Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

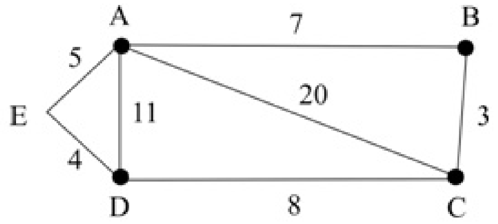


Table of least differences

	A	B	C	D	E
A	–	7			5
B	7	–	3		
C		3	–	8	12
D			8	–	4
E	5		12	4	–

Solution:

	A	B	C	D	E
A	–	7	**10**	**9**	5
B	7	–	3	**11**	**12**
C	**10**	3	–	8	12
D	**9**	**11**	8	–	4
E	5	**12**	12	4	–

AC – the shortest route is ABC length 10
AD – the shortest route is AED length 9
BD – the shortest route is BCD length 11
BE – the shortest route is BAE length 12

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

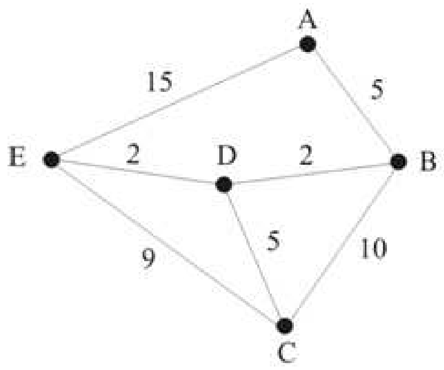


Table of least differences

	A	B	C	D	E
A	–	5		7	
B	5	–		2	
C			–	5	
D	7	2	5	–	2
E				2	–

Solution:

	A	B	C	D	E
A	–	5	**12**	7	**9**
B	5	–	7	2	**4**
C	**12**	7	–	5	7
D	7	2	5	–	2
E	**9**	**4**	7	2	–

AC – the shortest route is ABDC length 12
AE – the shortest route is ABDE length 9
BC – the shortest route is BDC length 7
BE – the shortest route is BDE length 4
CE – the shortest route is CDE length 7

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 3

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

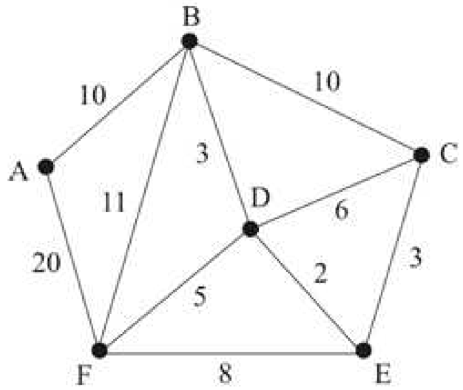


Table of least differences

	A	B	C	D	E	F
A	–	10		13	15	
B	10	–		3		
C			–		3	
D	13	3		–	2	5
E	15		3	2	–	
F				5		–

Solution:

	A	B	C	D	E	F
A	–	10	**18**	13	15	**18**
B	10	–	**8**	3	**5**	**8**
C	**18**	**8**	–	**5**	3	**10**
D	13	3	**5**	–	2	5
E	15	**5**	3	2	–	**7**
F	**18**	**8**	**10**	5	**7**	–

AC – the shortest route is ABDEC length 18
AF – the shortest route is ABDF length 18
BC – the shortest route is BDEC length 8
BE – the shortest route is BDE length 5
BF – the shortest route is BDF length 8
CD – the shortest route is CED length 5
CF – the shortest route is CEDF length 10
EF – the shortest route is EDF length 7

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 4

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

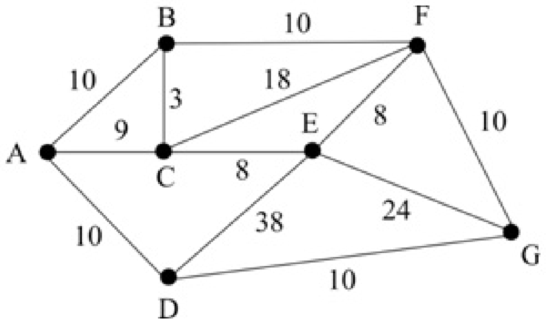


Table of least differences

	A	B	C	D	E	F	G
A	–	10	9	10	17		
B	10	–	3	20		10	20
C	9	3	–	19	8		
D	10	20	19	–		20	10
E	17		8		–	8	18
F		10		20	8	–	10
G		20		10	18	10	–

Solution:

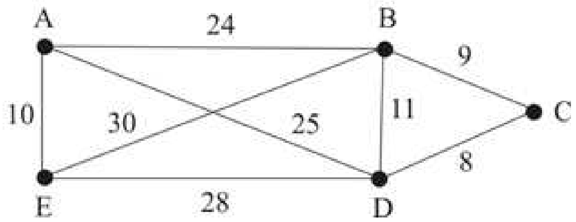
	A	B	C	D	E	F	G
A	–	10	9	10	17	**20**	**20**
B	10	–	3	20	**11**	10	20
C	9	3	–	19	8	**13**	**23**
D	10	20	19	–	**27**	20	10
E	17	**11**	8		–	8	18
F	**20**	10	**13**	20	8	–	10
G	**20**	20	**23**	10	18	10	–

AF – the shortest route is ABF length 20
AG – the shortest route is ADG length 20
BE – the shortest route is BCE length 11
CF – the shortest route is CBF length 13
CG – the shortest route is CBFG length 23
DE – the shortest route is DACE length 27

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 1

Question:



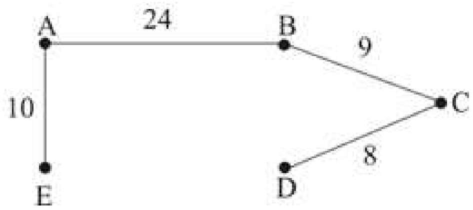
a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.

b Use a shortcut to find a better upper bound

c State the route given by your improved upper bound and state its length.

Solution:

a



$$\text{Initial upper bound} = 2 \times 51$$
$$= 102$$

b Use DE as a shortcut
Route length $= 51 + 28 = 79$

c Route A B C D E A length 79

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 2

Question:

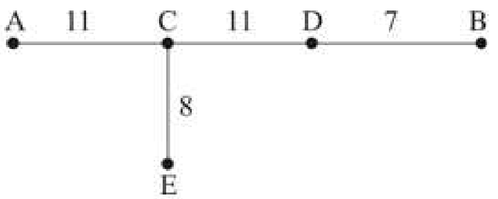
	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She will start and finish at A and wishes to minimise her total travelling distance.

a Find the minimum spanning tree for the network.
b Hence find an initial upper bound for the length of the employee's route.
c Use shortcuts to reduce the upper bound to a value below 65.
d State the route given by your improved upper bound and state its length.

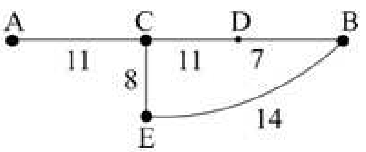
Solution:

a



b Initial upper bound $= 2 \times 37 = 74$

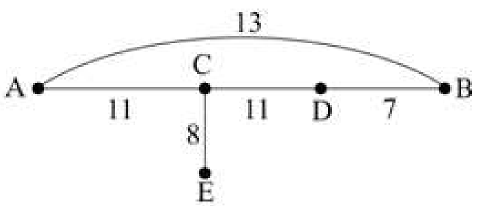
c For example **i** use BE as a shortcut



or

ii use AB as a shortcut

Other answers also possible



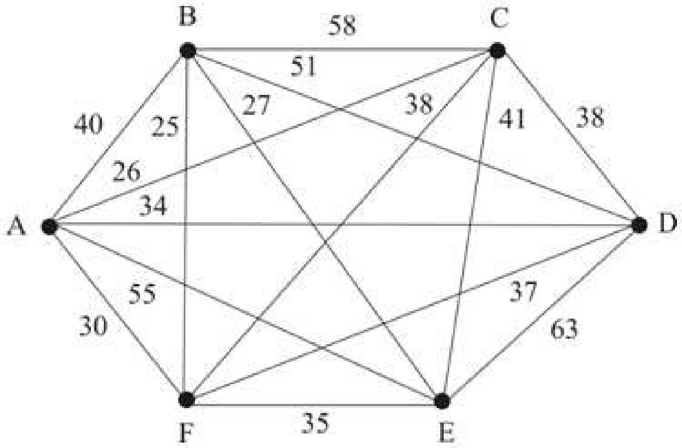
d i Using BE route is A C E B D C A length 62

ii Using AB route is A C E C D B A length 58

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

Question:



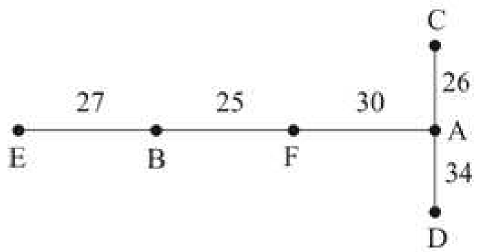
a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.

b Use shortcuts to reduce the upper bound to below 240.

c State the route given by your improved upper bound and state its length.

