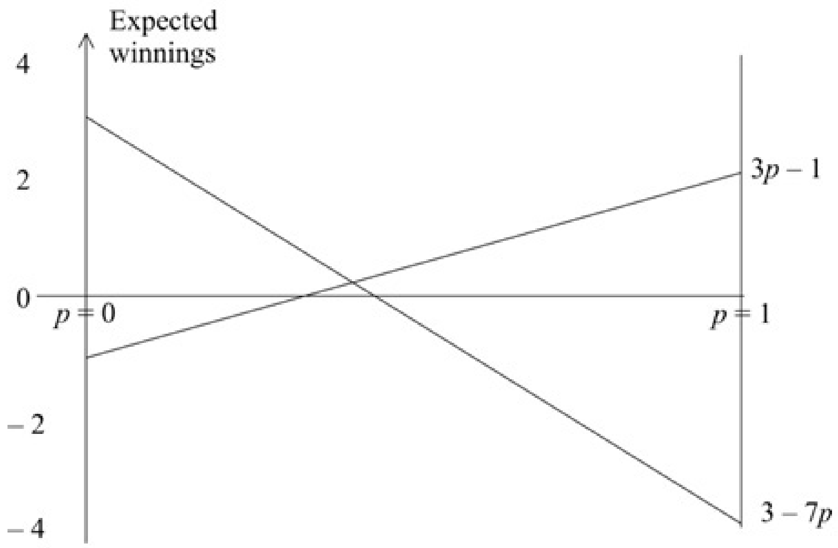
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	2	−4	−4	
A plays 2	−1	3	−1	←
Column max	2	3		
	↑			

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winning are $2p-1(1-p)=3p-1$
If B plays 2 A's expected winnings are $4p+3(1-p)=3-7p$



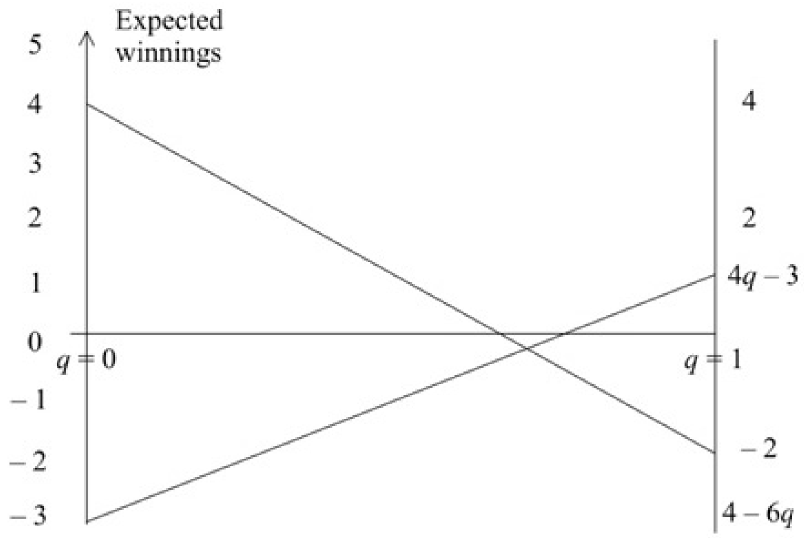
$3p-1=3-7p$
$10p=4$
$p=\frac{2}{5}$

A should play 1 with probability $\frac{2}{5}$

A should play 2 with probability $\frac{3}{5}$

The value of the game to A is $3(\frac{2}{5})-1=\frac{1}{5}$

c Let B play 1 with probability q
so B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[2q-4(1-q)]=4-6q$
If A plays 2 B's expected winnings are $-[-q+3(1-q)]=4q-3$



$$4-6q=4q-3$$
$$10q=7$$
$$q=\frac{7}{10}$$

B should play 1 with probability $\frac{7}{10}$

B should play 2 with probability $\frac{3}{10}$

The value of the game to B is $4(\frac{3}{10})-3=\frac{-1}{5}$

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Exercise B, Question 5

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−3	5
A plays 2	2	−4

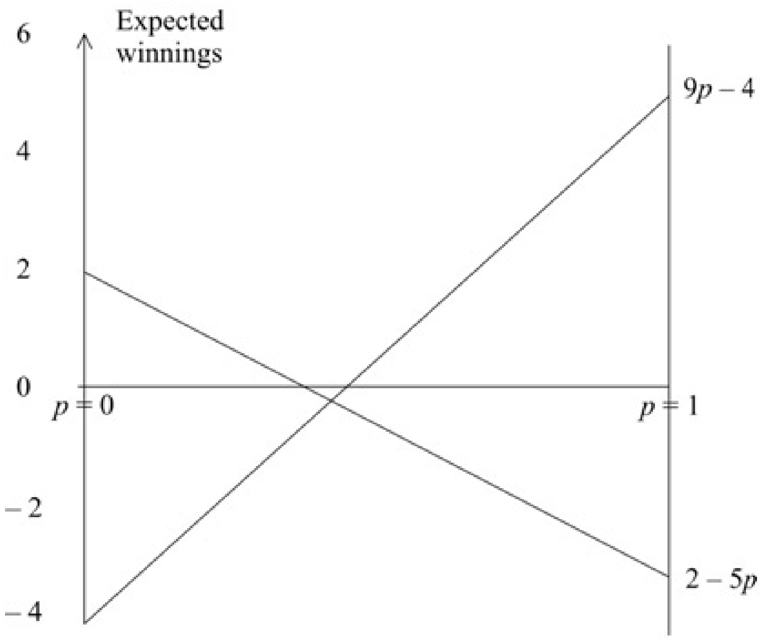
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−3	5	−3	←
B plays 2	2	−4	−4	
Column max	2	5		
	↑			

Since $2 \neq -3$ (column minimax ≠ row maximin) the game is not stable

b Let A play row 1 with probability p
So A plays row 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-3p+2(1-p)=2-5p$
If B plays 2 A's expected winnings are $5p-4(1-p)=9p-4$



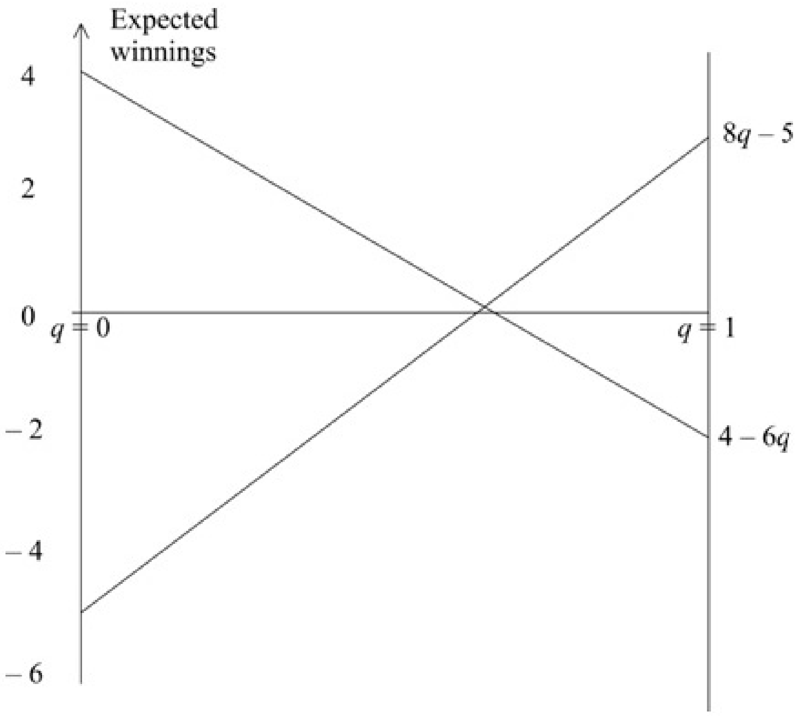
$$2-5p=9p-4$$
$$14p=6$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with proability $\frac{4}{7}$

The value of the game to A is $2-5\left(\frac{3}{7}\right)=\frac{-1}{7}$

c Let B play column 1 with probability q
So B plays column 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-3q+5(1-q)]=8q-5$
If A plays 2 B's expected winning are $-[2q-4(1-q)]=4-6q$



$8q-5=4-6q$

$14q=9$

$q=\frac{9}{14}$

B should play 1 with probability $\frac{9}{14}$

B should play 2 with probability $\frac{5}{14}$

The value of the game to B is $8(\frac{9}{14})-5=\frac{1}{7}$.

