Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise A, Question 1

Question:

Integrate the following with respect to x:

(a) $3\sec^2 x + \frac{5}{x} + \frac{2}{x^2}$

(b) $5e^x - 4\sin x + 2x^3$

(c) $2(\sin x - \cos x + x)$

(d) $3\sec x\tan x - \frac{2}{x}$

(e) $5e^x + 4\cos x - \frac{2}{x^2}$

(f) $\frac{1}{2x} + 2\operatorname{cosec}^2 x$

(g) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$

(h) $e^x + \sin x + \cos x$

(i) $2\operatorname{cosec} x\cot x - \sec^2 x$

(j) $e^x + \frac{1}{x} - \operatorname{cosec}^2 x$

Solution:

(a) $\int \left(3\sec^2 x + \frac{5}{x} + \frac{2}{x^2} \right) dx$

$= \int \left(3\sec^2 x + \frac{5}{x} + 2x^{-2} \right) dx$

$= 3\tan x + 5\ln|x| - \frac{2}{x} + C$

(b) $\int \left(5e^{x} - 4\sin x + 2x^{3} \right) dx$

$= 5e^{x} + 4\cos x + \frac{2x^{4}}{4} + C$

$= 5e^{x} + 4\cos x + \frac{x^{4}}{2} + C$

(c) $\int 2 \left(\sin x - \cos x + x \right) dx$

$= \int \left(2\sin x - 2\cos x + 2x \right) dx$

$= -2\cos x - 2\sin x + x^{2} + C$

(d) $\int \left(3\sec x\tan x - \frac{2}{x} \right) dx$

$= 3\sec x - 2\ln |x| + C$

(e) $\int \left(5e^{x} + 4\cos x - \frac{2}{x^{2}} \right) dx$

$= \int \left(5e^{x} + 4\cos x - 2x^{-2} \right) dx$

$= 5e^{x} + 4\sin x + \frac{2}{x} + C$

(f) $\int \left(\frac{1}{2x} + 2\operatorname{cosec}^{2} x \right) dx$

$= \int \left(\frac{1}{2} \times \frac{1}{x} + 2\operatorname{cosec}^{2} x \right) dx$

$= \frac{1}{2}\ln |x| - 2\cot x + C$

(g) $\int \left(\frac{1}{x} + \frac{1}{x^{2}} + \frac{1}{x^{3}} \right) dx$

$= \int \left(\frac{1}{x} + x^{-2} + x^{-3} \right) dx$

$= \ln |x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + C$

$= \ln |x| - \frac{1}{x} - \frac{1}{2x^{2}} + C$

(h) $\int \left(e^{x} + \sin x + \cos x \right) dx$

$= e^{x} - \cos x + \sin x + C$

(i) $\int (2\operatorname{cosec} x \cot x - \sec^2 x)\, dx$
$= -2\operatorname{cosec} x - \tan x + C$

(j) $\int \left(e^x + \frac{1}{x} - \operatorname{cosec}^2 x \right) dx$

$= e^x + \ln |x| + \cot x + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise A, Question 2

Question:

Find the following integrals:

(a) $\int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) \mathrm{d}x$

(b) $\int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) \mathrm{d}x$

(c) $\int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) \mathrm{d}x$

(d) $\int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) \mathrm{d}x$

(e) $\int \sin x \left(1 + \sec^2 x \right) \mathrm{d}x$

(f) $\int \cos x \left(1 + \operatorname{cosec}^2 x \right) \mathrm{d}x$

(g) $\int \operatorname{cosec}^2 x \left(1 + \tan^2 x \right) \mathrm{d}x$

(h) $\int \sec^2 x \left(1 - \cot^2 x \right) \mathrm{d}x$

(i) $\int \sec^2 x \left(1 + e^x \cos^2 x \right) \mathrm{d}x$

(j) $\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) \mathrm{d}x$

Solution:

(a) $\int \left(\frac{1}{\cos^2 x} + \frac{1}{x^2} \right) \mathrm{d}x$

$= \int \left(\sec^2 x + x^{-2} \right) \mathrm{d}x$

$= \tan x - \frac{1}{x} + C$

(b) $\int \left(\frac{\sin x}{\cos^2 x} + 2e^x \right) dx$

$= \int (\tan x \sec x + 2e^x) \, dx$

$= \sec x + 2e^x + C$

(c) $\int \left(\frac{1 + \cos x}{\sin^2 x} + \frac{1 + x}{x^2} \right) dx$

$= \int (\text{cosec}^2 x + \cot x \, \text{cosec} \, x + x^{-2} + x^{-1}) \, dx$

$= -\cot x - \text{cosec} \, x - \frac{1}{x} + \ln | x | + C$

(d) $\int \left(\frac{1}{\sin^2 x} + \frac{1}{x} \right) dx$

$= \int \left(\text{cosec}^2 x + \frac{1}{x} \right) dx$

$= -\cot x + \ln | x | + C$

(e) $\int \sin x (1 + \sec^2 x) \, dx$

$= \int (\sin x + \sin x \sec^2 x) \, dx$

$= \int (\sin x + \tan x \sec x) \, dx$

$= -\cos x + \sec x + C$

(f) $\int \cos x (1 + \text{cosec}^2 x) \, dx$

$= \int (\cos x + \cos x \, \text{cosec}^2 x) \, dx$

$= \int (\cos x + \cot x \, \text{cosec} \, x) \, dx$

$= \sin x - \text{cosec} \, x + C$

(g) $\int \text{cosec}^2 x (1 + \tan^2 x) \, dx$

$= \int (\text{cosec}^2 x + \text{cosec}^2 x \tan^2 x) \, dx$

$= \int (\text{cosec}^2 x + \sec^2 x) \, dx$

$= -\cot x + \tan x + C$

(h) $\int \sec^2 x (1 - \cot^2 x) \, dx$

$= \int (\sec^2 x - \sec^2 x \cot^2 x) \, dx$

$= \int (\sec^2 x - \text{cosec}^2 x) \, dx$

$= \tan x + \cot x + C$

(i) $\int \sec^2 x (1 + e^x \cos^2 x) \, dx$

$= \int (\sec^2 x + e^x \cos^2 x \sec^2 x) \, dx$

$= \int (\sec^2 x + e^x) \, dx$

$= \tan x + e^x + C$

(j) $\int \left(\frac{1 + \sin x}{\cos^2 x} + \cos^2 x \sec x \right) dx$

$= \int (\sec^2 x + \tan x \sec x + \cos x) \, dx$

$= \tan x + \sec x + \sin x + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 1

Question:

Integrate the following:

(a) $\sin(2x+1)$

(b) $3e^{2x}$

(c) $4e^{x+5}$

(d) $\cos(1-2x)$

(e) $\operatorname{cosec}^2 3x$

(f) $\sec 4x \tan 4x$

(g) $3\sin\left(\frac{1}{2}x+1\right)$

(h) $\sec^2(2-x)$

(i) $\operatorname{cosec} 2x \cot 2x$

(j) $\cos 3x - \sin 3x$

Solution:

(a) $\int \sin(2x+1)\,dx = -\frac{1}{2}\cos(2x+1) + C$

(b) $\int 3e^{2x}dx = \frac{3}{2}e^{2x} + C$

(c) $\int 4e^{x+5}dx = 4e^{x+5} + C$

(d) $\int \cos(1-2x)\,dx = -\frac{1}{2}\sin(1-2x) + C$

OR Let $y = \sin(1-2x)$

then $\frac{dy}{dx} = \cos\left(1 - 2x\right) \times \left(-2\right)$ (by chain rule)

$\therefore \int \cos\left(1 - 2x\right) dx = -\frac{1}{2}\sin\left(1 - 2x\right) + C$

(e) $\int \text{cosec}^2 3x \, dx = -\frac{1}{3}\cot 3x + C$

(f) $\int \sec 4x \tan 4x \, dx = \frac{1}{4}\sec 4x + C$

(g) $\int 3\sin\left(\frac{1}{2}x + 1\right) dx = -6\cos\left(\frac{1}{2}x + 1\right) + C$

(h) $\int \sec^2 (2 - x)\, dx = -\tan(2 - x) + C$

OR Let $y = \tan(2 - x)$

then $\frac{dy}{dx} = \sec^2\left(2 - x\right) \times \left(-1\right)$ (by chain rule)

$\therefore \int \sec^2 (2 - x)\, dx = -\tan(2 - x) + C$

(i) $\int \text{cosec}\, 2x \cot 2x \, dx = -\frac{1}{2}\text{cosec}\, 2x + C$

(j) $\int (\cos 3x - \sin 3x)\, dx$

$= \frac{1}{3}\sin 3x + \frac{1}{3}\cos 3x + C$

$= \frac{1}{3}\left(\sin 3x + \cos 3x\right) + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 2

Question:

Find the following integrals:

(a) $\int \left(e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right) dx$

(b) $\int (e^{x} + 1)^{2} dx$

(c) $\int \sec^{2} 2x (1 + \sin 2x) dx$

(d) $\int \left(\frac{3 - 2\cos (\frac{1}{2} x)}{\sin^{2} (\frac{1}{2} x)} \right) dx$

(e) $\int [e^{3 - x} + \sin (3 - x) + \cos (3 - x)] dx$

Solution:

(a) $\int \left[e^{2x} - \frac{1}{2} \sin \left(2x - 1 \right) \right] dx = \frac{1}{2} e^{2x} + \frac{1}{4} \cos \left(2x - 1 \right) + C$

(b) $\int (e^{x} + 1)^{2} dx$
$= \int (e^{2x} + 2e^{x} + 1) dx$
$= \frac{1}{2} e^{2x} + 2e^{x} + x + C$

(c) $\int \sec^{2} 2x (1 + \sin 2x) dx$
$= \int (\sec^{2} 2x + \sec^{2} 2x \sin 2x) dx$
$= \int (\sec^{2} 2x + \sec 2x \tan 2x) dx$
$= \frac{1}{2} \tan 2x + \frac{1}{2} \sec 2x + C$

(d) $\int \left[\dfrac{3 - 2\cos\left(\frac{1}{2}x\right)}{\sin^2\left(\frac{1}{2}x\right)} \right] dx$

$= \int \left(3\operatorname{cosec}^2 \frac{1}{2}x - 2\operatorname{cosec} \frac{1}{2}x \cot \frac{1}{2}x \right) dx$

$= -6\cot\left(\frac{1}{2}x\right) + 4\operatorname{cosec}\left(\frac{1}{2}x\right) + C$

(e) $\int \left[e^{3-x} + \sin(3-x) + \cos(3-x) \right] dx$

$= -e^{3-x} + \cos(3-x) - \sin(3-x) + C$

Note: extra minus signs from $-x$ terms and chain rule.