Solution:

a



Initial upper bound $= 2 \times 142$
$= 284$

b Many possibilities: for example
DE or EC or DF and EC

c DE gives A C A F B E D A length 231
EC gives A D A F B E C A length 217
DF and EC gives A C E D F D A length 190

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

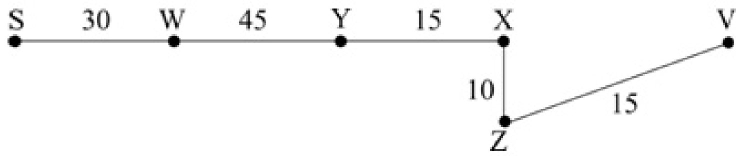
Question:

	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time. He will start and finish at the surgery, S.

a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.
b Use shortcuts to reduce the upper bound to below 200.
c State the route given by your improved upper bound and state its length.

Solution:

a 

Initial upper bound $2 \times 115 = 230$

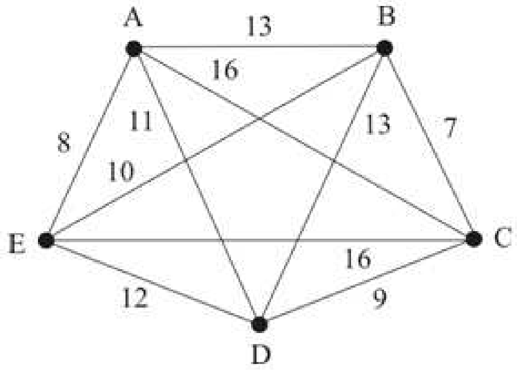
b For example arc VS

c Route S W Y X Z V X S length 190

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

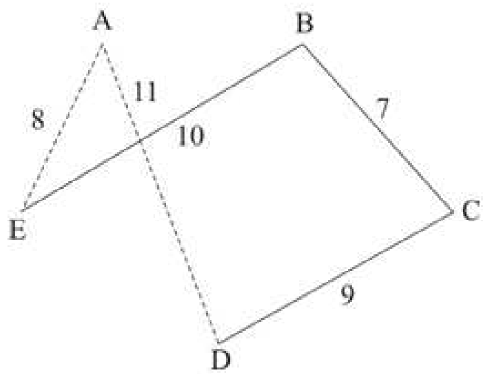


a By deleting vertex A, find a lower bound to the travelling salesman problem for the network above.

b Comment on your answer.

Solution:

a



Weight of residual minimum spanning tree = 26

Two shortest arcs from A, AE and AD

Lower bound = 26 + 8 + 11

= 45

b This is a route, so it is optimal

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

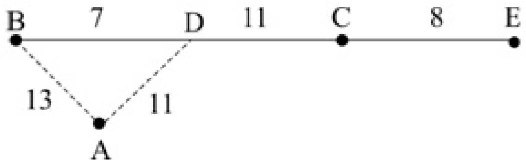
A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She will start and finish at A and wishes to minimise her total travelling distance.

a By deleting vertices A then B find two lower bounds for the employee's route.

b Select the better lower bound, giving a reason for your answer.

Solution:

a *Deleting A*



Weight of residual minimum spanning tree = 26

Two shortest arcs are AD and AB

Lower bound = 26+11+13

= 50

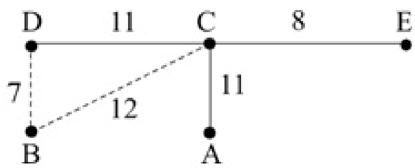
Deleting B

Weight of residual minimum spanning, tree = 30

Two shortest arcs are BD and BC

Lower bound = 30+7+12

= 49

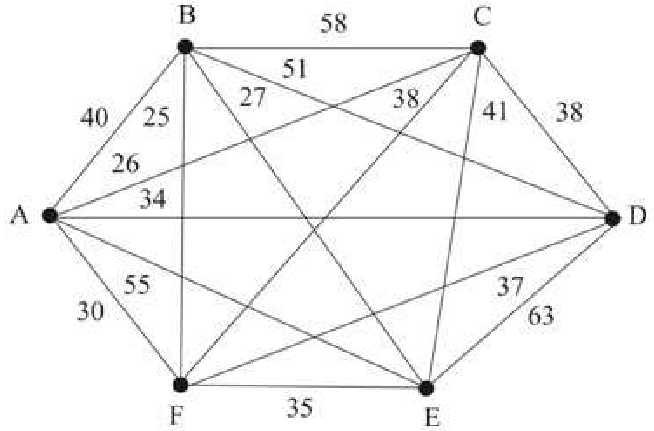


b The better lower bound is 50 since it is higher.