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Exercise B, Question 6

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	5	−1
A plays 2	−2	1

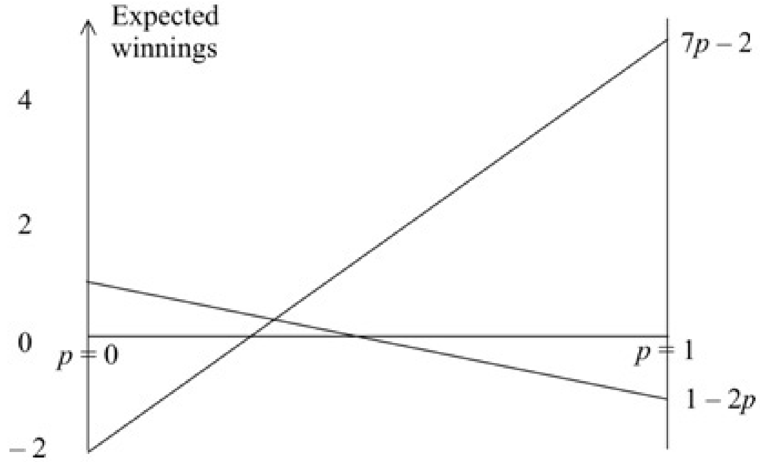
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	5	−1	−1	←
A plays 2	−2	1	−2	
Column max	5	1		
		↑		

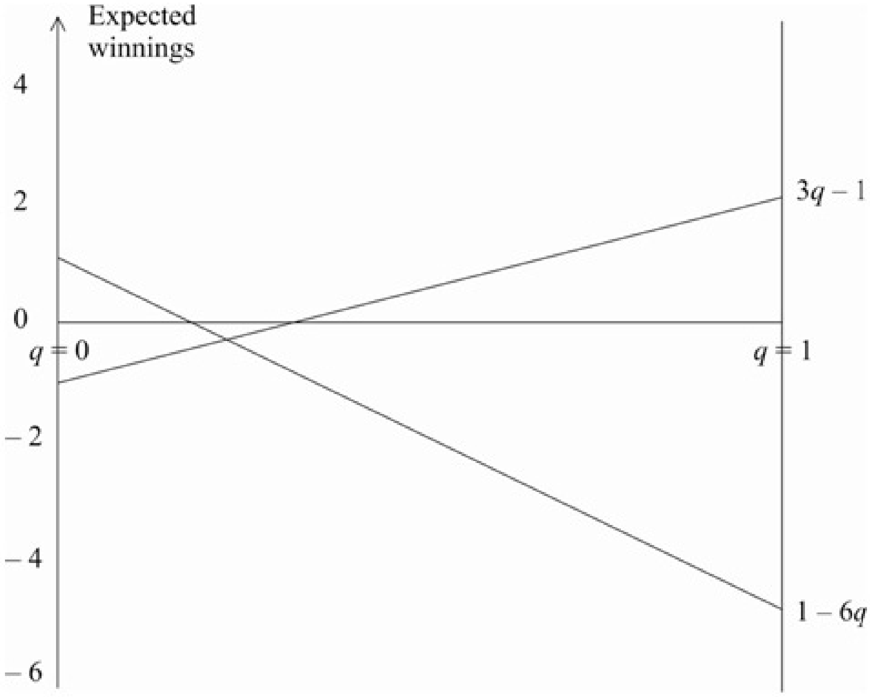
Since $-1 \neq 1$ (column minimax ≠ row maximin) the game is not stable

b Let A play row 1 with probability p
So A plays rows 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $5p-2(1-p)=7p-2$
If B plays 2 A's expected winnings are $-p+1(1-p)=1-2p$



$7p-2=1-2p$
$9p=3$
$p=\frac{1}{3}$
A should play 1 with probability $\frac{1}{3}$
A should play 2 with probability $\frac{2}{3}$
The value of the game to A is $7(\frac{1}{3})-2=\frac{1}{3}$

c Let B play column 1 with probability q
so B plays column 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[5q-1(1-q)]=1-6q$
If A plays 2 B's expected winning are $-[-2q+1(1-q)]=3q-1$



$3q - 1 = 1 - 6q$

$9q = 2$

$q = \frac{2}{9}$

B should play 1 with probability $\frac{2}{9}$

B should play 2 with probability $\frac{7}{9}$

The value of the game to B is

$1 - 6(\frac{2}{9}) = -\frac{1}{3}$

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Exercise B, Question 7

Question:

a Verify that there is no stable solution.

b Determine the optimal mixed strategy and the value of the game to A.

c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−1	3
A plays 2	1	−2

Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−1	3	−1	←
A plays 2	1	−2	−2	
Column max	1	3		
	↑			

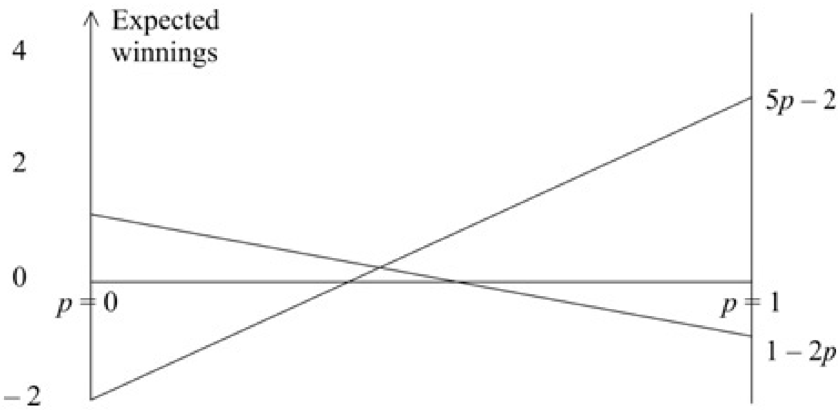
Since $1 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p

So A plays 2 with probability $(1-p)$

If B plays 1 A's expected winnings are $-p+(1-p)=1-2p$

If B plays 2 A's expected winnings are $3p-2(1-p)=5p-2$



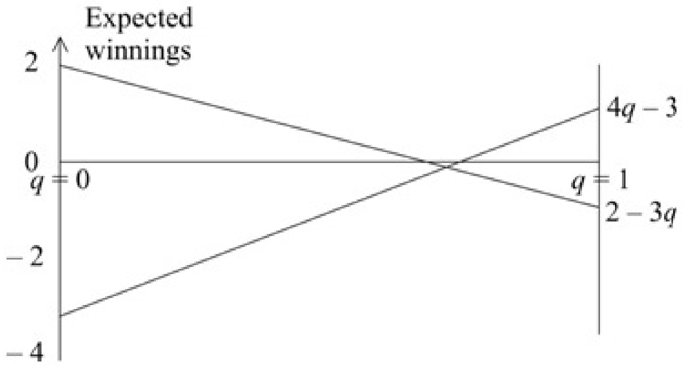
$$5p-2=1-2p$$
$$7p=3$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with probability $\frac{4}{7}$

The value of the game to A is $1-2(\frac{3}{7})=\frac{1}{7}$

c Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-q+3(1-q)]=4q-3$
If A plays 2 B's expected winnings are $-[q-2(1-q)]=2-3q$



$$4q-3=2-3q$$
$$7q=5$$
$$q=\frac{5}{7}$$

B should play 1 with probability $\frac{5}{7}$

B should play 2 with probability $\frac{2}{7}$

The value of the game to B is $4\left(\frac{5}{7}\right)-3=-\frac{1}{7}$

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Exercise C, Question 1

Question:

a Verify that there is no stable solution.

b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	2	2
A plays 2	1	−3	−4

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	−5	2	2	−5	
A plays 2	1	−3	−4	−4	←
Column max	1	2	2		
	↑				

Since $1 \neq -4$ (column minimax $\neq$ row maximin) the game is not stable

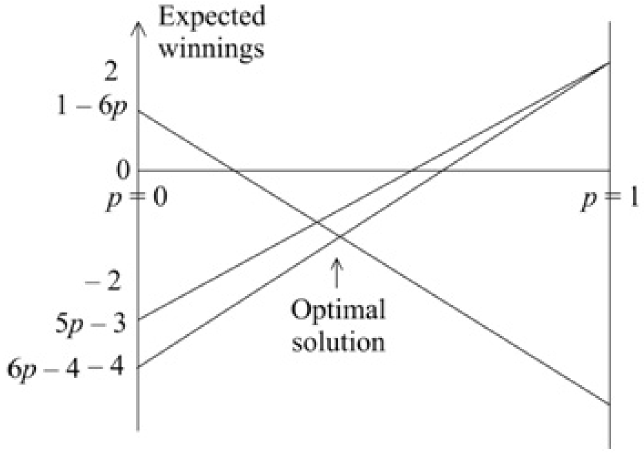
b Let A play 1 with probability p

So A plays 2 with probability $(1-p)$

If B plays 1 A's expected winnings are $-5p+1(1-p)=1-6p$

If B plays 2 A's expected winnings are $2p-3(1-p)=5p-3$

If B plays 3 A's expected winnings are $2p-4(1-p)=6p-4$



$$6p-4=1-6p$$
$$12p=5$$
$$p=\frac{5}{12}$$

A should play 1 with probability $\frac{5}{12}$

A should play 2 with probability $\frac{7}{12}$

The value of the game to A is

$$1-6\left(\frac{5}{12}\right)=-\frac{3}{2}$$

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Exercise C, Question 2

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	2	6	−2
A plays 2	−1	−4	3

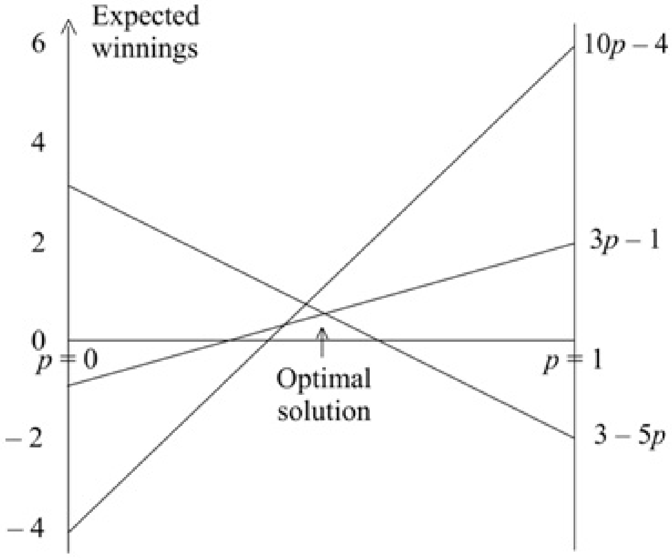
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	6	−2	−2	←
A plays 2	−1	−4	3	−4	
Column max	2	6	3		
	↑				