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise B, Question 3

Question:

Integrate the following:

(a) $\frac{1}{2x+1}$

(b) $\frac{1}{(2x+1)^2}$

(c) $(2x+1)^2$

(d) $\frac{3}{4x-1}$

(e) $\frac{3}{1-4x}$

(f) $\frac{3}{(1-4x)^2}$

(g) $(3x+2)^5$

(h) $\frac{3}{(1-2x)^3}$

(i) $\frac{6}{(3-2x)^4}$

(j) $\frac{5}{3-2x}$

Solution:

(a) $\int \frac{1}{2x+1} dx = \frac{1}{2} \ln |2x+1| + C$

(b) $\int \frac{1}{(2x+1)^2}\,dx$

$= \int (2x+1)^{-2}\,dx$

$= \frac{(2x+1)^{-1}}{-1} \times \frac{1}{2} + C$

$= -\frac{1}{2(2x+1)} + C$

(c) $\int (2x+1)^2\,dx$

$= \frac{(2x+1)^3}{3} \times \frac{1}{2} + C$

$= \frac{(2x+1)^3}{6} + C$

(d) $\int \frac{3}{4x-1}\,dx = \frac{3}{4}\ln|4x-1| + C$

(e) $\int \frac{3}{1-4x}\,dx$

$= -\int \frac{3}{4x-1}\,dx$

$= -\frac{3}{4}\ln|4x-1| + C$

OR Let $y = \ln|1-4x|$

then $\frac{dy}{dx} = \frac{1}{1-4x} \times \left(-4\right)$ (by chain rule)

$\therefore \int \frac{3}{1-4x}\,dx = -\frac{3}{4}\ln|1-4x| + C$

Note: $\ln|1-4x| = \ln|4x-1|$ because of $|\quad|$ sign.

(f) $\int \frac{3}{(1-4x)^2}\,dx$

$= \int 3(1-4x)^{-2}\,dx$

$= \frac{3}{-4} \times \frac{(1-4x)^{-1}}{-1}$

$= \frac{3}{4(1-4x)} + C$

(g) $\int (3x+2)^5 dx = \frac{(3x+2)^6}{18} + C$

(h) $\int \frac{3}{(1-2x)^3} dx = \frac{3}{-2} \times \frac{(1-2x)^{-2}}{-2} + C = \frac{3}{4(1-2x)^2} + C$

OR Let $y = (1-2x)^{-2}$

then $\frac{dy}{dx} = -2(1-2x)^{-3} \times \left(-2\right)$ (by chain rule)

$\therefore \int \frac{3}{(1-2x)^3} dx = \frac{3}{4}(1-2x)^{-2} + C$

(i) $\int \frac{6}{(3-2x)^4} dx = \frac{6}{-2} \times \frac{(3-2x)^{-3}}{-3} + C = \frac{1}{(3-2x)^3} + C$

OR Let $y = (3-2x)^{-3}$

then $\frac{dy}{dx} = -3(3-2x)^{-4} \times \left(-2\right)$

$\therefore \int \frac{6}{(3-2x)^4} dx = \frac{1}{(3-2x)^3} + C$

(j) $\int \frac{5}{(3-2x)} dx = -\frac{5}{2} \ln|3-2x| + C$

OR Let $y = \ln|3-2x|$

then $\frac{dy}{dx} = \frac{1}{3-2x} \times \left(-2\right)$ (by chain rule)

$\therefore \int \frac{5}{3-2x} dx = -\frac{5}{2} \ln|3-2x| + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise B, Question 4

Question:

Find the following integrals

(a) $\int \left(3\sin\left(2x+1\right) + \frac{4}{2x+1} \right) dx$

(b) $\int \left[e^{5x} + (1-x)^5 \right] dx$

(c) $\int \left(\frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2} \right) dx$

(d) $\int \left[(3x+2)^2 + \frac{1}{(3x+2)^2} \right] dx$

Solution:

(a) $\int \left[3\sin\left(2x+1\right) + \frac{4}{2x+1} \right] dx$

$= -\frac{3}{2}\cos\left(2x+1\right) + \frac{4}{2}\ln|2x+1| + C$

$= -\frac{3}{2}\cos\left(2x+1\right) + 2\ln|2x+1| + C$

(b) $\int \left[e^{5x} + (1-x)^5 \right] dx$

$= \int e^{5x} dx + \int (1-x)^5 dx$

$= \frac{1}{5}e^{5x} - \frac{1}{6}(1-x)^6 + C$ (from ⑪ and ⑩)

OR Let $y = (1-x)^6$

then $\frac{dy}{dx} = 6(1-x)^5 \times \left(-1\right)$ (by chain rule)

$\therefore \int (1-x)^5 dx = -\frac{1}{6}(1-x)^6 + C$

(c) $\int \left[\frac{1}{\sin^2 2x} + \frac{1}{1+2x} + \frac{1}{(1+2x)^2} \right] dx$

$$= \int \left[\operatorname{cosec}^2 2x + \frac{1}{1+2x} + (1+2x)^{-2} \right] dx$$

$$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| + \frac{(1+2x)^{-1}}{-1} \times \frac{1}{2} + C$$

$$= -\frac{1}{2}\cot 2x + \frac{1}{2}\ln|1+2x| - \frac{1}{2(1+2x)} + C$$

(d) $\int \left[(3x+2)^2 + \frac{1}{(3x+2)^2} \right] dx$

$$= \int [(3x+2)^2 + (3x+2)^{-2}]\, dx$$

$$= \frac{(3x+2)^3}{9} - \frac{(3x+2)^{-1}}{3} + C$$

$$= \frac{(3x+2)^3}{9} - \frac{1}{3(3x+2)} + C$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 1

Question:

Integrate the following:

(a) $\cot^2 x$

(b) $\cos^2 x$

(c) $\sin 2x \cos 2x$

(d) $(1 + \sin x)^2$

(e) $\tan^2 3x$

(f) $(\cot x - \operatorname{cosec} x)^2$

(g) $(\sin x + \cos x)^2$

(h) $\sin^2 x \cos^2 x$

(i) $\dfrac{1}{\sin^2 x \quad \cos^2 x}$

(j) $(\cos 2x - 1)^2$

Solution:

(a) $\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$
$= - \cot x - x + C$

(b) $\int \cos^2 x \, dx = \int \frac{1}{2} \left(1 + \cos 2x \right) dx$

$= \frac{1}{2} x + \frac{1}{4} \sin 2x + C$

(c) $\int \sin 2x \cos 2x \, dx = \int \frac{1}{2} \sin 4x \, dx$

$= -\frac{1}{8}\cos 4x + C$

(d) $\int (1+\sin x)^2 \, dx = \int (1 + 2\sin x + \sin^2 x)\, dx$

But $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\therefore \int (1+\sin x)^2 \, dx = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right) dx$

$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + C$

(e) $\int \tan^2 3x \, dx = \int (\sec^2 3x - 1)\, dx$

$= \frac{1}{3}\tan 3x - x + C$

(f) $\int (\cot x - \operatorname{cosec} x)^2 \, dx = \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x)\, dx$

$= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x)\, dx$

$= -2\cot x - x + 2\operatorname{cosec} x + C$

(g) $\int (\sin x + \cos x)^2 \, dx = \int (\sin^2 x + 2\sin x \cos x + \cos^2 x)\, dx$

$= \int (1 + \sin 2x)\, dx$

$= x - \frac{1}{2}\cos 2x + C$

(h) $\int \sin^2 x \cos^2 x \, dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \int \frac{1}{4}\sin^2 2x \, dx$

$= \int \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$

$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) dx$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

(i) $\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\left(\frac{1}{2}\sin 2x\right)^2} = 4\operatorname{cosec}^2 2x$

$$\therefore \int \frac{1}{\sin^2 x \cos^2 x} \, dx = \int 4 \operatorname{cosec}^2 2x \, dx$$
$$= -2 \cot 2x + C$$

(j) $\int (\cos 2x - 1)^2 dx = \int (\cos^2 2x - 2\cos 2x + 1) dx$

$$= \int \left(\frac{1}{2} \cos 4x + \frac{1}{2} - 2\cos 2x + 1 \right) dx$$
$$= \int \left(\frac{1}{2} \cos 4x + \frac{3}{2} - 2\cos 2x \right) dx$$
$$= \frac{1}{8} \sin 4x + \frac{3}{2} x - \sin 2x + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 2

Question:

Find the following integrals:

(a) $\int \left(\frac{1 - \sin x}{\cos^2 x} \right) dx$

(b) $\int \left(\frac{1 + \cos x}{\sin^2 x} \right) dx$

(c) $\int \frac{\cos 2x}{\cos^2 x} dx$

(d) $\int \frac{\cos^2 x}{\sin^2 x} dx$

(e) $\int \frac{(1 + \cos x)^2}{\sin^2 x} dx$

(f) $\int \frac{(1 + \sin x)^2}{\cos^2 x} dx$

(g) $\int (\cot x - \tan x)^2 dx$

(h) $\int (\cos x - \sin x)^2 dx$

(i) $\int (\cos x - \sec x)^2 dx$

(j) $\int \frac{\cos 2x}{1 - \cos^2 2x} dx$

Solution:

(a) $\int \left(\frac{1 - \sin x}{\cos^2 x} \right) dx = \int \left(\sec^2 x - \tan x \sec x \right) dx$

$= \tan x - \sec x + C$

(b) $\int \left(\frac{1 + \cos x}{\sin^2 x} \right) dx = \int \left(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x \right) dx$

$= - \cot x - \operatorname{cosec} x + C$

(c) $\int \frac{\cos 2x}{\cos^2 x} dx = \int \frac{2\cos^2 x - 1}{\cos^2 x} dx$

$= \int (2 - \sec^2 x) dx$

$= 2x - \tan x + C$

(d) $\int \frac{\cos^2 x}{\sin^2 x} dx = \int \cot^2 x \, dx$

$= \int (\operatorname{cosec}^2 x - 1) dx$

$= - \cot x - x + C$

(e) $I = \int \frac{(1 + \cos x)^2}{\sin^2 x} dx = \int \frac{1 + 2\cos x + \cos^2 x}{\sin^2 x} dx$

$= \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x) dx$

But $\operatorname{cosec}^2 x = 1 + \cot^2 x \Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

$\therefore I = \int (2\operatorname{cosec}^2 x - 1 + 2\cot x \operatorname{cosec} x) dx$

$= - 2\cot x - x - 2\operatorname{cosec} x + C$

(f) $I = \int \frac{(1 + \sin x)^2}{\cos^2 x} dx = \int \frac{1 + 2\sin x + \sin^2 x}{\cos^2 x} dx$

$= \int (\sec^2 x + 2\tan x \sec x + \tan^2 x) dx$

But $\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$\therefore I = \int (2\sec^2 x - 1 + 2\tan x \sec x) dx$

$= 2\tan x - x + 2\sec x + C$

(g) $\int (\cot x - \tan x)^2 dx = \int (\cot^2 x - 2\cot x \tan x + \tan^2 x) dx$

$= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1) dx$

$= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x) dx$

$= - \cot x - 4x + \tan x + C$

(h) $\int (\cos x - \sin x)^2 dx = \int (\cos^2 x - 2\cos x \sin x + \sin^2 x) dx$

$= \int (1 - \sin 2x) dx$

$= x + \frac{1}{2} \cos 2x + C$

(i) $\int (\cos x - \sec x)^2 dx = \int (\cos^2 x - 2\cos x \sec x + \sec^2 x)\, dx$

$= \int \left(\frac{1}{2}\cos 2x + \frac{1}{2} - 2 + \sec^2 x \right) dx$

$= \int \left(\frac{1}{2}\cos 2x - \frac{3}{2} + \sec^2 x \right) dx$

$= \frac{1}{4}\sin 2x - \frac{3}{2}x + \tan x + C$

(j) $\int \frac{\cos 2x}{1 - \cos^2 2x} dx = \int \frac{\cos 2x}{\sin^2 2x} dx$

$= \int \cot 2x \operatorname{cosec} 2x\, dx$

$= -\frac{1}{2}\operatorname{cosec} 2x + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise C, Question 3

Question:

Find the following integrals:

(a) $\int \cos 2x \cos x \, dx$

(b) $\int 2 \sin 5x \cos 3x \, dx$

(c) $\int 2 \sin 3x \cos 5x \, dx$

(d) $\int 2 \sin 2x \sin 5x \, dx$

(e) $4 \int \cos 3x \cos 7x \, dx$

(f) $\int 2 \cos 4x \cos 4x \, dx$

(g) $\int 2 \cos 4x \sin 4x \, dx$

(h) $\int 2 \sin 4x \sin 4x \, dx$

Solution:

(a) $\cos 3x + \cos x = 2 \cos \frac{3x + x}{2} \cos \frac{3x - x}{2} = 2 \cos 2x \cos x$

$\therefore \int \cos 2x \cos x \, dx = \frac{1}{2} \int \left(\cos 3x + \cos x \right) dx$

$= \frac{1}{2} \left(\frac{1}{3} \sin 3x + \sin x \right) + C$

$= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + C$

(b) $\sin 8x + \sin 2x = 2 \sin 5x \cos 3x$

$\therefore \int 2 \sin 5x \cos 3x \, dx = \int (\sin 8x + \sin 2x) \, dx$

$= - \frac{1}{8} \cos 8x - \frac{1}{2} \cos 2x + C$

(c) $\sin 8x - \sin 2x = 2 \sin 3x \cos 5x$

$\therefore \int 2 \sin 3x \cos 5x \, dx = \int (\sin 8x - \sin 2x) \, dx$

$= -\frac{1}{8}\cos 8x + \frac{1}{2}\cos 2x + C$

(d) $\cos 7x - \cos 3x = -2\sin 5x \sin 2x$

$\therefore \int 2\sin 2x \sin 5x \, dx = \int (\cos 3x - \cos 7x)\, dx$

$= \frac{1}{3}\sin 3x - \frac{1}{7}\sin 7x + C$

(e) $\cos 10x + \cos 4x = 2\cos 7x \cos 3x$

$\therefore \int 4\cos 3x \cos 7x \, dx = 2\int (\cos 10x + \cos 4x)\, dx$

$= 2\left(\frac{1}{10}\sin 10x + \frac{1}{4}\sin 4x\right) + C$

$= \frac{1}{5}\sin 10x + \frac{1}{2}\sin 4x + C$

(f) $\cos 8x + \cos 0x = 2\cos 4x \cos 4x$

i.e. $\cos 8x + 1 = 2\cos 4x \cos 4x$

$\therefore \int 2\cos 4x \cos 4x \, dx = \int (1 + \cos 8x)\, dx$

$= x + \frac{1}{8}\sin 8x + C$

(g) $\sin 8x + \sin 0x = 2\sin 4x \cos 4x$

$\therefore \int 2\cos 4x \sin 4x \, dx = \int \sin 8x \, dx$

$= -\frac{1}{8}\cos 8x + C$

(h) $\cos 8x - \cos 0x = -2\sin 4x \sin 4x$

i.e. $\cos 8x - 1 = -2\sin 4x \sin 4x$

$\therefore \int 2\sin 4x \sin 4x \, dx = \int (1 - \cos 8x)\, dx$

$= x - \frac{1}{8}\sin 8x + C$

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Integration
Exercise D, Question 1

Question:

Use partial fractions to integrate the following:

(a) $\frac{3x+5}{(x+1)(x+2)}$

(b) $\frac{3x-1}{(2x+1)(x-2)}$

(c) $\frac{2x-6}{(x+3)(x-1)}$

(d) $\frac{3}{(2+x)(1-x)}$

(e) $\frac{4}{(2x+1)(1-2x)}$

(f) $\frac{3(x+1)}{9x^2-1}$

(g) $\frac{3-5x}{(1-x)(2-3x)}$

(h) $\frac{x^2-3}{(2+x)(1+x)^2}$

(i) $\frac{5+3x}{(x+2)(x+1)^2}$

(j) $\frac{17-5x}{(3+2x)(2-x)^2}$

Solution:

(a) $\frac{3x+5}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$

$\Rightarrow \quad 3x + 5 \equiv A(x+2) + B(x+1)$

$x = -1 \quad \Rightarrow \quad 2 = A$

$x = -2 \quad \Rightarrow \quad -1 = -B \quad \Rightarrow \quad B = 1$

$\therefore \int \frac{3x+5}{(x+1)(x+2)} dx = \int \left(\frac{2}{x+1} + \frac{1}{x+2} \right) dx$

$= 2\ln|x+1| + \ln|x+2| + C$

$= \ln[|x+1|^2] + \ln|x+2| + C$

$= \ln|(x+1)^2(x+2)| + C$

(b) $\frac{3x-1}{(2x+1)(x-2)} \equiv \frac{A}{2x+1} + \frac{B}{x-2}$

$\Rightarrow \quad 3x - 1 \equiv A(x-2) + B(2x+1)$

$x = 2 \quad \Rightarrow \quad 5 = 5B \quad \Rightarrow \quad B = 1$

$x = -\frac{1}{2} \quad \Rightarrow \quad -\frac{5}{2} = -\frac{5}{2}A \quad \Rightarrow \quad A = 1$

$\therefore \int \frac{3x-1}{(2x+1)(x-2)} dx = \int \left(\frac{1}{2x+1} + \frac{1}{x-2} \right) dx$

$= \frac{1}{2}\ln|2x+1| + \ln|x-2| + C$

$= \ln|(x-2)\sqrt{2x+1}| + C$

(c) $\frac{2x-6}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$

$\Rightarrow \quad 2x - 6 \equiv A(x-1) + B(x+3)$

$x = 1 \quad \Rightarrow \quad -4 = 4B \quad \Rightarrow \quad B = -1$

$x = -3 \quad \Rightarrow \quad -12 = -4A \quad \Rightarrow \quad A = 3$

$\therefore \int \frac{2x-6}{(x+3)(x-1)} dx = \int \left(\frac{3}{x+3} - \frac{1}{x-1} \right) dx$