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 3

Question:



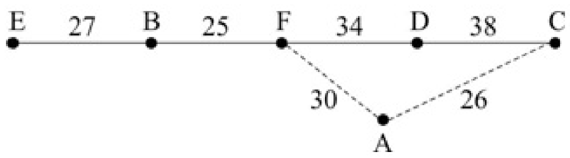
a By deleting vertices A then B, find two lower bounds for the travelling salesman problem.

b Select the better lower bound, giving a reason for your answer.

c Use inequalities, your answer to **b** and the better upper bound found in Exercise 3B Question 3, to write down the smallest interval containing the optimal route

Solution:

a *Deleting A*



Weight of residual minimum spanning tree $= 124$

Two shortest arcs are AC and AF

Lower bound $= 124 + 26 + 30 = 180$

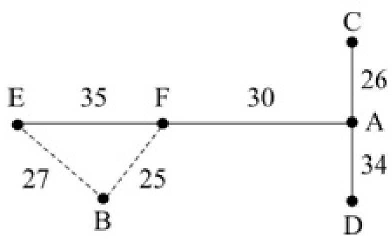
a *Deleting B*

Weight of residual minimum spanning tree $= 125$

Two least arcs BF and BE

lower bound $= 125 + 25 + 27$

$= 177$



b The better lower bound is 180 because it is higher

c $180 < \text{ optimal value } \leq 190$ (or your answer to Exercise 3B question 3**c**)

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 4

Question:

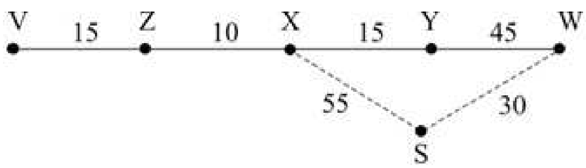
	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time. He will start and finish at the surgery, S.

a By deleting vertices S then V, find two lower bounds for the vet's route.
b Select the better lower bound, giving a reason for your answer.
c Use inequalities, your answer to **b** and the better upper bound found in Exercise 3B Question 4, to write down the smallest interval containing the optimal route.

Solution:

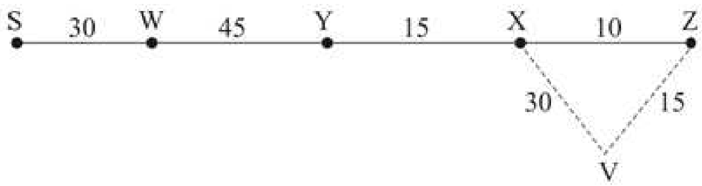
a *Deleting S*



Weight of residual minimum spanning tree = 85
Two least arcs SW and SX

Lower bound = 85 + 30 + 55
= 170

Deleting V



Weight of residual minimum spanning tree = 100
Two least arcs VZ and VX
Lower bound = 100 + 15 + 30
= 145

b The better lower bound is 170 because it is higher

c $170 < \text{optimal value} \leq 190$ (or your answer to Exercise 3B question 4**c**)

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 1

Question:

(This is the same problem as described in Exercise 3C Question 2)

	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She wishes to minimise her total travelling distance.

a Starting at D, find a nearest neighbour route to give an upper bound for the council employee's route.
b Show that there are two nearest neighbour routes starting from E.
c Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $D_7 B_{12} C_8 E_{14} A_{19} D = 60$

b $E_8 C_{11} A_{13} B_7 D_{14} E = 53$

or

$E_8 C_{11} D_7 B_{13} A_{14} E = 53$

c The better upper bound is 53 since this is lower.