Since $2 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $2p-(1-p)=3p-1$
If B plays 2 A's expected winnings are $6p-4(1-p)=10p-4$
If B plays 3 A's expected winnings are $-2p+3(1-p)=3-5p$



$$3p-1=3-5p$$
$$8p=4$$
$$p=\frac{1}{2}$$

A should play 1 with probability $\frac{1}{2}$
A should play 2 with probability $\frac{1}{2}$
The value of the game to A is $3(\frac{1}{2})-1=\frac{1}{2}$

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Exercise C, Question 3

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	−2	3	6
A plays 2	5	1	−4

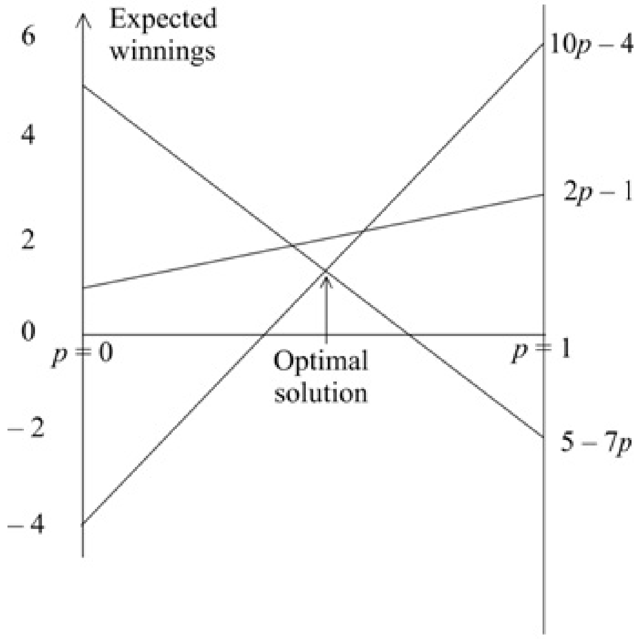
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	−2	3	6	−2	←
A plays 2	5	1	−4	−4	
Column max	5	3	6		
		↑			

Since $3 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-2p+5(1-p)=5-7p$
If B plays 2 A's expected winnings are $3p+1(1-p)=2p+1$
If B plays 3 A's expected winnings are $6p-4(1-p)=10p-4$



$10p-4=5-7p$
$17p=9$
$p=\frac{9}{17}$

A should play 1 with probability $\frac{9}{17}$
A should play 2 with probability $\frac{8}{17}$
The value of the game to A is
$10(\frac{9}{17})-4=\frac{22}{17}$

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Exercise C, Question 4

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	5	−2	−4
A plays 2	−3	1	6

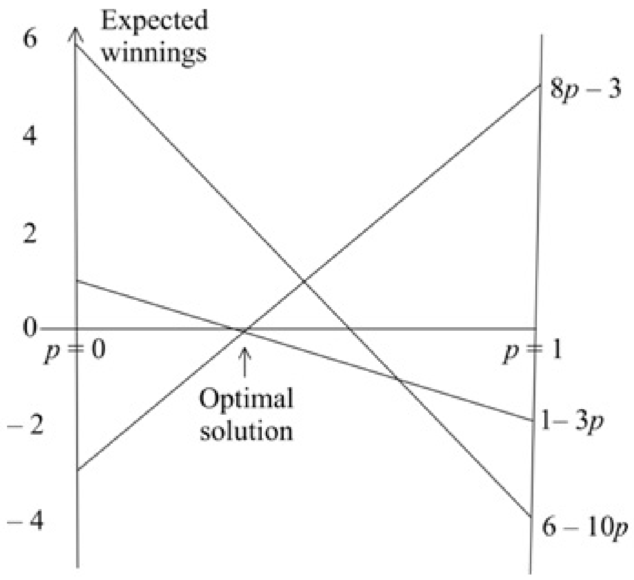
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	5	−2	−4	−4	
A plays 2	−3	1	6	−3	←
Column max	5	1	6		
		↑			

Since $1 \neq -3$ (column minimax ≠ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $5p-3(1-p)=8p-3$
If B plays 2 A's expected winnings are $-2p+1(1-p)=1-3p$
If B plays 3 A's expected winnings are $-4p+6(1-p)=6-10p$



$8p-3=1-3p$
$11p=4$
$p=\frac{4}{11}$

A should play 1 with probability $\frac{4}{11}$
A should play 2 with probability $\frac{7}{11}$
The value of the game to A is $1-3(\frac{4}{11})=-\frac{1}{11}$

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Exercise C, Question 5

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−1	1
A plays 2	3	−4
A plays 3	−2	2

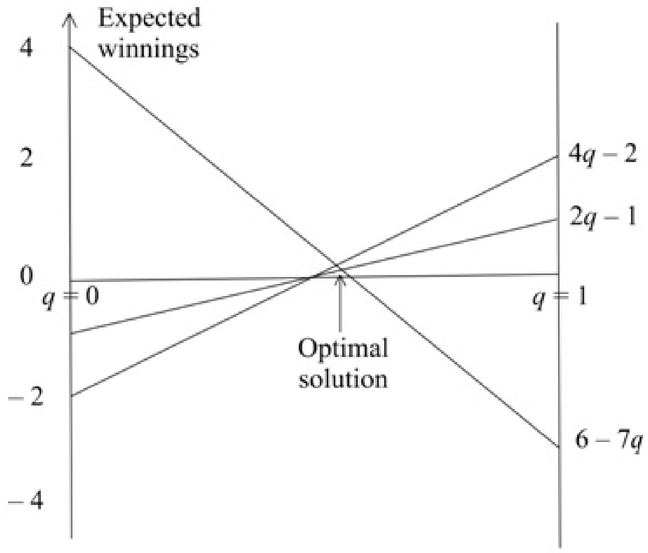
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−1	1	−1	←
A plays 2	3	−4	−4	
A plays 3	−2	2	−2	
Column max	3	2		
		↑		

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-q+1(1-q)]=2q-1$
If A plays 2 B's expected winnings are $-[3q-4(1-q)]=4-7q$
If A plays 3 B's expected winnings are $-[-2q+2(1-q)]=4q-2$



$$2q-1=4-7q$$
$$9q=5$$
$$q=\frac{5}{9}$$

B should play 1 with probability $\frac{5}{9}$

B should play 2 with probability $\frac{4}{9}$

The value of the game to B is $2(\frac{5}{9})-1=\frac{1}{9}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 6

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−5	4
A plays 2	3	−3
A plays 3	1	−2

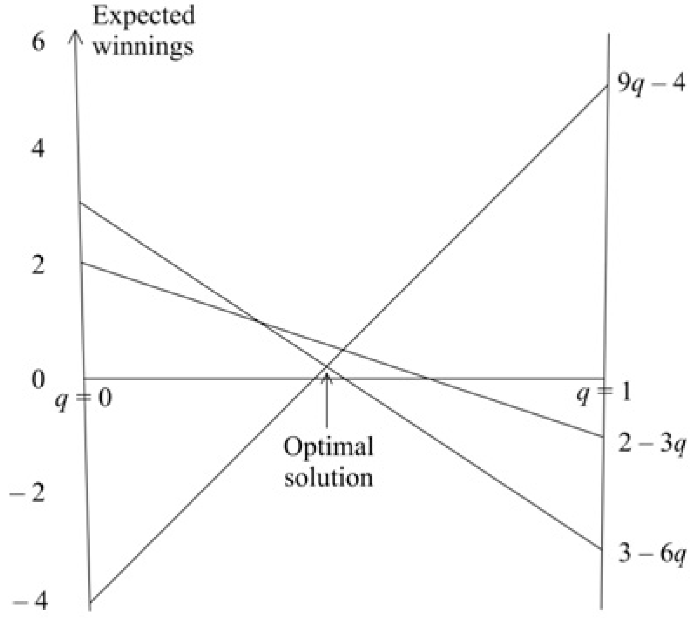
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−5	4	−5	
A plays 2	3	−3	−3	
A plays 3	1	−2	−2	←
Column max	3	4		
	↑			