$= 3\ln|x+3| - \ln|x-1| + C$

$= \ln\left|\frac{(x+3)^3}{x-1}\right| + C$

(d) $\frac{3}{(2+x)(1-x)} \equiv \frac{A}{(2+x)} + \frac{B}{1-x}$

$\Rightarrow \quad 3 \equiv A(1-x) + B(2+x)$

$x = 1 \quad \Rightarrow \quad 3 = 3B \quad \Rightarrow \quad B = 1$

$x = -2 \quad \Rightarrow \quad 3 = 3A \quad \Rightarrow \quad A = 1$

$$\therefore \int \frac{3}{(2+x)(1-x)}dx = \int \left(\frac{1}{2+x} + \frac{1}{1-x} \right) dx.$$

$$= \ln|2+x| - \ln|1-x| + C$$

$$= \ln\left|\frac{2+x}{1-x}\right| + C$$

(e) $\frac{4}{(2x+1)(1-2x)} \equiv \frac{A}{2x+1} + \frac{B}{1-2x}$

$\Rightarrow \quad 4 \equiv A(1-2x) + B(2x+1)$

$x = \frac{1}{2} \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2$

$x = -\frac{1}{2} \quad \Rightarrow \quad 4 = 2A \quad \Rightarrow \quad A = 2$

$$\therefore \int \frac{4}{(2x+1)(1-2x)}dx = \int \left(\frac{2}{2x+1} + \frac{2}{1-2x} \right) dx$$

$$= \ln|2x+1| - \ln|1-2x| + C$$

$$= \ln\left|\frac{2x+1}{1-2x}\right| + C$$

(f) $\frac{3(x+1)}{9x^2-1} \equiv \frac{3(x+1)}{(3x-1)(3x+1)} \equiv \frac{A}{3x-1} + \frac{B}{3x+1}$

$\Rightarrow \quad 3x+3 \equiv A(3x+1) + B(3x-1)$

$x = -\frac{1}{3} \quad \Rightarrow \quad 2 = -2B \quad \Rightarrow \quad B = -1$

$x = \frac{1}{3} \quad \Rightarrow \quad 4 = 2A \quad \Rightarrow \quad A = 2$

$$\therefore \int \frac{3(x+1)}{9x^2-1}dx = \int \left(\frac{2}{3x-1} - \frac{1}{3x+1} \right) dx$$

$$= \frac{2}{3}\ln|3x-1| - \frac{1}{3}\ln|3x+1| + C$$

$$= \frac{1}{3}\ln\left|\frac{(3x-1)^2}{3x+1}\right| + C$$

(g) $\frac{3-5x}{(1-x)(2-3x)} \equiv \frac{A}{1-x} + \frac{B}{2-3x}$

$\Rightarrow \quad 3-5x \equiv A(2-3x) + B(1-x)$

$x = \frac{2}{3} \quad \Rightarrow \quad -\frac{1}{3} = \frac{1}{3}B \quad \Rightarrow \quad B = -1$

$x = 1 \Rightarrow -2 = -A \Rightarrow A = 2$

$$\therefore \int \frac{3-5x}{(1-x)(2-3x)}dx = \int \left(\frac{2}{1-x} - \frac{1}{2-3x} \right) dx$$

$$= -2\ln|1-x| + \frac{1}{3}\ln|2-3x| + C$$

$$= \ln\left| \frac{(2-3x)^{\frac{1}{3}}}{(1-x)^2} \right| + C$$

(h) $\dfrac{x^2-3}{(2+x)(1+x)^2} \equiv \dfrac{A}{2+x} + \dfrac{B}{1+x} + \dfrac{C}{(1+x)^2}$

$\Rightarrow x^2 - 3 \equiv A(1+x)^2 + B(2+x)(1+x) + C(2+x)$

$x = -1 \Rightarrow -2 = C \Rightarrow C = -2$

$x = -2 \Rightarrow 1 = 1A \Rightarrow A = 1$

Coefficient of $x^2 \Rightarrow 1 = A + B \Rightarrow B = 0$

$$\therefore \int \frac{x^2-3}{(2+x)(1+x)^2}dx = \int \left(\frac{1}{2+x} - \frac{2}{(1+x)^2} \right) dx$$

$$= \ln|2+x| - 2\frac{(1+x)^{-1}}{-1} + C$$

$$= \ln|2+x| + \frac{2}{1+x} + C$$

(i) $\dfrac{5+3x}{(x+2)(x+1)^2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$\Rightarrow 5 + 3x \equiv A(x+1)^2 + B(x+2)(x+1) + C(x+2)$

$x = -1 \Rightarrow 2 = C \Rightarrow C = 2$

$x = -2 \Rightarrow -1 = A \Rightarrow A = -1$

Coefficient of $x^2 \Rightarrow 0 = A + B \Rightarrow B = 1$

$$\therefore \int \frac{5+3x}{(x+2)(x+1)^2}dx = \int \left(-\frac{1}{x+2} + \frac{1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= -\ln|x+2| + \ln|x+1| - \frac{2}{x+1} + C$$

$$= \ln\left| \frac{x+1}{x+2} \right| - \frac{2}{x+1} + C$$

(j) $\frac{17-5x}{(3+2x)(2-x)^2} \equiv \frac{A}{3+2x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$

$\Rightarrow \quad 17-5x \equiv A(2-x)^2 + B(3+2x)(2-x) + C(3+2x)$

$x = 2 \quad \Rightarrow \quad 7 = 7C \quad \Rightarrow \quad C = 1$

$x = -\frac{3}{2} \quad \Rightarrow \quad \frac{49}{2} = \frac{49}{4}A \quad \Rightarrow \quad A = 2$

Coefficient of $x^2 \quad \Rightarrow \quad 0 = A - 2B \quad \Rightarrow \quad B = 1$

$\therefore \int \frac{17-5x}{(3+2x)(2-x)^2}\mathrm{d}x = \int \left(\frac{2}{3+2x} + \frac{1}{2-x} + \frac{1}{(2-x)^2} \right) \mathrm{d}x$

$= \frac{2}{2}\ln|3+2x| - \ln|2-x| + \frac{1}{2-x} + C$

$= \ln\left|\frac{3+2x}{2-x}\right| + \frac{1}{2-x} + C$

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Integration
Exercise D, Question 2

Question:

Find the following integrals:

(a) $\int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\mathrm{d}x$

(b) $\int \frac{x^3+2x^2+2}{x(x+1)}\mathrm{d}x$

(c) $\int \frac{x^2}{x^2-4}\mathrm{d}x$

(d) $\int \frac{x^2+x+2}{3-2x-x^2}\mathrm{d}x$

(e) $\int \frac{6+3x-x^2}{x^3+2x^2}\mathrm{d}x$

Solution:

(a) $\frac{2(x^2+3x-1)}{(x+1)(2x-1)} \equiv 1 + \frac{A}{x+1} + \frac{B}{2x-1}$

$\Rightarrow \quad 2x^2+6x-2 \equiv (x+1)(2x-1) + A(2x-1) + B(x+1)$

$x = -1 \quad \Rightarrow \quad -6 = -3A \quad \Rightarrow \quad A = 2$

$x = \frac{1}{2} \quad \Rightarrow \quad \frac{3}{2} = \frac{3}{2}B \quad \Rightarrow \quad B = 1$

$\therefore \int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\mathrm{d}x = \int \left(1 + \frac{2}{x+1} + \frac{1}{2x-1}\right)\mathrm{d}x$

$= x + 2\ln|x+1| + \frac{1}{2}\ln|2x-1| + C$

$= x + \ln|(x+1)^2\sqrt{2x-1}| + C$

(b) $\frac{x^3+2x^2+2}{x(x+1)} \quad \Rightarrow$

$$\begin{array}{r} x+1 \\ x^2+x \overline{\smash{)}\, x^3+2x^2+2} \\ \underline{x^3+x^2} \\ x^2+2 \\ \underline{x^2+x} \\ 2-x \end{array}$$

$$\frac{x^3+2x^2+2}{x(x+1)} \equiv x+1+\frac{2-x}{x(x+1)}$$

$$\equiv x+1+\frac{A}{x}+\frac{B}{x+1}$$

$$\Rightarrow \quad x^3+2x^2+2 \equiv (x+1)x(x+1)+A(x+1)+Bx$$

$x=0 \quad \Rightarrow \quad 2=A \quad \Rightarrow \quad A=2$

$x=-1 \quad \Rightarrow \quad 3=-B \quad \Rightarrow \quad B=-3$

$$\therefore \int \frac{x^3+2x^2+2}{x(x+1)}\mathrm{d}x = \int \left(x+1+\frac{2}{x}-\frac{3}{x+1} \right) \mathrm{d}x$$

$$= \frac{x^2}{2}+x+2\ln|x| - 3\ln|x+1| + C$$

$$= \frac{x^2}{2}+x+\ln\left|\frac{x^2}{(x+1)^3}\right| + C$$

(c) $\frac{x^2}{x^2-4} \equiv 1+\frac{A}{x-2}+\frac{B}{x+2}$

$$\Rightarrow \quad x^2 \equiv (x-2)(x+2)+A(x+2)+B(x-2)$$

$x=2 \quad \Rightarrow \quad 4=4A \quad \Rightarrow \quad A=1$

$x=-2 \quad \Rightarrow \quad 4=-4B \quad \Rightarrow \quad B=-1$

$$\therefore \int \frac{x^2}{x^2-4}\mathrm{d}x = \int \left(1+\frac{1}{x-2}-\frac{1}{x+2} \right) \mathrm{d}x$$

$$= x+\ln|x-2| - \ln|x+2| + C$$

$$= x+\ln\left|\frac{x-2}{x+2}\right| + C$$

(d) $\frac{x^2+x+2}{3-2x-x^2} \equiv \frac{x^2+x+2}{(3+x)(1-x)} \equiv -1+\frac{A}{3+x}+\frac{B}{1-x}$

$$\Rightarrow \quad x^2+x+2 \equiv -1(3+x)(1-x)+A(1-x)+B(3+x)$$

$x=1 \quad \Rightarrow \quad 4=4B \quad \Rightarrow \quad B=1$

$x=-3 \quad \Rightarrow \quad 8=4A \quad \Rightarrow \quad A=2$

$$\therefore \int \frac{x^2+x+2}{3-2x-x^2}\mathrm{d}x = \int \left(-1+\frac{2}{3+x}+\frac{1}{1-x} \right) \mathrm{d}x$$

$$= -x+2\ln|3+x| - \ln|1-x| + C$$

$$= -x+\ln\left|\frac{(3+x)^2}{1-x}\right| + C$$

(e) $\frac{6+3x-x^2}{x^3+2x^2} \equiv \frac{6+3x-x^2}{x^2(x+2)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$