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 2

Question:

(This is the same problem as described in Exercise 3C Question 4)

	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time.

a Starting at Z, find a nearest neighbour route.
b Find two further nearest neighbour routes starting at X then V.
c Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $Z_{10}X_{15}Y_{40}V_{55}W_{30}S_{70}Z = 220$

b $X_{10}Z_{15}V_{40}Y_{45}W_{30}S_{55}X = 195$

$V_{15}Z_{10}X_{15}Y_{45}W_{30}S_{75}V = 190$

c The better upper bound is 190 because it is lower

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 3

Question:

	R	S	T	U	V	W
R	–	150	210	150	120	240
S	150	–	210	120	210	240
T	210	210	–	120	150	180
U	150	120	120	–	180	270
V	120	210	150	180	–	300
W	240	240	180	270	300	–

A printing company prints six magazines R, S, T, U, V and W, each week. The printing equipment needs to be set up differently for each magazine and the table shows the time, in minutes, needed to set up the equipment from one magazine to another. The printer must print magazine R at the start of the first day each week so the equipment is already set up to print magazine R, and must be left set up for magazine R at the end of the week. The other magazines can be printed in any order.

a If the magazines were printed in the order RSTUVWR, how long would it take in total to set up the equipment?

b Show that there are two nearest neighbour routes starting from U.

c Show that there are three nearest neighbour routes starting from V.

d Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $R_{150}S_{210}T_{120}U_{180}V_{300}W_{240}R = 1200$ minutes

b $U_{120}S_{150}R_{120}V_{150}T_{180}W_{270}U = 990$

and

$U_{120}T_{150}V_{120}R_{150}S_{240}W_{270}U = 1050$

c $V_{120}R_{150}S_{120}U_{120}T_{180}W_{300}V = 990$

and

$V_{120}R_{150}U_{120}S_{210}T_{180}W_{300}V = 1080$

and

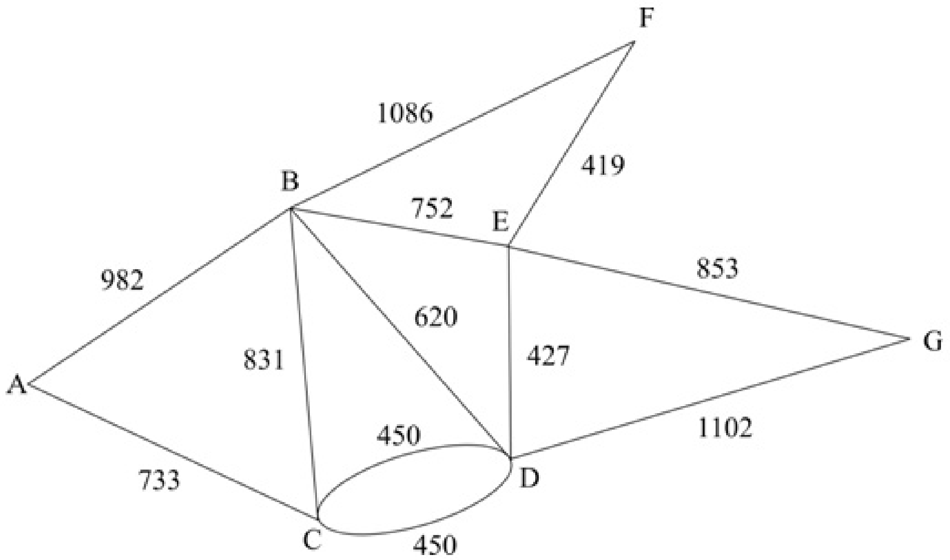
$V_{120}R_{150}U_{120}T_{180}W_{240}S_{210}V = 1020$

d The better upper bound is 990 because it is lower.

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 1

Question:



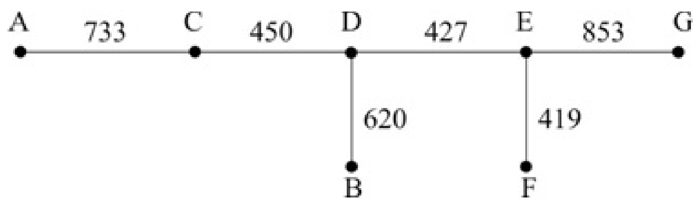
a Use an efficient algorithm to find a minimum connector for the network above. You must make your method clear.

b Hence find an initial upper bound for the travelling salesman problem.

c Use the method of short cuts to find an upper bound below 6100.