Since $3 \neq -2$ (column minimax ≠ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-5q+4(1-q)] = 9q-4$
If A plays 2 B's expected winnings are $-[3q-3(1-q)] = 3-6q$
If A plays 3 B's expected winnings are $-[q-2(1-q)] = 2-3q$



$$9q-4 = 3-6q$$
$$15q = 7$$
$$q = \frac{7}{15}$$

B should play 1 with probability $\frac{7}{15}$
B should play 2 with probability $\frac{8}{15}$
The value of the game to B is $9(\frac{7}{15})-4 = \frac{3}{15}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 7

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−3	2
A plays 2	−1	−2
A plays 3	2	−4

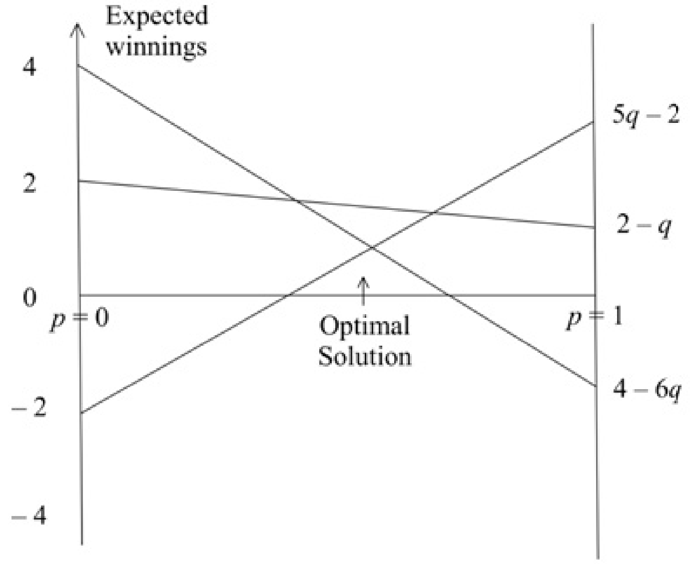
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−3	2	−3	
A plays 2	−1	−2	−2	←
A plays 3	2	−4	−4	
Column max	2	2		
	↑	↑		

Since $2 \neq -2$ (column minimax ≠ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-3q+2(1-q)] = 5q-2$
If A plays 2 B's expected winnings are $-[-q-2(1-q)] = 2-q$
If A plays 3 B's expected winnings are $-[2q-4(1-q)] = 4-6q$



$5q-2 = 4-6q$
$11q = 6$
$q = \frac{6}{11}$

B should play 1 with probability $\frac{6}{11}$
B should play 2 with probability $\frac{5}{11}$
The value of the game to B is $5(\frac{6}{11})-2 = \frac{8}{11}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 8

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	2	−3
A plays 2	−2	4
A plays 3	1	−1

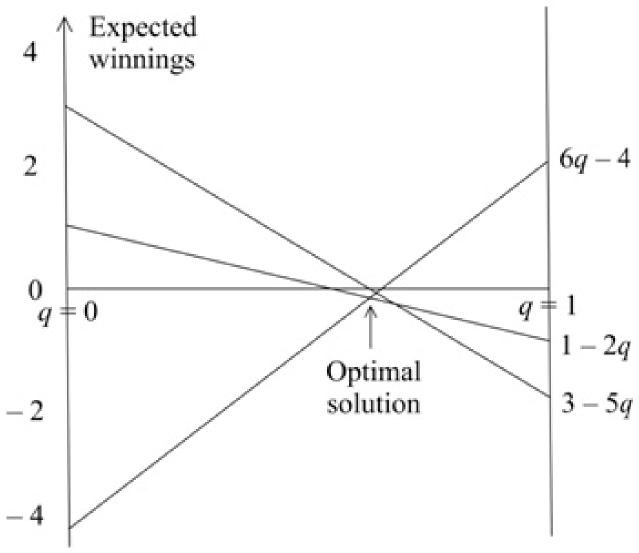
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	2	−3	−3	
A plays 2	−2	4	−2	
A plays 3	1	−1	−1	←
Column max	2	4		
	↑			

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[2q-3(1-q)]=3-5q$
If A plays 2 B's expected winnings are $-[-2q+4(1-q)]=6q-4$
If A plays 3 B's expected winnings are $-[q-1(1-q)]=1-2q$



$6q-4=1-2q$
$8q=5$
$q=\frac{5}{8}$

B should play 1 with probability $\frac{5}{8}$

B should play 2 with probability $\frac{3}{8}$

The value of the game to B is $6(\frac{5}{8})-4=-\frac{1}{4}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 1

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2
A plays 1	−1	1
A plays 2	3	−4
A plays 3	−2	2

Solution:

Add 5 to all elements

	B plays 1	B plays 2
A plays 1	4	6
A plays 2	8	1
A plays 3	3	7

Let A play 1 with probability p_1
and A play 2 with probability p_2
and A play 3 with probability p_3
Let the value of the game to A be v and $V = v + 5$

Maximise $P = V$

Subject to $4p_1 + 8p_2 + 3p_3 \geq V \Rightarrow V - 4p_1 - 8p_2 - 3p_3 + r = 0$

$6p_1 + p_2 + 7p_3 \geq V \Rightarrow V - 6p_2 - p_2 - 7p_3 + s = 0$

$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + t = 1$

$p_1, p_2, p_3, r, s, t \geq 0$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 2

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	4	1
A plays 2	3	−3	2
A plays 3	1	−2	−1

Solution:

Add 6 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	1	10	7
A plays 2	9	3	8
A plays 3	7	4	5

Let A play 1 with probability p_1
and A play 2 with probability p_2
and A play 3 with probability p_3
Let the value of the game to A be v and $V = v + 6$

Maximise $P = V$

Subject to

$$p_1 + 9p_2 + 7p_3 \geq V \Rightarrow V - p_1 - 9p_2 - 7p_2 + r = 0$$
$$10p_1 + 3p_2 + 4p_3 \geq V \Rightarrow V - 10p_1 - 3p_2 - 4p_3 + s = 0$$
$$7p_1 + 8p_2 + 5p_3 \geq V \Rightarrow V - 7p_1 - 8p_2 - 5p_3 + t = 0$$
$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$
$$p_1, p_2, p_3, r, s, t, u \geq 0$$

Solutionbank D2

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Exercise D, Question 3

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−3	2	−1
A plays 2	−1	−2	1
A plays 3	2	−4	−2

Solution:

Add 5 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	2	7	4
A plays 2	4	3	6
A plays 3	7	1	3

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v and $V = v + 5$

Maximise $P = V$

Subject to

$$2p_1 + 4p_2 + 7p_3 \geq V \Rightarrow V - 2p_1 - 4p_2 - 7p_3 + r = 0$$

$$7p_1 + 3p_2 + p_3 \geq V \Rightarrow V - 7p_1 - 3p_2 - p_3 + s = 0$$

$$4p_1 + 6p_2 + 3p_3 \geq V \Rightarrow V - 4p_1 - 6p_2 - 3p_3 + t = 0$$

$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$

$$p_1, p_2, p_3, r, s, t, u \geq 0$$

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Exercise D, Question 4

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−3	−1
A plays 2	−2	4	1
A plays 3	1	−1	0

Solution:

Add 4 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	6	1	3
A plays 2	2	8	5
A plays 3	5	3	4

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v and $V = v + 4$

Maximise $P = V$

Subject to

$$6p_1 + 2p_2 + 5p_3 \geq V \Rightarrow V - 6p_1 - 2p_2 - 5p_3 + r = 0$$

$$p_1 + 8p_2 + 3p_3 \geq V \Rightarrow V - p_1 - 8p_2 - 3p_3 + s = 0$$

$$3p_1 + 5p_2 + 4p_3 \geq V \Rightarrow V - 3p_1 - 5p_2 - 4p_3 + t = 0$$

$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$

$$p_1, p_2, p_3, r, s, t, u \geq 0$$

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Exercise D, Question 5

Question:

Formulate the game below as a linear programming problem for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	2	3
A plays 2	1	−3	−4

Solution:

	A plays 1	A plays 2			A plays 1	A plays 2
B plays 1	5	−1	Adding	B plays 1	9	3
B plays 2	−2	3	4 to all	B plays 2	2	7
B plays 3	−3	4	elements	B plays 3	1	8

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 4$

Maximise $P = V$

Subject to

$$9q_1 + 2q_2 + q_3 \geq V \quad V - 9q_1 - 2q_2 - q_3 + r = 0$$

$$3q_1 + 7q_2 + 8q_3 \geq V \quad V - 3q_1 - 7q_2 - 8q_3 + s = 0$$

$$q_1 + q_2 + q_3 \leq 1 \quad q_1 + q_2 + q_3 + t = 1$$

$$q_1, q_2, q_3, r, s, t \geq 0$$

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Exercise D, Question 6

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	4	1
A plays 2	3	−3	2
A plays 3	1	−2	−1

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	5	−3	−1	Adding 5	B plays 1	10	2	4
B plays 2	−4	3	2	to all	B plays 2	1	8	7
B plays 3	−1	−2	1	elements	B plays 3	4	3	6