$\Rightarrow \quad 6+3x-x^2 \equiv Ax(x+2) + B(x+2) + Cx^2$

$x=0 \quad \Rightarrow \quad 6=2B \quad \Rightarrow \quad B=3$

$x=-2 \quad \Rightarrow \quad -4=4C \quad \Rightarrow \quad C=-1$

Coefficient of $x^2 \quad \Rightarrow \quad -1=A+C \quad \Rightarrow \quad A=0$

$\therefore \int \frac{6+3x-x^2}{x^3+2x^2}dx = \int \left(\frac{3}{x^2} - \frac{1}{x+2} \right) dx$

$= -\frac{3}{x} - \ln|x+2| + C$

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Integration
Exercise E, Question 1

Question:

Integrate the following functions:

(a) $\frac{x}{x^2 + 4}$

(b) $\frac{e^{2x}}{e^{2x} + 1}$

(c) $\frac{x}{(x^2 + 4)^3}$

(d) $\frac{e^{2x}}{(e^{2x} + 1)^3}$

(e) $\frac{\cos 2x}{3 + \sin 2x}$

(f) $\frac{\sin 2x}{(3 + \cos 2x)^3}$

(g) xe^{x^2}

(h) $\cos 2x (1 + \sin 2x)^4$

(i) $\sec^2 x \tan^2 x$

(j) $\sec^2 x (1 + \tan^2 x)$

Solution:

(a) $y = \ln | x^2 + 4 |$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^2 + 4} \times 2x$ (chain rule)

$$\therefore \int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4| + C$$

(b) $y = \ln|e^{2x}+1|$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^{2x}+1} \times e^{2x} \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\ln|e^{2x}+1| + C$$

(c) $y = (x^2+4)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2(x^2+4)^{-3} \times 2x \quad \text{(chain rule)}$$

$$\therefore \int \frac{x}{(x^2+4)^3}\,dx = -\frac{1}{4}(x^2+4)^{-2} + C$$

$$\text{or} \quad -\frac{1}{4(x^2+4)^2} + C$$

(d) $y = (e^{2x}+1)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2(e^{2x}+1)^{-3} \times e^{2x} \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{e^{2x}}{(e^{2x}+1)^3}\,dx = -\frac{1}{4}(e^{2x}+1)^{-2} + C$$

$$\text{or} \quad -\frac{1}{4(e^{2x}+1)^2} + C$$

(e) $y = \ln|3+\sin 2x|$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3+\sin 2x} \times \cos 2x \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{\cos 2x}{3+\sin 2x}\,dx = \frac{1}{2}\ln|3+\sin 2x| + C$$

(f) $y = (3+\cos 2x)^{-2}$

$$\Rightarrow \frac{dy}{dx} = -2(3+\cos 2x)^{-3} \times \left(-\sin 2x\right) \times 2 \quad \text{(chain rule)}$$

$$\therefore \int \frac{\sin 2x}{(3+\cos 2x)^3}\,dx = \frac{1}{4}(3+\cos 2x)^{-2} + C$$

$$\text{or} \quad \frac{1}{4(3+\cos 2x)^2} + C$$

(g) $y = e^{x^2}$

$\Rightarrow \quad \frac{dy}{dx} = e^{x^2} \times 2x \quad$ (chain rule)

$\therefore \int x e^{x^2} dx = \frac{1}{2} e^{x^2} + C$

(h) $y = (1 + \sin 2x)^5$

$\Rightarrow \quad \frac{dy}{dx} = 5(1 + \sin 2x)^4 \times \cos 2x \times 2 \quad$ (chain rule)

$\therefore \int \cos 2x (1 + \sin 2x)^4 dx = \frac{1}{10} (1 + \sin 2x)^5 + C$

(i) $y = \tan^3 x$

$\Rightarrow \quad \frac{dy}{dx} = 3 \tan^2 x \times \sec^2 x \quad$ (chain rule)

$\therefore \int \sec^2 x \tan^2 x \, dx = \frac{1}{3} \tan^3 x + C$

(j) $\sec^2 x (1 + \tan^2 x) = \sec^2 x + \sec^2 x \tan^2 x$

$\therefore \int \sec^2 x (1 + \tan^2 x) dx = \int (\sec^2 x + \sec^2 x \tan^2 x) dx$

$= \tan x + \frac{1}{3} \tan^3 x + C$

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Integration
Exercise E, Question 2

Question:

Find the following integrals:

(a) $\int (x+1)(x^2+2x+3)^4 \, dx$

(b) $\int \text{cosec}^2 2x \cot 2x \, dx$

(c) $\int \sin^5 3x \cos 3x \, dx$

(d) $\int \cos x \, e^{\sin x} \, dx$

(e) $\int \frac{e^{2x}}{e^{2x}+3} \, dx$

(f) $\int x (x^2+1)^{\frac{3}{2}} \, dx$

(g) $\int (2x+1)\sqrt{x^2+x+5} \, dx$

(h) $\int \frac{2x+1}{\sqrt{x^2+x+5}} \, dx$

(i) $\int \frac{\sin x \cos x}{\sqrt{\cos 2x+3}} \, dx$

(j) $\int \frac{\sin x \cos x}{\cos 2x+3} \, dx$

Solution:

(a) $y = (x^2+2x+3)^5$

$\Rightarrow \quad \frac{dy}{dx} = 5(x^2+2x+3)^4 \times \left(2x+2\right)$

$= 5(x^2+2x+3)^4 \times 2(x+1)$

$\therefore \int \left(x+1\right)(x^2+2x+3)^4 dx = \frac{1}{10}(x^2+2x+3)^5 + C$

(b) $y = \cot^2 2x$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cot 2x \times \left(-\operatorname{cosec}^2 2x \right) \times 2$$

$$= -4\operatorname{cosec}^2 2x \cot 2x$$

$$\therefore \int \operatorname{cosec}^2 2x \cot 2x \, dx = -\frac{1}{4}\cot^2 2x + C$$

(c) $y = \sin^6 3x$

$$\Rightarrow \quad \frac{dy}{dx} = 6\sin^5 3x \times \cos 3x \times 3$$

$$\therefore \int \sin^5 3x \cos 3x \, dx = \frac{1}{18}\sin^6 3x + C$$

(d) $y = e^{\sin x}$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sin x} \times \cos x$$

$$\therefore \int \cos x \, e^{\sin x} \, dx = e^{\sin x} + C$$

(e) $y = \ln | e^{2x} + 3 |$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^{2x} + 3} \times e^{2x} \times 2$$

$$\therefore \int \frac{e^{2x}}{e^{2x} + 3} dx = \frac{1}{2}\ln | e^{2x} + 3 | + C$$

(f) $y = (x^2 + 1)^{\frac{5}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{5}{2}(x^2 + 1)^{\frac{3}{2}} \times 2x = 5x (x^2 + 1)^{\frac{3}{2}}$$

$$\therefore \int x (x^2 + 1)^{\frac{3}{2}} dx = \frac{1}{5}(x^2 + 1)^{\frac{5}{2}} + C$$

(g) $y = (x^2 + x + 5)^{\frac{3}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}(x^2 + x + 5)^{\frac{1}{2}} \times \left(2x + 1 \right)$$

$$\therefore \int \left(2x + 1 \right) \sqrt{x^2 + x + 5} \, dx = \frac{2}{3}(x^2 + x + 5)^{\frac{3}{2}} + C$$

(h) $y = (x^2 + x + 5)^{\frac{1}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(x^2 + x + 5\right)^{-\frac{1}{2}} \times \left(2x + 1\right)$$

$$= \frac{1}{2}\frac{(2x + 1)}{\sqrt{x^2 + x + 5}}$$

$$\therefore \int \frac{2x + 1}{\sqrt{x^2 + x + 5}} dx = 2\left(x^2 + x + 5\right)^{\frac{1}{2}} + C$$

(i) $y = \left(\cos 2x + 3\right)^{\frac{1}{2}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(\cos 2x + 3\right)^{-\frac{1}{2}} \times \left(-\sin 2x\right) \times 2$$

$$= -\frac{\sin 2x}{\sqrt{\cos 2x + 3}}$$

$$= -\frac{2\sin x \cos x}{\sqrt{\cos 2x + 3}}$$

$$\therefore \int \frac{\sin x \cos x}{\sqrt{\cos 2x + 3}} dx = -\frac{1}{2}\left(\cos 2x + 3\right)^{\frac{1}{2}} + C$$