Solution:

a Either Kruskal: EF, DE, CD, BD, AC, EG or Prim (e.g.): AC, CD, DE, EF, BD, EG



b $2 \times 3502 = 7004$

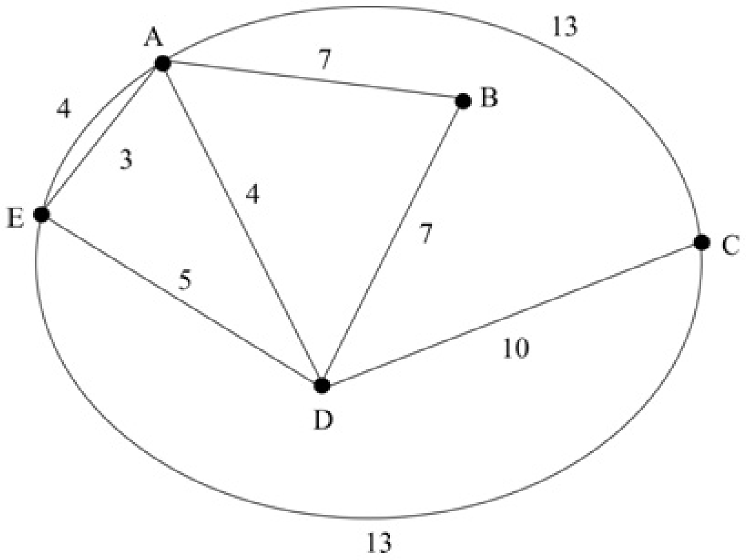
c For example use AB and DG

Route A C D E F E G D B A length 6005

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 2

Question:



The network above shows a number of hostels in a national park and the possible paths joining them. The numbers on the edges give the lengths, in km, of the paths.

a Draw a complete network showing the shortest distances between the hostels. (You may do this by inspection. The application of an algorithm is not required.)

b Use the nearest neighbour algorithm on the complete network to obtain an upper bound to the length of a tour in this network which starts and finishes at A and visits each hostel exactly once.

c Interpret your result in part **b** in terms of the original network.

Solution:

a

	A	B	C	D	E
A	–	7	13	4	3
B	7	–	17	7	10
C	13	17	–	10	13
D	4	7	10	–	5
E	3	10	13	5	–

b $A_3E_5D_7B_{17}C_{13}A = 45$

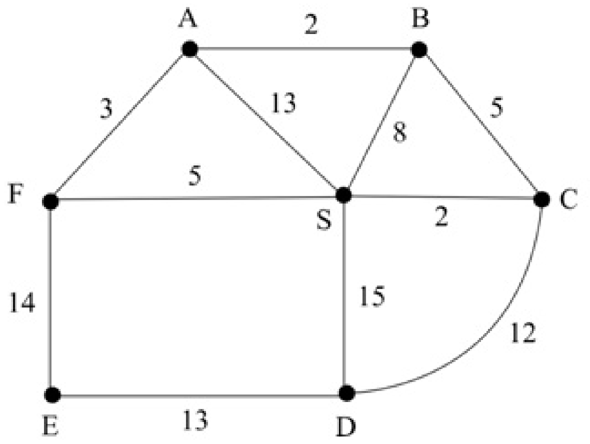
c A E D B D C A (BC is not on the original network)

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 3

Question:

(This is the network given in example 2)



The table of least distances below was formed from the network, N, above.

	S	A	B	C	D	E	F
S	–	8	7	2	14	19	5
A	8	–	2	7	19	17	3
B	7	2	–	5	17	19	5
C	2	7	5	–	12	21	7
D	14	19	17	12	–	13	19
E	19	17	19	21	13	–	14
F	5	3	5	7	19	14	–

The table shows the distances, in km, between the central sorting office at S and six post offices A, B, C, D, E and F.

A postal worker will leave the sorting office, go to each post office to collect mail and return to the sorting office. He wishes to minimise his route.

a Use Prim's algorithm, starting at S, to obtain two minimum spanning trees. State the order in which you select the arcs.
b Hence find an initial upper bound for the postal worker's route.
c Starting from this upper bound, use shortcuts to reduce the upper bound to a value below 60 km. You must state the shortcuts you use.
d Starting at C, and then at D, find two nearest neighbour routes stating their lengths.
e Select the better upper bound from your answers to **c** and **d**, give a reason for your answer.

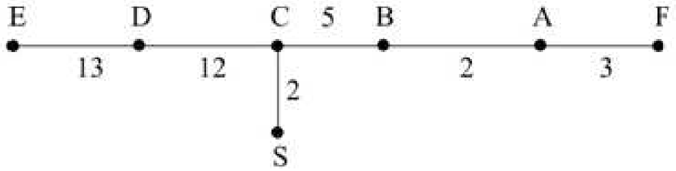
f Interpret your answer to **e** in terms of the original network, N, of roads.
g Using the table of least distances, and by deleting C, find a lower bound for the postal worker's route.