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 5$

Maximise $P = V$

Subject to

$$10q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 10q_1 - q_2 - 4q_3 + r = 0$$

$$2q_1 + 8q_2 + 3q_3 \geq V \Rightarrow V - 2q_1 - 8q_2 - 3q_3 + s = 0$$

$$4q_1 + 7q_2 + 6q_3 \geq V \Rightarrow V - 4q_1 - 7q_2 - 6q_3 + t = 0$$

$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$

$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 7

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−3	2	−1
A plays 2	−1	−2	1
A plays 3	2	−4	−2

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	3	1	−2	Adding 3	B plays 1	6	4	1
B plays 2	−2	2	4	to all	B plays 2	1	5	7
B plays 3	1	−1	2	elements	B plays 3	4	2	5

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 3$

Maximise $P = V$

Subject to:

$$6q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 6q_1 - q_2 - 4q_3 + r = 0$$
$$4q_1 + 5q_2 + 2q_3 \geq V \Rightarrow V - 4q_1 - 5q_2 - 2q_3 + s = 0$$
$$q_1 + 7q_2 + 5q_3 \geq V \Rightarrow V - q_1 - 7q_2 - 5q_3 + t = 0$$
$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$
$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 8

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−3	−1
A plays 2	−2	4	1
A plays 3	1	−1	0

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	−2	2	−1	Adding	B plays 1	3	7	4
B plays 2	3	−4	1	5 to all	B plays 2	8	1	6
B plays 3	1	−1	0	elements	B plays 3	6	4	5

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 5$

Maximise $P = V$

Subject to:

$$3q_1 + 8q_2 + 6q_3 \geq v \Rightarrow V - 3q_1 - 8q_2 - 6q_3 + r = 0$$

$$7q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 7q_1 - q_2 - 4q_3 + s = 0$$

$$4q_1 + 6q_2 + 5q_3 \geq 1v \Rightarrow V - 4q_1 - 6q_2 - 5q_3 + t = 0$$

$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$

$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 9

Question:

Using your answer to question **1**,

a write down an initial simplex tableau to solve the zero-sum game below, for player A,

b use the simplex algorithm to determine A's best strategy.

	B plays 1	B plays 2
A plays 1	−1	1
A plays 2	3	−4
A plays 3	−2	2

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	value
r	(1)	−4	−8	−3	1	0	0	0
s	1	−6	−1	−7	0	1	0	0
t	0	1	1	1	0	0	1	1
P	−1	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	value	
V	1	−4	−8	−3	1	0	0	0	R1÷1
s	0	−2	7	−4	−1	1	0	0	R2−R1
t	0	(1)	1	1	0	0	1	1	R3 no change
P	0	−4	−8	−3	1	0	0	0	R4+R1

b.v.	V	p_1	p_2	p_3	r	s	t	values	
V	1	0	−4	1	1	0	3	3	R1+4R3
s	0	0	(9)	−2	−1	1	2	2	R2+2R3
p_1	0	1	1	1	0	0	1	1	R3 no change
P	0	0	−4	1	1	0	4	4	R4+4R3

b.v.	V	p_1	p_2	p_3	r	s	t	value	
V	1	0	0	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{35}{9}$	$\frac{44}{9}$	R1+4R2
p_2	0	0	1	$\frac{-2}{9}$	$\frac{-1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	R2÷9
p_1	0	1	0	$\frac{11}{9}$	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{7}{9}$	$\frac{7}{9}$	R3−R2
P	0	0	0	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{35}{9}$	$\frac{44}{9}$	R4+4R2

$V = \frac{44}{9}$ so $v = \frac{44}{9} - 5 = \frac{-1}{9}$ $\quad p_1 = \frac{7}{9}$ $\quad p_2 = \frac{2}{9}$ $\quad p_3 = 0$

A should play 1 with probability $\frac{7}{9}$, play 2 with probability $\frac{2}{9}$ and play 3 never

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Exercise D, Question 10

Question:

Using your answer to question **5**,

a write down an initial simplex tableau to solve the zero-sum game below, for player B,

b use the simplex algorithm to determine B's best strategy.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	2	3
A plays 2	1	−3	−4

Solution:

a

b.v.	V	q_1	q_2	q_3	r	s	t	value
r	(1)	−9	−2	−1	1	0	0	0
s	1	−3	−7	−8	0	1	0	0
t	0	1	1	1	0	0	1	1
P	−1	0	0	0	0	0	0	0

b

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
V	1	−9	−2	−1	1	0	0	0	R2 ÷ 1
s	0	(6)	−5	−7	−1	1	0	0	R2 − R1
t	0	1	1	1	0	0	1	1	R3 no change
P	0	−9	−2	−1	1	0	0	0	R4 + R1

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
v	1	0	$\frac{-19}{2}$	$\frac{-23}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	R1 + 9R2
q_1	0	1	$\frac{-5}{6}$	$\frac{-7}{6}$	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	R2 ÷ 6
t	0	0	$\frac{11}{6}$	($\frac{13}{6}$)	$\frac{1}{6}$	$\frac{-1}{6}$	1	1	R3 − R2
P	0	0	$\frac{-19}{2}$	$\frac{-23}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	R4 + 9R2

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
v	1	0	$\frac{3}{13}$	0	$\frac{5}{13}$	$\frac{8}{13}$	$\frac{69}{13}$	$\frac{69}{13}$	$R1+\frac{23}{2}R3$
q_1	0	1	$\frac{2}{13}$	0	$\frac{-1}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	$\frac{7}{13}$	$R2+\frac{7}{6}R3$
q_3	0	0	$\frac{11}{13}$	1	$\frac{1}{13}$	$\frac{-1}{13}$	$\frac{6}{13}$	$\frac{6}{13}$	$R3\div\frac{15}{6}$
P	0	0	$\frac{3}{13}$	0	$\frac{5}{13}$	$\frac{8}{13}$	$\frac{69}{13}$	$\frac{69}{13}$	$R4+\frac{23}{2}R3$

$V=\frac{69}{13}$ so $V=\frac{69}{13}-4=\frac{17}{13}$ $\quad q_1=\frac{7}{13}$ $\quad q_2=0$ $\quad q_3=\frac{6}{13}$

B should play 1 with probability $\frac{7}{13}$, play 2 never and play 3 with probability $\frac{6}{13}$

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Exercise D, Question 11

Question:

	B plays 1	B plays 2	B plays 3
A plays 1	−5	4	1
A plays 2	3	−3	2
A plays 3	1	−2	−1

Using your answer to question **2**,

a write down an initial simplex tableau to solve the zero-sum game, for player A,

b use the simplex algorithm to determine A's best strategy.

Using your answer to question **6**,

c write down an initial simplex tableau to solve the zero-sum game, for player B,

d use the simplex algorithm to determine B's best strategy.

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	u	value
r	(1)	−1	−9	−7	1	0	0	0	0
s	1	−10	−3	−4	0	1	0	0	0
t	1	−7	−8	−5	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	−1	−9	−7	1	0	0	0	0	R1 ÷ 1
s	0	−9	(6)	3	−1	1	0	0	0	R2 − R1
t	0	−6	1	2	−1	0	1	0	0	R3 − R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−1	−9	−7	1	0	0	0	0	R5 + R1

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	$\frac{-29}{2}$	0	$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	0	R1 + 9R2
p_2	0	$\frac{-3}{2}$	1	$\frac{1}{2}$	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	0	R2 ÷ 6
t	0	$\frac{-9}{2}$	0	$\frac{3}{2}$	$\frac{-5}{6}$	$\frac{-1}{6}$	1	0	0	R3 − R2
u	0	($\frac{5}{2}$)	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{-1}{6}$	0	1	1	R4 − R2
P	0	$\frac{-29}{2}$	0	$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	0	R5 + 9R2

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{8}{15}$	0	$\frac{29}{5}$	$\frac{29}{5}$	$R1+\frac{29}{2}R4$
p_2	0	0	1	$\frac{4}{5}$	$\frac{-1}{15}$	$\frac{1}{15}$	0	$\frac{3}{5}$	$\frac{3}{5}$	$R2+\frac{3}{2}R4$
t	0	0	0	$\frac{12}{5}$	$\frac{-8}{15}$	$\frac{-7}{15}$	1	$\frac{9}{5}$	$\frac{9}{5}$	$R3+\frac{9}{2}R4$
p_1	0	1	0	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{-1}{15}$	0	$\frac{2}{5}$	$\frac{2}{5}$	$R4\div\frac{5}{2}$
P	0	0	0	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{8}{15}$	0	$\frac{29}{5}$	$\frac{29}{5}$	$R5+\frac{29}{2}R4$

$V=\frac{29}{5}$, so $v=\frac{29}{5}-6=\frac{-1}{5}$, $p_1=\frac{2}{5}$ $p_2=\frac{3}{5}$ $p_3=0$

A should play 1 with probability $\frac{2}{5}$

A should play 2 with probability $\frac{3}{5}$

A should play 3 never

c

b.v.	V	q_1	q_2	q_3	r	s	t	u	value
r	(1)	−10	−1	−4	1	0	0	0	0
s	1	−2	−8	−3	0	1	0	0	0
t	1	−4	−7	−6	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

d

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	−10	−1	−4	1	0	0	0	0	R1 ÷ 1
s	0	(8)	−7	1	−1	1	0	0	0	R2 − R1
t	0	6	−6	−2	−1	0	1	0	0	R3 − R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−10	−1	−4	1	0	0	0	0	R5 + R1