(j) $y = \ln\left|\cos 2x + 3\right|$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos 2x + 3} \times \left(-\sin 2x\right) \times 2$$

$$= -\frac{2\sin 2x}{\cos 2x + 3}$$

$$= -\frac{4\sin x \cos x}{\cos 2x + 3}$$

$$\therefore \int \frac{\sin x \cos x}{\cos 2x + 3} dx = -\frac{1}{4}\ln\left|\cos 2x + 3\right| + C$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 1

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{1+x}\,dx$; $u = 1 + x$

(b) $\int \frac{x}{\sqrt{1+x}}dx$; $u = 1 + x$

(c) $\int \frac{1+\sin x}{\cos x}dx$; $u = \sin x$

(d) $\int x(3+2x)^5 dx$; $u = 3 + 2x$

(e) $\int \sin^3 x\,dx$; $u = \cos x$

Solution:

(a) $u = 1 + x \quad \Rightarrow \quad du = dx$ and $x = u - 1$

$$\therefore \int x(1+x)^{\frac{1}{2}}dx = \int (u-1)u^{\frac{1}{2}}du$$

$$= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C$$

$$\text{OR} = \frac{2}{15}(1+x)^{\frac{3}{2}}\left[3\left(1+x\right) - 5\right] + C$$

$$= \frac{2}{15}(1+x)^{\frac{3}{2}}\left(3x-2\right) + C$$

(b) $u = 1 + x \quad \Rightarrow \quad du = dx$ and $x = u - 1$

$$\therefore \int \frac{x}{\sqrt{1+x}}dx = \int \frac{u-1}{u^{\frac{1}{2}}}du$$

$= \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\, du$

$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}(1 + x)^{\frac{3}{2}} - 2(1 + x)^{\frac{1}{2}} + C$

OR $= \frac{2}{3}(1 + x)^{\frac{1}{2}} \left[1 + x - 3 \right] + C$

$= \frac{2}{3}(1 + x)^{\frac{1}{2}} \left(x - 2 \right) + C$

(c) $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

$\therefore \int \frac{1 + \sin x}{\cos x} dx = \int \frac{1 + u}{\cos x} \frac{du}{\cos x}$

$= \int \frac{1 + u}{1 - \sin^2 x} du$

$= \int \frac{1 + u}{1 - u^2} du$

$= \int \frac{(1 + u)}{(1 - u)(1 + u)} du$

$= \int \frac{1}{1 - u} du$

$= -\ln | 1 - u | + C$

$= -\ln | 1 - \sin x | + C$

(d) $u = 3 + 2x \Rightarrow du = 2\, dx$ and $x = \frac{u - 3}{2}$

$\therefore \int x(3 + 2x)^5 dx = \int \frac{u - 3}{2} u^5 \frac{du}{2}$

$= \int \left(\frac{u^6}{4} - \frac{3u^5}{4} \right) du$

$= \frac{u^7}{28} - \frac{3u^6}{24} + C$

$= \frac{u^7}{28} - \frac{u^6}{8} + C$

$= \frac{(3 + 2x)^7}{28} - \frac{(3 + 2x)^6}{8} + C$

(e) $u = \cos x \quad \Rightarrow \quad du = -\sin x \, dx$

$\therefore \int \sin^3 x \, dx = \int -(1 - u^2) \, du$

$= \int (u^2 - 1) \, du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cos^3 x}{3} - \cos x + C$

OR $= \frac{\cos x}{3} \left(\cos^2 x - 3 \right) + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 2

Question:

Use the given substitution to find the following integrals:

(a) $\int x\sqrt{2+x}\,dx$; $u^2 = 2 + x$

(b) $\int \frac{2}{\sqrt{x}(x-4)}dx$; $u = \sqrt{x}$

(c) $\int \sec^2 x \tan x\sqrt{1+\tan x}\,dx$; $u^2 = 1 + \tan x$

(d) $\int \frac{\sqrt{x^2+4}}{x}dx$; $u^2 = x^2 + 4$

(e) $\int \sec^4 x\,dx$; $u = \tan x$

Solution:

(a) $u^2 = 2 + x \quad \Rightarrow \quad 2u\,du = dx$ and $x = u^2 - 2$

$\therefore \int x\sqrt{2+x}\,dx = \int (u^2 - 2) \times u \times 2u\,du$

$= \int (2u^4 - 4u^2)\,du$

$= \frac{2u^5}{5} - \frac{4u^3}{3} + C$

$= \frac{2}{5}(2+x)^{\frac{5}{2}} - \frac{4}{3}(2+x)^{\frac{3}{2}} + C$

(b) $u = x^{\frac{1}{2}} \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{dx}{\sqrt{x}} = 2\,du$

and $x - 4 = u^2 - 4$

$\therefore I = \int \frac{2}{\sqrt{x}(x-4)}dx = \int \frac{2}{u^2-4} \times 2\,du = \int \frac{4}{u^2-4}du$

$\frac{4}{u^2-4} = \frac{A}{u-2} + \frac{B}{u+2}$

$\Rightarrow \quad 4 = A(u+2) + B(u-2)$

$u = 2 \quad \Rightarrow \quad 4 = 4A \quad \Rightarrow \quad A = 1$

$u = -2 \Rightarrow 4 = -4B \Rightarrow B = -1$

$\therefore I = \int \left(\frac{1}{u-2} - \frac{1}{u+2} \right) du$

$= \ln | u-2 | - \ln | u+2 | + C$

$= \ln \left| \frac{\sqrt{x}-2}{\sqrt{x}+2} \right| + C$

(c) $u^2 = 1 + \tan x \Rightarrow 2u\,du = \sec^2 x\,dx$

$\therefore \int \sec^2 x \tan x \sqrt{1+\tan x}\,dx$

$= \int (u^2 - 1) \times u \times 2u\,du$

$= \int (2u^4 - 2u^2)\,du$

$= \frac{2u^5}{5} - \frac{2u^3}{3} + C$

$= \frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$

(d) $u^2 = x^2 + 4 \Rightarrow 2u\,du = 2x\,dx \Rightarrow \frac{u du}{x} = dx$

$\therefore \int \frac{\sqrt{x^2+4}}{x} dx = \int \frac{u}{x} \times \frac{u du}{x}$

$= \int \frac{u^2}{x^2} du$

$= \int \frac{u^2}{u^2-4} du$

$= \int \left(1 + \frac{4}{u^2-4}\right) du$ but $\frac{4}{u^2-4} \equiv \frac{A}{u+2} + \frac{B}{u+2}$

$4 \equiv A(u+2) + B(u-2)$

$u = 2 : 4 = 4A,\ A = 1$

$u = -2 : 4 = -4B,\ B = -1$

$= \int \left(1 + \frac{1}{u-2} - \frac{1}{u+2} \right) du$

$= u + \ln | u-2 | - \ln | u+2 | + C$

$= \sqrt{x^2+4} + \ln \left| \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2} \right| + C$

(e) $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$\therefore \int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$

$= \int (1 + u^2) \, du$

$= u + \frac{u^3}{3} + C$

$= \tan x + \frac{\tan^3 x}{3} + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise F, Question 3

Question:

Evaluate the following:

(a) $\int_0^5 x\sqrt{x+4}\,dx$

(b) $\int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2}\,dx$

(c) $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,dx$; use $u^2 = x - 1$

(d) $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,d\theta$; let $u = 1 + \cos\theta$

(e) $\int_0^1 x(2+x)^3\,dx$

(f) $\int_1^4 \frac{1}{\sqrt{x}(4x-1)}\,dx$; let $u = \sqrt{x}$

Solution:

(a) $u^2 = x + 4 \Rightarrow 2u\,du = dx$ and $x = u^2 - 4$

Also $u = 3$ when $x = 5$

and $u = 2$ when $x = 0$.

$\therefore \int_0^5 x\sqrt{x+4}\,dx = \int_2^3 (u^2 - 4) \times u \times 2u\,du$

$= \int_2^3 (2u^4 - 8u^2)\,du$

$= \left[\frac{2}{5}u^5 - \frac{8}{3}u^3 \right]_2^3$

$= \left(\frac{2}{5} \times 243 - \frac{8}{3} \times 27 \right) - \left(\frac{64}{5} - \frac{64}{3} \right)$

$= 25.2 - -8.53$

$= 33.73$

$= 33.7$ (3 s.f.)

(b) $u^2 = \sec x + 2 \Rightarrow 2u\,du = \sec x \tan x\,dx$

Also $u = 2$ when $x = \frac{\pi}{3}$

and $u = \sqrt{3}$ when $x = 0$.

$\therefore \int_0^{\frac{\pi}{3}} \sec x \tan x \sqrt{\sec x + 2}\, dx = \int_{\sqrt{3}}^{2} u \times 2u\, du$

$= \int_{\sqrt{3}}^{2} 2u^2\, du$

$= \left[\frac{2}{3} u^3 \right]_{\sqrt{3}}^{2}$

$= \left(\frac{16}{3} \right) - \left(\frac{2}{3} \times 3\sqrt{3} \right)$

$= \frac{16}{3} - 2\sqrt{3}$

(c) $u^2 = x - 1 \quad \Rightarrow \quad 2u\, du = dx$

Also $u = 2$ when $x = 5$

and $u = 1$ when $x = 2$.

$\therefore \int_2^5 \frac{1}{1 + \sqrt{x - 1}} dx = \int_1^2 \frac{1}{1 + u} \times 2u\, du$

$= \int_1^2 \frac{2u}{u + 1} du$

$= \int_1^2 \left(2 - \frac{2}{u + 1} \right) du$

$= \left[2u - 2\ln | u + 1 | \right]_1^2$

$= (4 - 2\ln 3) - (2 - 2\ln 2)$

$= 2 + 2\ln \frac{2}{3}$

(d) $u = 1 + \cos\theta \quad \Rightarrow \quad du = -\sin\theta\, d\theta$ or $-du = \sin\theta\, d\theta$

Also $u = 1$ when $\theta = \frac{\pi}{2}$

and $u = 2$ when $\theta = 0$.