Solution:

a SC SF FA AB CD DE – tree 1

and

SC CB BA AF CD DE – tree 2



b Weight of each tree is 37
So initial upper bound is $2 \times 37 = 74$

c From tree 1
Use BE as a shortcut
(Route is S C D E B A F S) length 56

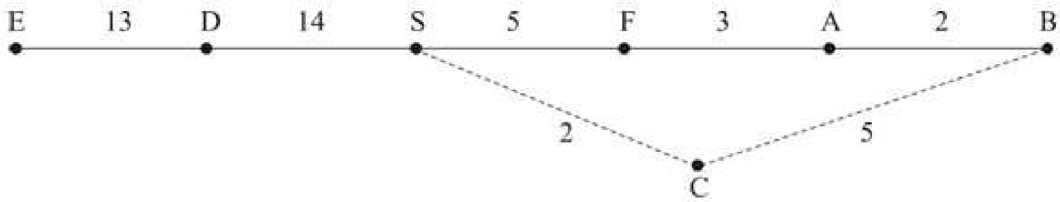
From tree 2
Use EF as a shortcut
(Route is S C B A F E D C S) length 53

d $C_2S_5F_3A_2B_{17}D_{13}E_{21}C = 63$
$D_{12}C_2S_5F_3A_2B_{19}E_{13}D = 56$

e The better upper bound is 53 since it is smaller

f The route is S C B A F E D C S

g



Weight of residual minimum spanning tree = 37
Two least arcs from C are CS and CB

Lower bound $= 37 + 2 + 5$
$= 44$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 4

Question:

a Explain the difference between the classical and practical travelling salesman problem.

	P	Q	R	S	T	U	V
P	–	19	30	45	38	33	29
Q	19	–	28	27	50	23	55
R	30	28	–	51	29	49	50
S	45	27	51	–	77	21	71
T	38	50	29	77	–	69	37
U	33	23	49	21	69	–	56
V	29	55	50	71	37	56	–

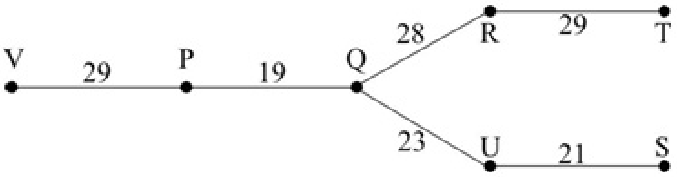
The table shows the travel time, in minutes, between seven town halls P, Q, R, S, T, U and V. Kim works at P and must visit each of the other town halls to deliver leaflets. She wishes to minimise her route.

b Find a minimum connector for the network. You must make your method clear by listing the arcs in order of selection.
c Use the minimum connector and shortcuts to find an upper bound below 220. You must list the shortcuts you use and your final route.
d Starting at P, find a nearest neighbour route and state its length.
e Find a lower bound for the length of the route by deleting P.
f Looking at your answers to **c**, **d** and **e**, use inequalities to write down the smallest interval containing the optimal solution.

Solution:

a In the classical problem each vertex must be visited *exactly* once before returning to the start.
In the practical problem each vertex must be visited *at least* once before returning to the start.

b

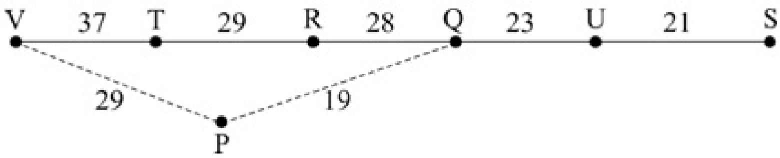


Order of arcs: PQ, QU, US, QR, $\begin{Bmatrix} \text{TR} \\ \text{VP} \end{Bmatrix}$

c Use VT and QS as shortcuts giving a length of 213
(Route P Q U S Q R T V P)

d $P_{19}Q_{22}U_{21}S_{51}R_{29}T_{37}V_{29}P = 209$

e



Weight of residual minimum spanning tree = 140
Two least arcs PQ and PV
$$\begin{aligned}\text{Lower bound} &= 138 + 19 + 29 \\ &= 186\end{aligned}$$

f $186 < \text{optimal value} \leq 209$

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 5

Question:

	A	B	C	D	E	F	G
A	–	103	89	42	54	143	153
B	103	–	60	98	56	99	59
C	89	60	–	65	38	58	77
D	42	98	65	–	45	111	139
E	54	56	38	45	–	95	100
F	143	99	58	111	95	–	75
G	153	59	77	139	100	75	–

A computer supplier has outlets in seven cities A, B, C, D, E, F and G. The table shows the distances, in km, between each of these seven cities. John lives in city A and has to visit each of these cities to advise on displays. He wishes to plan a route starting and finishing at A, visiting each city and covering a minimum distance.

a Obtain a minimum spanning tree for this network explaining briefly how you applied the algorithm that you used. (Start with A and state the order in which you selected the arcs used in your tree.)

b Hence determine an initial upper bound for the length of the route travelled by John.

c Explain why the upper bound found in this way is unlikely to give the minimum route length.