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-39}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{5}{4}$	0	0	0	R1 + 10R2
q_1	0	1	$\frac{-7}{8}$	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{1}{8}$	0	0	0	R2 ÷ 8
t	0	0	$\frac{-3}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{-3}{4}$	1	0	0	R3 − 6R2
u	0	0	($\frac{15}{8}$)	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{-1}{8}$	0	1	1	R4 − R2
P	0	0	$\frac{-39}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{5}{4}$	0	0	0	R5 + 10R2

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{9}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{26}{5}$	$\frac{26}{5}$	R1 + $\frac{39}{4}$R4
q_1	0	1	0	$\frac{8}{15}$	$\frac{-1}{15}$	$\frac{1}{15}$	0	$\frac{7}{15}$	$\frac{7}{15}$	R2 + $\frac{7}{8}$R4
t	0	0	0	$\frac{-6}{5}$	$\frac{-1}{5}$	$\frac{-4}{5}$	1	$\frac{2}{5}$	$\frac{6}{15}$	R3 + $\frac{3}{4}$R4
q_2	0	0	1	$\frac{7}{15}$	$\frac{1}{15}$	$\frac{-1}{15}$	0	$\frac{8}{15}$	$\frac{8}{15}$	R4 ÷ $\frac{15}{8}$
P	0	0	0	$\frac{9}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{26}{5}$	$\frac{26}{5}$	R5 + $\frac{39}{4}$R4

$V = \frac{26}{5}$, so $v = \frac{26}{5} - 5 = \frac{1}{5}$ $\quad q_1 = \frac{7}{15} \quad q_2 = \frac{8}{15} \quad q_3 = 0$

B should play 1 with probability $\frac{7}{15}$

B should play 2 with probability $\frac{8}{15}$

B should play 3 never

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Exercise D, Question 12

Question:

	B plays 1	B plays 2	B plays 3
A plays 1	−3	2	−1
A plays 2	−1	−2	1
A plays 3	2	−4	−2

Using your answer to question **3**,

a write down an initial simplex tableau to solve the zero-sum game, for player A,

b use the simplex algorithm to determine A's best strategy.

Using your answer to question 7,

c write down an initial simplex tableau to solve the zero-sum game, for player B,

d use the simplex algorithm to determine B's best strategy.

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	u	value
r	(1)	−2	−4	−7	1	0	0	0	0
s	1	−7	−3	−1	0	1	0	0	0
t	1	−4	−6	−3	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	−2	−4	−7	1	0	0	0	0	R1 ÷ 1
s	0	−5	1	(6)	−1	1	0	0	0	R2 − R1
t	0	−2	−2	4	−1	0	1	0	0	R3 − R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−2	−4	−7	1	0	0	0	0	R5 + R1

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	$\frac{-47}{6}$	$\frac{-17}{6}$	0	$\frac{-1}{6}$	$\frac{7}{6}$	0	0	0	R1 + 7R2
p_3	0	$\frac{-5}{6}$	$\frac{1}{6}$	1	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	0	R2 ÷ 6
t	0	($\frac{4}{3}$)	$\frac{-8}{3}$	0	$\frac{-1}{3}$	$\frac{-2}{3}$	1	0	0	R3 − 4R2
u	0	$\frac{11}{6}$	$\frac{5}{6}$	0	$\frac{1}{6}$	$\frac{-1}{6}$	0	1	1	R4 − R2
P	0	$\frac{-47}{6}$	$\frac{-17}{6}$	0	$\frac{-1}{6}$	$\frac{7}{6}$	0	0	0	R5 + 7R2

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-37}{2}$	0	$\frac{-17}{8}$	$\frac{-11}{4}$	$\frac{47}{8}$	0	0	R1 ÷ $\frac{47}{6}$R3
p_3	0	0	$\frac{-3}{2}$	1	$\frac{-3}{8}$	$\frac{-1}{4}$	$\frac{5}{8}$	0	0	R2 + $\frac{5}{6}$R3
p_1	0	1	−2	0	$\frac{-1}{4}$	$\frac{-1}{2}$	$\frac{3}{4}$	0	0	R3 ÷ $\frac{4}{3}$
u	0	0	($\frac{9}{2}$)	0	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{-11}{8}$	1	1	R4 − $\frac{11}{6}$R3
P	0	0	$\frac{-37}{2}$	0	$\frac{-17}{8}$	$\frac{-11}{4}$	$\frac{47}{8}$	0	0	R5 + $\frac{47}{6}$R3

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	0	0	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{37}{9}$	$\frac{37}{9}$	$R1+\frac{37}{9}R4$
p_3	0	0	0	1	$\frac{-1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$R2+\frac{3}{2}R4$
p_1	0	1	0	0	$\frac{1}{36}$	$\frac{-1}{6}$	$\frac{5}{36}$	$\frac{4}{9}$	$\frac{4}{9}$	$R3+2R4$
p_2	0	0	1	0	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{-11}{36}$	$\frac{2}{9}$	$\frac{2}{9}$	$R4\div\frac{9}{2}$
P	0	0	0	0	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{37}{9}$	$\frac{37}{9}$	$R5+\frac{37}{2}R4$

$V=\frac{37}{9}$ so $v=\frac{37}{9}-5=\frac{-8}{9}$ $\quad p_1=\frac{4}{9}\quad p_2=\frac{2}{9}\quad p_3=\frac{3}{9}$

A should play 1 with probability $\frac{4}{9}$

A should play 2 with probability $\frac{2}{9}$

A should play 3 with probability $\frac{3}{9}$

c

b.v.	V	q_1	q_2	q_3	r	s	t	u	value
r	(1)	−6	−1	−4	1	0	0	0	0
s	1	−4	−5	−2	0	1	0	0	0
t	1	−1	−7	−5	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

d

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	−6	−1	−4	1	0	0	0	0	$R1\div 1$
s	0	2	−4	2	−1	1	0	0	0	$R2-R1$
t	0	(5)	−6	−1	−1	0	1	0	0	$R3-R1$
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−6	−1	−4	1	0	0	0	0	$R5+R1$

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-41}{5}$	$\frac{-26}{5}$	$\frac{-1}{5}$	0	$\frac{6}{5}$	0	0	R1+6R3
s	0	0	$\frac{-8}{5}$	$\frac{12}{5}$	$\frac{-3}{5}$	1	$\frac{-2}{5}$	0	0	R3−2R3
q_1	0	1	$\frac{-6}{5}$	$\frac{-1}{5}$	$\frac{-1}{5}$	0	$\frac{1}{5}$	0	0	R3÷5
u	0	0	$\frac{11}{5}$	$\frac{6}{5}$	$\frac{1}{5}$	0	$\frac{-1}{5}$	1	1	R4−R3
P	0	0	$\frac{-41}{5}$	$\frac{-26}{5}$	$\frac{-1}{5}$	0	$\frac{6}{5}$	0	0	R5+6R3

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{-8}{11}$	$\frac{6}{11}$	0	$\frac{5}{11}$	$\frac{41}{11}$	$\frac{41}{11}$	$R1+\frac{41}{5}R4$
s	0	0	0	$\frac{36}{11}$	$\frac{-5}{11}$	1	$\frac{-6}{11}$	$\frac{8}{11}$	$\frac{8}{11}$	$R2+\frac{8}{5}R4$
q_1	0	1	0	$\frac{5}{11}$	$\frac{-1}{11}$	0	$\frac{1}{11}$	$\frac{6}{11}$	$\frac{6}{11}$	$R3+\frac{6}{5}R4$
q_2	0	0	1	$\frac{6}{11}$	$\frac{1}{11}$	0	$\frac{-1}{11}$	$\frac{5}{11}$	$\frac{5}{11}$	$R4\div\frac{11}{5}$
P	0	0	0	$\frac{-8}{11}$	$\frac{6}{11}$	0	$\frac{5}{11}$	$\frac{41}{11}$	$\frac{41}{11}$	$R5+\frac{41}{5}R4$

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{35}{9}$	$\frac{35}{9}$	$R1+\frac{8}{11}R2$
q_3	0	0	0	1	$\frac{-5}{36}$	$\frac{11}{36}$	$\frac{-1}{6}$	$\frac{2}{9}$	$\frac{2}{9}$	$R2\div\frac{36}{11}$
q_1	0	1	0	0	$\frac{-1}{36}$	$\frac{-5}{36}$	$\frac{1}{6}$	$\frac{4}{9}$	$\frac{4}{9}$	$R3-\frac{5}{11}R2$
q_2	0	0	1	0	$\frac{1}{6}$	$\frac{-1}{6}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$R4-\frac{6}{11}R2$
P	0	0	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{35}{9}$	$\frac{35}{9}$	$R5+\frac{8}{11}R2$

$V=\frac{35}{9}$ so $v=\frac{35}{9}-3=\frac{8}{9}$ $\quad q_1=\frac{4}{9}$ $\quad q_2=\frac{3}{9}$ $\quad q_3=\frac{2}{9}$

B should play 1 with probability $\frac{4}{9}$

B should play 2 with probability $\frac{3}{9}$

B should play 3 with probability $\frac{2}{9}$

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Exercise E, Question 1

Question:

A two-person zero-sum game is represented by the following pay-off matrix for player A. Find the best strategy for each player and the value of the game.