$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1 + \cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{2\sin\theta\cos\theta}{1 + \cos\theta} d\theta = \int_2^1 - \frac{2(u - 1)}{u} du$

Use ' − ' to reverse limits:

$I = \int_1^2 \frac{2u - 2}{u} du$

$= \int_1^2 \left(2 - \frac{2}{u} \right) du$

$= \left[2u - 2\ln | u | \right]_1^2$

$= (4 - 2\ln 2) - (2 - 2\ln 1)$
$= 2 - 2\ln 2$

(e) $u = 2 + x \Rightarrow \mathrm{d}u = \mathrm{d}x$ and $x = u - 2$

Also $u = 3$ when $x = 1$
and $u = 2$ when $x = 0$.

$\therefore \int_0^1 x(2 + x)^3 \mathrm{d}x = \int_2^3 (u - 2)u^3 \mathrm{d}u$

$= \int_2^3 (u^4 - 2u^3)\,\mathrm{d}u$

$= \left[\frac{u^5}{5} - \frac{2}{4}u^4\right]_2^3$

$= \left(\frac{243}{5} - \frac{81}{2}\right) - \left(\frac{32}{5} - \frac{16}{2}\right)$

$= \frac{211}{5} - 32.5$

$= 42.2 - 32.5$
$= 9.7$

(f) $u = x^{\frac{1}{2}} \Rightarrow \mathrm{d}u = \frac{1}{2}x^{-\frac{1}{2}}\mathrm{d}x \Rightarrow \frac{\mathrm{d}x}{\sqrt{x}} = 2\,\mathrm{d}u$

and $4x - 1 = 4u^2 - 1$
Also $u = 2$ when $x = 4$
and $u = 1$ when $x = 1$.

$\therefore I = \int_1^4 \frac{1}{\sqrt{x}(4x - 1)}\mathrm{d}x = \int_1^2 \frac{1}{4u^2 - 1} \times 2\,\mathrm{d}u$

$\frac{2}{4u^2 - 1} = \frac{A}{2u - 1} + \frac{B}{2u + 1}$

$\Rightarrow 2 = A(2u + 1) + B(2u - 1)$

$u = \frac{1}{2} \Rightarrow 2 = 2A \Rightarrow A = 1$

$u = -\frac{1}{2} \Rightarrow 2 = -2B \Rightarrow B = -1$

$\therefore I = \int_1^2 \left(\frac{1}{2u - 1} - \frac{1}{2u + 1}\right)\mathrm{d}u$

$= \left[\frac{1}{2}\ln|2u - 1| - \frac{1}{2}\ln|2u + 1|\right]_1^2$

$= \left[\frac{1}{2}\ln\left|\frac{2u - 1}{2u + 1}\right|\right]_1^2$

$$= \left(\frac{1}{2}\ln\left|\frac{3}{5}\right|\right) - \left(\frac{1}{2}\ln\left|\frac{1}{3}\right|\right)$$

$$= \frac{1}{2}\ln\frac{9}{5}$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 1

Question:

Find the following integrals:

(a) $\int x \sin x \, dx$

(b) $\int x e^{x} \, dx$

(c) $\int x \sec^{2} x \, dx$

(d) $\int x \sec x \tan x \, dx$

(e) $\int \frac{x}{\sin^{2} x} dx$

Solution:

(a) $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore \int x \sin x \, dx = -x \cos x - \int -\cos x \times 1 \, dx$
$= -x \cos x + \int \cos x \, dx$
$= -x \cos x + \sin x + C$

(b) $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{x} \Rightarrow v = e^{x}$

$\therefore \int x e^{x} dx = x e^{x} - \int e^{x} \times 1 \, dx$

$= x e^{x} - e^{x} + C$

(c) $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^{2} x \Rightarrow v = \tan x$

$\therefore \int x \sec^{2} x \, dx = x \tan x - \int \tan x \times 1 \, dx$

$= x\tan x - \ln|\sec x| + C$

(d) $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec x \tan x \Rightarrow v = \sec x$

$\therefore \int x \sec x \tan x \, dx = x \sec x - \int \sec x \times 1 \, dx$

$= x \sec x - \ln|\sec x + \tan x| + C$

(e) $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \text{cosec}^2 x \Rightarrow v = -\cot x$

$\therefore \int \frac{x}{\sin^2 x} dx = \int x \, \text{cosec}^2 x \, dx$

$= -x \cot x - \int -\cot x \times 1 \, dx$

$= -x \cot x + \int \cot x \, dx$

$= -x \cot x + \ln|\sin x| + C$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 2

Question:

Find the following integrals:

(a) $\int x^2 \ln x \, dx$

(b) $\int 3 \ln x \, dx$

(c) $\int \frac{\ln x}{x^3} dx$

(d) $\int (\ln x)^2 dx$

(e) $\int (x^2 + 1) \ln x \, dx$

Solution:

(a) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$\therefore \int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \times \frac{1}{x} dx$

$= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$

(b) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 3 \Rightarrow v = 3x$

$\therefore \int 3 \ln x \, dx = 3x \ln x - \int 3x \times \frac{1}{x} dx$

$= 3x \ln x - \int 3 \, dx$

$= 3x \ln x - 3x + C$

(c) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-3} \Rightarrow v = \frac{x^{-2}}{-2}$

$\therefore \int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2} \ln x - \int -\frac{1}{2x^2} \times \frac{1}{x} dx$

$= -\frac{\ln x}{2x^2} + \int \frac{1}{2} x^{-3} dx$

$= -\frac{\ln x}{2x^2} + \frac{x^{-2}}{2 \times (-2)} + C$

$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$

(d) $u = (\ln x)^2 \Rightarrow \frac{du}{dx} = 2 \ln x \times \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore I = \int (\ln x)^2 dx = x(\ln x)^2 - \int x \times 2 \ln x \times \frac{1}{x} dx$

$= x(\ln x)^2 - \int 2 \ln x dx$

Let $J = \int 2 \ln x dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 2 \Rightarrow v = 2x$

$\therefore J = 2x \ln x - \int 2x \times \frac{1}{x} dx = 2x \ln x - 2x + C$

$\therefore I = x(\ln x)^2 - 2x \ln x + 2x + C$

(e) $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 + 1 \Rightarrow v = \frac{x^3}{3} + x$

$\therefore \int \left(x^2 + 1 \right) \ln x dx = \ln x \left(\frac{x^3}{3} + x \right) - \int \left(\frac{x^3}{3} + x \right) \times \frac{1}{x} dx$

$= \left(\frac{x^3}{3} + x \right) \ln x - \int \left(\frac{x^2}{3} + 1 \right) dx$

$$= \left(\frac{x^3}{3} + x \right) \ln x - \frac{x^3}{9} - x + C$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 3

Question:

Find the following integrals:

(a) $\int x^2 e^{-x} dx$

(b) $\int x^2 \cos x \, dx$

(c) $\int 12x^2 (3 + 2x)^5 dx$

(d) $\int 2x^2 \sin 2x \, dx$

(e) $\int x^2 2 \sec^2 x \tan x \, dx$

Solution:

(a) $u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore I = \int x^2 e^{-x} dx = -x^2 e^{-x} - \int -e^{-x} \times 2x \, dx$

$= -x^2 e^{-x} + \int 2x e^{-x} dx$

Let $J = \int 2x e^{-x} dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore J = -e^{-x} \ 2x - \int (-e^{-x}) \times 2 \, dx$

$= -2x e^{-x} + \int 2 e^{-x} dx$

$= -2x e^{-x} - 2 e^{-x} + C$

$\therefore I = -x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} + C'$

(b) $u = x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore I = \int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$

Let $J = \int 2x \sin x \, dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore J = -2x \cos x - \int (-\cos x) \times 2 \, dx$

$= -2x \cos x + \int 2 \cos x \, dx$

$= -2x \cos x + 2 \sin x + C$

$\therefore I = x^2 \sin x + 2x \cos x - 2 \sin x + C'$

(c) $u = 12x^2 \Rightarrow \frac{du}{dx} = 24x$

$\frac{dv}{dx} = (3 + 2x)^5 \Rightarrow v = \frac{(3 + 2x)^6}{12}$

$\therefore I = \int 12x^2 (3 + 2x)^5 dx = 12x^2 \frac{(3 + 2x)^6}{12} - \int 24x \frac{(3 + 2x)^6}{12} dx$

$= x^2 (3 + 2x)^6 - \int 2x (3 + 2x)^6 dx$

Let $J = \int 2x (3 + 2x)^6 dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$v = \frac{(3 + 2x)^7}{14} \Rightarrow \frac{dv}{dx} = (3 + 2x)^6$

$\therefore J = 2x \frac{(3 + 2x)^7}{14} - \int \frac{(3 + 2x)^7}{14} \times 2 \, dx$

$= x \frac{(3 + 2x)^7}{7} - \int \frac{(3 + 2x)^7}{7} dx$

$= x \frac{(3 + 2x)^7}{7} - \frac{(3 + 2x)^8}{7 \times 16} + C$

$\therefore I = x^2 (3 + 2x)^6 - x \frac{(3 + 2x)^7}{7} + \frac{(3 + 2x)^8}{112} + C'$

(d) $u = 2x^2 \Rightarrow \frac{du}{dx} = 4x$

$\frac{dv}{dx} = \sin 2x \Rightarrow v = -\frac{1}{2} \cos 2x$

$\therefore I = \int 2x^2 \sin 2x \, dx = -\frac{2x^2}{2}\cos 2x - \int \left(-\frac{1}{2}\cos 2x\right) \times 4x \, dx$