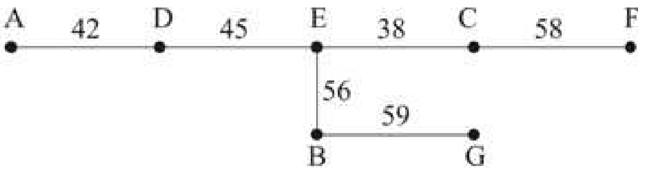
d Starting from your initial upper bound and using an appropriate method, find an upper bound for the length of the route which is less than 430 km.

e By deleting city A, determine a lower bound for the length of John's route.

f Explain under what circumstances a lower bound obtained by this method might be an optimum solution.

Solution:

a



order of arcs: AD, DE, EC, EB, CF, BG

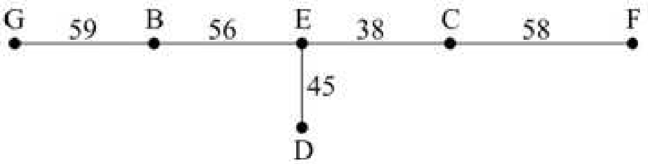
b Initial upper bound $= 2 \times 298$

$= 596$

c The minimum connector has been doubled and each arc in it repeated

d Use AE and GF as shortcuts - length 427
(route is A D E B G F C E A)

e



Weight of residual minimum spanning tree = 256
Two least arcs from A are AD (42) and AE (54)
Lower bound = 256 + 42 + 54 = 352 km

f The lower bound will give the optimal solution if it is a tour.
If the minimum spanning tree has no 'branches' – so the two end vertices have valency 1, and all other vertices have valency 2, then if the two least arcs are incident on the 2 vertices of valency 1 an optimal solution cannot be found.

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise E, Question 6

Question:

	L	C	O	B	N	E
London (L)	–	80	56	120	131	200
Cambridge (C)	80	–	100	98	87	250
Oxford (O)	56	100	–	68	103	154
Birmingham (B)	120	98	68	–	54	161
Nottingham (N)	131	87	103	54	–	209
Exeter (E)	200	250	154	161	209	–

A sales representative, Sheila, has to visit clients in six cities, London, Cambridge, Oxford, Birmingham, Nottingham and Exeter. The table shows the distances, in miles, between these six cities. Sheila lives in London and plans a route starting and finishing in London. She wishes to visit each city and drive the minimum distance.

a Starting from London, use Prim's algorithm to obtain a minimum spanning tree. Show your working. State the order in which you selected the arcs and draw the tree.

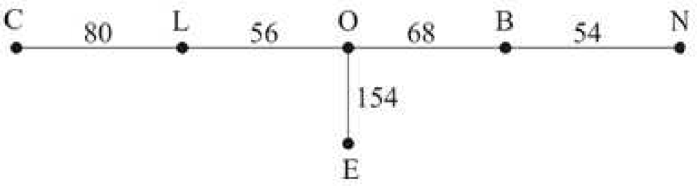
b **i** Hence determine an initial upper bound for the length of the route planned by Sheila.

ii Starting from your initial upper bound and using shortcuts, obtain a route which is less than 660 miles.

c By deleting Exeter from the table determine a lower bound for the length of Sheila's route.

Solution:

a



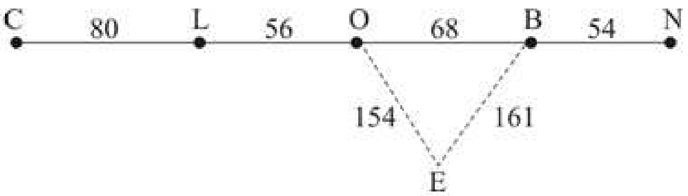
order of selection: LO, OB, BN, LC, OE

b **i** Initial upper bound $= 2 \times 412$

$= 824$ miles

ii Use NC as a shortcut – length is 653
(Route is L O E O B N C L)

c



Weight of residual minimum spanning tree = 258
Two least arcs are EO and EB

Lower bound $= 258 + 154 + 161$

$= 573$