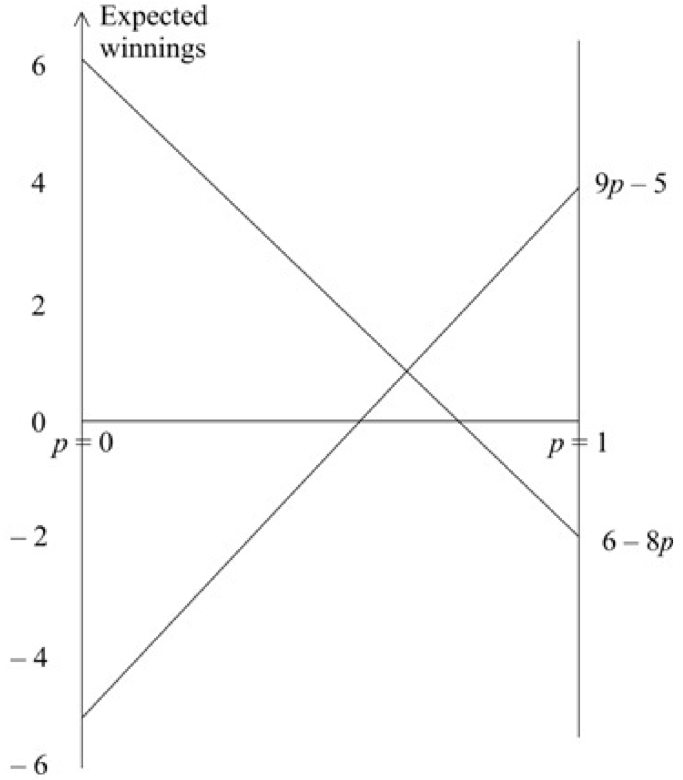
B

A		I	II
	I	4	−2
	II	−5	6

Solution:

	B plays 1	B plays 2	Row min	
A plays 1	4	−2	−2	←
A plays 2	−5	6	−5	
Column max	4	6		
	↑			

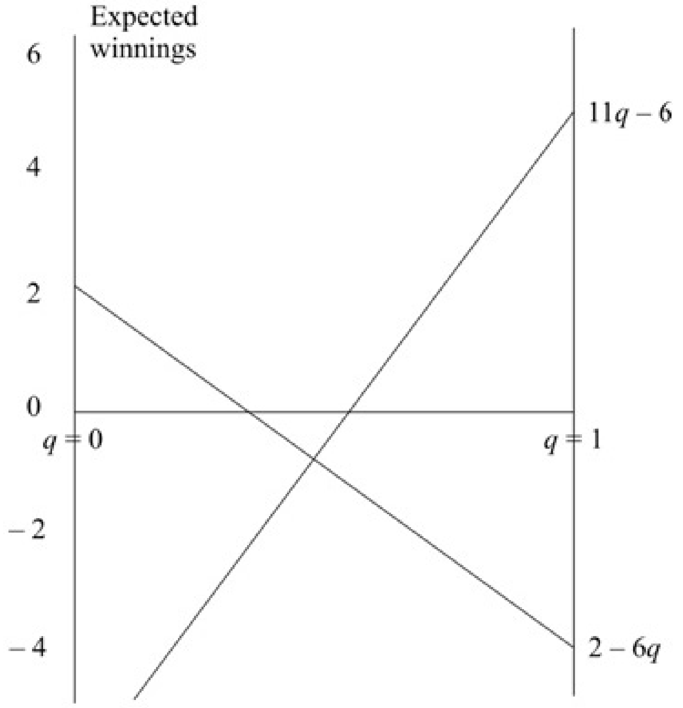
No stable solution since $4 \neq -2$ (column minimax $\neq$ row maximin)
Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $4p-5(1-p)=9p-5$
If B plays 2 A's expected winnings are $-2p+6(1-p)=6-8p$



$$9p-5=6-8p$$
$$17p=11$$
$$p=\frac{11}{17}$$

A should play 1 with probability $\frac{11}{17}$
A should play 2 with probability $\frac{6}{17}$
The value of the game to A is $\frac{14}{17}$

Let B play 1 with probability q
Let B play 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[4q-2(1-q)]=2-6q$
If A plays 2 B's expected winnings are $-[-5q+6(1-q)]=11q-6$



$$11q - 6 = 2 - 6q$$
$$17q = 8$$
$$q = \frac{8}{17}$$

B should play 1 with probability $\frac{8}{17}$

B should play 2 with probability $\frac{9}{17}$

The value of the game to B is $\frac{-14}{17}$

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Exercise E, Question 2

Question:

Ben and Greg play a zero-sum game, represented by the following pay-off matrix for Ben.

	Greg plays 1	Greg plays 2	Greg plays 3
Ben plays 1	−5	4	3
Ben plays 2	1	−1	−4

a Explain why this matrix might be reduced to

$$\begin{vmatrix} -5 & 3 \\ 1 & -4 \end{vmatrix}$$

b Hence find the best strategy for each player and the value of the game.

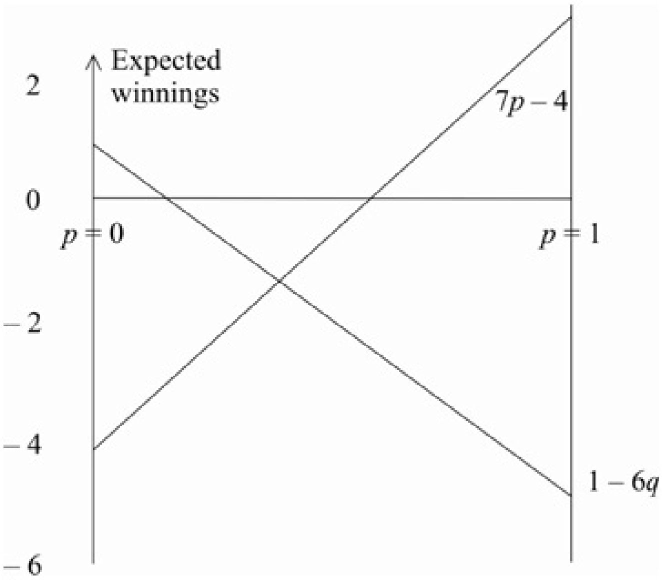
Solution:

a Column 3 dominates column 2 (since $3 < 4$ and $-4 < -1$)

b

	A play 1	A play 2	Row min	
B plays 1	−5	3	−5	
B plays 2	1	−4	−4	←
Col max	1	3		
	↑			

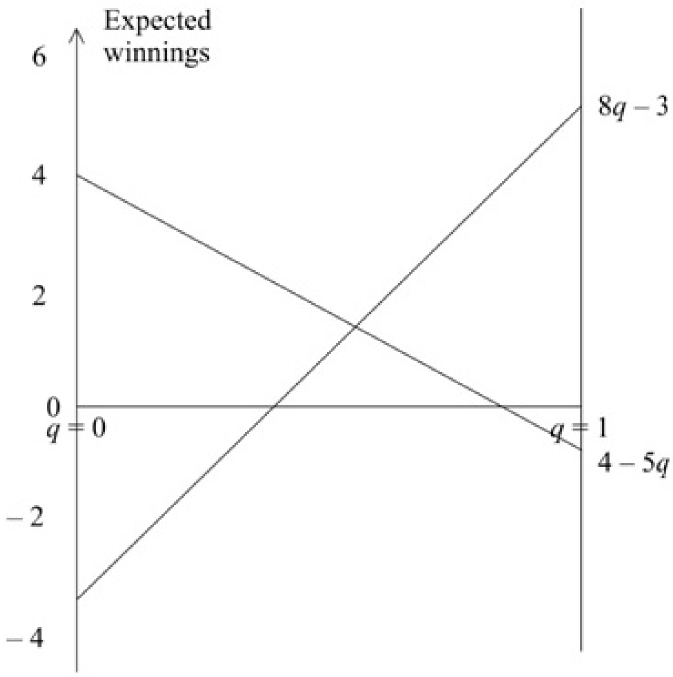
Since $1 \neq -4$ (column minimax $\neq$ row maximin) game is not stable
Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-5p+1(1-p)=1-6p$
If B plays 2 A's expected winnings are $3p-4(1-p)=7p-4$



$7p-4=1-6p$
$13p=5$
$p=\frac{5}{13}$

A should play 1 with probability $\frac{5}{13}$
A should play 2 with probability $\frac{8}{13}$
The value of the game is $\frac{-17}{13}$

Let B play 1 with probability q
Let B play 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-5q+3(1-q)]=8q-3$
If A plays 2 B's expected winnings are $-[q-4(1-q)]=4-5q$



$$8q - 3 = 4 - 5q$$
$$13q = 7$$
$$q = \frac{7}{13}$$

B should play 1 with probability $\frac{7}{13}$

B should play 2 with probability $\frac{6}{13}$

The value of the game is $\frac{17}{13}$

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Exercise E, Question 3

Question:

Cait and Georgi play a zero-sum game, represented by the following pay-off matrix for Cait.

	Georgi plays 1	Georgi plays 2	Georgi plays 3
Cait plays 1	−5	2	3
Cait plays 2	1	−3	−4
Cait plays 3	−7	0	1

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Use dominance to reduce the game to a 2×3 game, explaining your reasoning.
d Find Cait's best strategy and the value of the game to her.
e Write down the value of the game to Georgi.

Solution:

	G plays 1	G plays 2	G plays 3	Row min	
C plays 1	−5	2	3	−5	
C plays 2	1	−3	−4	−4	←
C plays 3	−7	0	1	−7	
Column max	1	2	3		
	↑				

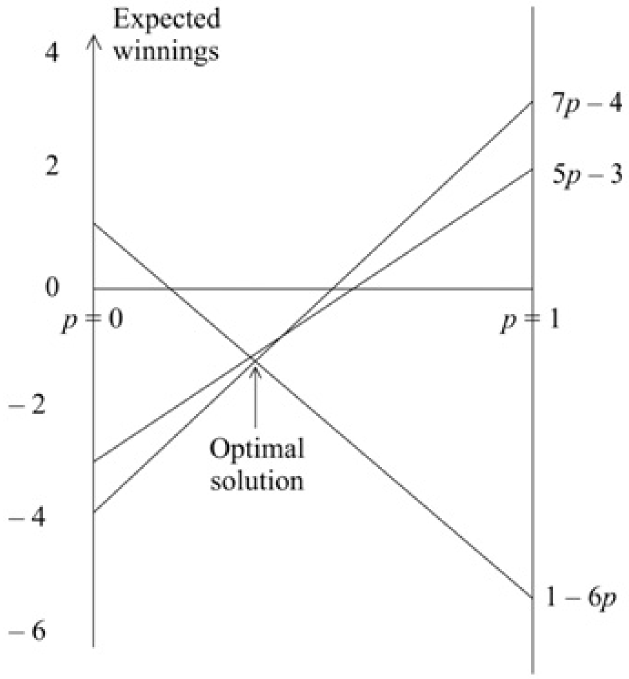
a Play safe: Cait plays 2 Georgi plays 1

b $1 \neq -4$ (column minimax ≠ row maximin) so no stable solution

c Row 1 dominates row 3 (since $-5 > -7 \quad 2 > 0 \quad 3 > 1$)

	G plays 1	G plays 2	G plays 3
C plays 1	−5	2	3
C plays 2	1	−3	−4

d Let C play 1 with probability p
So C plays 2 with probability $(1-p)$
If G plays 1 C's expected winnings are $-5p+1(1-p)=1-6p$
If G plays 2 C's expected winnings are $2p-3(1-p)=5p-3$
If G plays 3 C's expected winnings are $3p-4(1-p)=7p-4$



$$7p-4=1-6p$$
$$13p=5$$
$$p=\frac{5}{13}$$

Cait should play 1 with probability $\frac{5}{13}$
Cait should play 2 with probability $\frac{8}{13}$
Cait should play 3 never
The value of the game is $\frac{-17}{13}$

e The value of the game to Georgi is $\frac{17}{13}$

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Exercise E, Question 4

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−1	−3
A plays 2	−2	1	4
A plays 3	−3	1	−3
A plays 4	−1	2	−2

a Verify that there is no stable solution to this game

b Explain the circumstances under which a row, x, dominates a row, y.

c Reduce the game to a 3×3 game, explaining your reasoning.

d Formulate the 3×3 game as a linear programming problem for player A. Write the constraints as inequalities and define your variables.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	−1	−3	−3	
A plays 2	−2	1	4	−2	←
A plays 3	−3	1	−3	−3	
A plays 4	−1	2	−2	−2	←
Column max	2	2	4		
	↑	↑			

Since $2 \neq -2$ (column minimax $\neq$ row maximin) there is no stable solution.

b A row x dominates a row y, if, in each column, the element in row $x \geq$ the element in row y.

c Row 4 dominates row 3

	B plays 1	B plays 2	B plays 3
A plays 1	2	−1	−3
A plays 2	−2	1	4
A plays 3	−1	2	−2

d Add 4 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	6	3	1
A plays 2	2	5	8
A plays 3	3	6	2

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v so $V = v + 4$

Maximise $P = V$

Subject to:

$6p_1 + 2p_2 + 3p_3 \geq V$

$3p_1 + 5p_2 + 6p_2 \geq V$

$p_1 + 8p_2 + 2p_3 \geq V$

$p_1 + p_2 + p_3 \leq 1$

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Exercise E, Question 5

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	5	−1	1
A plays 2	−1	−4	4
A plays 3	3	−2	−1

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Use dominance to reduce the game to a 3×2 game, explaining your reasoning.
d Write down the pay-off matrix for player B.
e Find B's best strategy and the value of the game.

Solution:

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	5	−3	1	−3	
A plays 2	−1	−4	4	−4	
A plays 3	3	2	−1	−1	←
Column max	5	2	4		
		↑			

a Play safe (A plays 1, B plays 2)

b Since $2 \neq -1$ (column minimax ≠ row maximin) there is no stable solution

c Column 2 dominates column 1 $(-3 < 5, -4 < -1, 2 < 3)$ B would always choose to minimise A's winnings by playing 2 rather than 1

	B plays 2	B plays 3
A plays 1	−3	1
A plays 2	−4	4
A plays 3	2	−1

d

	A plays 1	A plays 2	A plays 3
B plays 2	3	4	−2
B plays 3	−1	−4	1

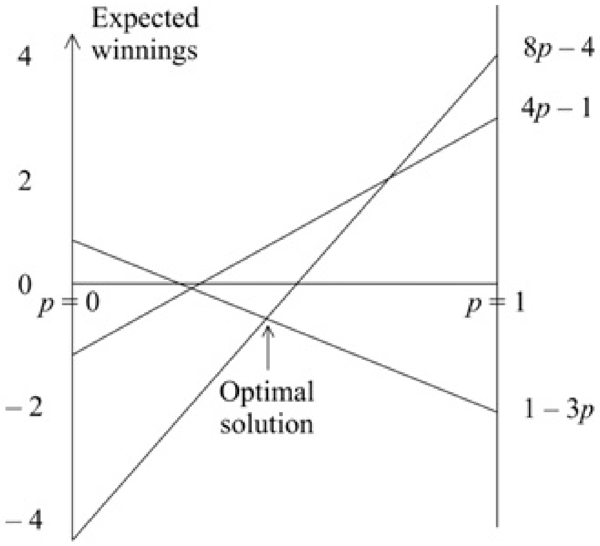
e Let B play 2 with probability p

So B plays 3 with probability $(1-p)$

If A plays 1 B's expected winnings are $3p-1(1-p)=4p-1$

If A plays 2 B's expected winnings are $4p-4(1-p)=8p-4$

If A plays 3 B's expected winnings are $-2p+1(1-p)=1-3p$



$$8p-4=1-3p$$
$$11p=5$$
$$p=\frac{5}{11}$$

B should play 2 with probability $\frac{5}{11}$

B should play 3 with probability $\frac{6}{11}$

The value of the game is $\frac{-4}{11}$

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Exercise E, Question 6

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays1	B plays2	B plays3
A plays1	2	7	−1
A plays2	5	0	8
A plays3	−2	3	5

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Write down the pay-off matrix for player B
d Formulate the game for player B as a linear programming problem. Define your variables and write your constraints as equations.
e Write down an initial tableau that you could use to solve the game for player B.

Solution:

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	7	−1	−1	
A plays 2	5	0	8	0	←
A plays 3	−2	3	5	−2	
Column max	5	7	8		
	↑				

a Play safe is (A plays 2, B plays 1)

b Since $5 \neq 0$ (column minimax $\neq$ row maximin) there is no stable solution

c

	A plays 1	A plays 2	A plays 3
B plays 1	−2	−5	2
B plays 2	−7	0	−3
B plays 3	1	−8	−5

d Adding 9 to all elements

	A plays 1	A plays 2	A plays 3
B plays 1	7	4	11
B plays 2	2	9	6
B plays 3	10	1	4

Let B play 1 with probability p_1, play 2 with probability p_2 and play 3 with probability p_3.

Let v = value of the game to B and $V = v + 9$

Maximise $P = V$

Subject to:

$$7p_1 + 2p_2 + 10p_3 \geq V \Rightarrow V - 7p_1 - 2p_2 - 10p_3 + r = 0$$
$$4p_1 + 9p_2 + p_3 \geq V \Rightarrow V - 4p_1 - 9p_2 - p_3 + s = 0$$
$$11p_1 + 6p_2 + 4p_3 \geq V \Rightarrow V - 11p_1 - 6p_2 - 4p_3 + t = 0$$
$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$
$$\text{where } p_1, p_2, p_3, r, s, t, u \geq 0$$

e

b.v.	V	P_1	P_2	P_3	r	s	t	u	value
r	1	−7	−2	−10	1	0	0	0	0
s	1	−4	−9	−1	0	1	0	0	0
t	1	−11	−6	−4	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0