$= -x^2 \cos 2x + \int 2x \cos 2x \, dx$

Let $J = \int 2x \cos 2x \, dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = 2\cos 2x \Rightarrow v = \sin 2x$

$\therefore J = x \sin 2x - \int \sin 2x \, dx$

$= x \sin 2x + \frac{1}{2}\cos 2x + C$

$\therefore I = -x^2 \cos 2x + x \sin 2x + \frac{1}{2}\cos 2x + C'$

(e) $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = 2\sec x \sec x \tan x \Rightarrow v = \sec^2 x$

$\therefore I = \int x^2 \times 2\sec^2 x \tan x \, dx = x^2 \sec^2 x - \int 2x \sec^2 x \, dx$

Let $J = \int 2x \sec^2 x \, dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$\therefore J = 2x \tan x - \int 2 \tan x \, dx$

$= 2x \tan x - 2 \ln | \sec x | + C$

$\therefore I = x^2 \sec^2 x - 2x \tan x + 2 \ln | \sec x | + C'$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise G, Question 4

Question:

Evaluate the following:

(a) $\int_0^{\ln 2} x e^{2x} \, dx$

(b) $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

(c) $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

(d) $\int_1^2 \frac{\ln x}{x^2} dx$

(e) $\int_0^1 4x(1+x)^3 dx$

(f) $\int_0^{\pi} x \cos\left(\frac{1}{4}x\right) dx$

(g) $\int_0^{\frac{\pi}{3}} \sin x \ln|\sec x| \, dx$

Solution:

(a) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x} \quad \Rightarrow \quad v = \frac{1}{2}e^{2x}$

$\therefore \int_0^{\ln 2} x e^{2x} dx = \left[\frac{1}{2}e^{2x} \times x\right]_0^{\ln 2} - \int_0^{\ln 2} \frac{1}{2}e^{2x} dx$

$= \left(\frac{1}{2}e^{2\ln 2}\ln 2\right) - \left(0\right) - \left[\frac{1}{4}e^{2x}\right]_0^{\ln 2}$

$= \frac{4}{2}\ln 2 - \left[\left(\frac{1}{4}e^{2\ln 2}\right) - \left(\frac{1}{4}e^0\right)\right]$

$= 2\ln 2 - \frac{4}{4} + \frac{1}{4}$

$= 2\ln 2 - \frac{3}{4}$

(b) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$

$\therefore \int_0^{\frac{\pi}{2}} x \sin x \, dx = [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x) \, dx$

$= \left(-\frac{\pi}{2}\cos\frac{\pi}{2}\right) - (0) + \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= 0 + [\sin x]_0^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2}\right) - (\sin 0)$

$= 1$

(c) $u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore \int_0^{\frac{\pi}{2}} x \cos x \, dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= \left(\frac{\pi}{2}\sin\frac{\pi}{2}\right) - (0) - [-\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2}\right) - (\cos 0)$

$= \frac{\pi}{2} - 1$

(d) $u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-2} \quad \Rightarrow \quad v = -x^{-1}$

$\therefore \int_1^2 \frac{\ln x}{x^2} \, dx = \left[-\frac{\ln x}{x}\right]_1^2 - \int_1^2 \frac{1}{x} \times (-x^{-1}) \, dx$

$= \left(-\frac{\ln 2}{2}\right) - \left(-\frac{\ln 1}{1}\right) + \int_1^2 \frac{1}{x^2} \, dx$

$= -\frac{1}{2}\ln 2 + [-x^{-1}]_1^2$

$$= -\frac{1}{2}\ln 2 + \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$$

$$= \frac{1}{2}\left(1 - \ln 2\right)$$

(e) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = 4(1+x)^3 \Rightarrow v = (1+x)^4$$

$$\therefore \int_0^1 4x(1+x)^3 dx = \left[x(1+x)^4\right]_0^1 - \int_0^1 (1+x)^4 dx$$

$$= \left(1 \times 2^4\right) - \left(0\right) - \left[\frac{(1+x)^5}{5}\right]_0^1$$

$$= 16 - \left[\left(\frac{2^5}{5}\right) - \left(\frac{1}{5}\right)\right]$$

$$= 16 - \frac{31}{5}$$

$$= 16 - 6.2$$
$$= 9.8$$

(f) $u = x \Rightarrow \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \cos\left(\frac{1}{4}x\right) \Rightarrow v = 4\sin\left(\frac{1}{4}x\right)$$

$$\therefore \int_0^\pi x \cos\left(\frac{1}{4}x\right) dx = \left[4x\sin\frac{x}{4}\right]_0^\pi - \int_0^\pi 4\sin\left(\frac{1}{4}x\right) dx$$

$$= \left(4\pi\sin\frac{\pi}{4}\right) - \left(0\right) + \left[16\cos\frac{1}{4}x\right]_0^\pi$$

$$= \frac{4\pi}{\sqrt{2}} + \left(16\cos\frac{\pi}{4}\right) - \left(16\cos 0\right)$$

$$= \frac{4\pi}{\sqrt{2}} + \frac{16}{\sqrt{2}} - 16$$

OR $= \frac{4\pi\sqrt{2}}{2} + \frac{16\sqrt{2}}{2} - 16$

$$= 2\pi\sqrt{2} + 8\sqrt{2} - 16$$

(g) $u = \ln|\sec x| \Rightarrow \frac{du}{dx} = \tan x$

$$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$$

$$\therefore \int_0^{\frac{\pi}{3}} \sin x \ln|\sec x| \, dx = \left[-\cos x \ln|\sec x| \right]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \cos x \tan x \, dx$$

$$= \left(-\cos \frac{\pi}{3} \ln \left| \sec \frac{\pi}{3} \right| \right) - \left(-\cos 0 \ln \left| \sec 0 \right| \right) + \int_0^{\frac{\pi}{3}} \sin x \, dx$$

$$= -\frac{1}{2} \ln 2 + 0 + \left[-\cos x \right]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2} \ln 2 + \left(-\frac{1}{2} \right) - \left(-1 \right)$$

$$= \frac{1}{2} \left(1 - \ln 2 \right)$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise H, Question 1

Question:

Use the trapezium rule with n strips to estimate the following:

(a) $\int_0^3 \ln(1+x^2)\,dx$; $n=6$

(b) $\int_0^{\frac{\pi}{3}} \sqrt{(1+\tan x)}\,dx$; $n=4$

(c) $\int_0^2 \frac{1}{\sqrt{(e^x+1)}}\,dx$; $n=4$

(d) $\int_{-1}^1 \mathrm{cosec}^2(x^2+1)\,dx$; $n=4$

(e) $\int_{0.1}^{1.1} \sqrt{\cot x}\,dx$; $n=5$

Solution:

(a)

x	0	0.5	1	1.5	2	2.5	3
$\ln(1+x^2)$	0	0.223	0.693	1.179	1.609	1.981	2.303

$I=\int_0^3 \ln(1+x^2)\,dx$

$\therefore I \approx \frac{1}{2}\times 0.5\left[0+2.303+2\left(0.223+0.693+1.179+1.609+1.981\right)\right]$

$=\frac{1}{4}\left(13.673\right)$

$=3.41825$

$=3.42$ (3 s.f.)

(b)

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{\pi}{3}$
$\sqrt{1+\tan x}$	1	1.126	1.256	1.414	1.653

$I=\int_0^{\frac{\pi}{3}} \sqrt{1+\tan x}\,dx$

$\therefore I - \frac{1}{2} \times \frac{\pi}{12} \left[1 + 1.653 + 2 \left(1.126 + 1.256 + 1.414 \right) \right]$

$= \frac{\pi}{24} \left(10.245 \right)$

$= 1.3410...$

$= 1.34$ (3 s.f.)

(c)

x	0	0.5	1	1.5	2
$\frac{1}{\sqrt{e^x + 1}}$	0.707	0.614	0.519	0.427	0.345

$I = \int_0^2 \frac{1}{\sqrt{e^x + 1}} dx$

$\therefore I \simeq \frac{1}{2} \times 0.5 \left[0.707 + 0.345 + 2 \left(0.614 + 0.519 + 0.427 \right) \right]$

$= \frac{1}{4} \left(4.172 \right)$

$= 1.043$

$= 1.04$ (3 s.f.)

(d)

x	-1	-0.5	0	0.5	1
$\text{cosec}^2 (x^2 + 1)$	1.209	1.110	1.412	1.110	1.209

$I = \int_{-1}^{1} \text{cosec}^2 (x^2 + 1) dx$

$\therefore I \simeq \frac{1}{2} \times 0.5 \left[1.209 \times 2 + 2 \left(1.110 + 1.412 + 1.110 \right) \right]$

$= \frac{1}{4} \left(9.682 \right)$

$= 2.42$ (3 s.f.)

(e)

x	0.1	0.3	0.5	0.7	0.9	1.1
$\sqrt{\cot x}$	3.157	1.798	1.353	1.090	0.891	0.713

$I = \int_{0.1}^{1.1} \sqrt{\cot x} dx$

$\therefore I \simeq \frac{1}{2} \times 0.2 \left[3.157 + 0.713 + 2 \left(1.798 + 1.353 + 1.090 + 0.891 \right) \right]$

$= \frac{1}{10} \left(14.134 \right)$

$= 1.41$ (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise H, Question 2

Question:

(a) Find the exact value of $I = \int_1^4 x \ln x \, dx$.

(b) Find approximate values for I using the trapezium rule with
(i) 3 strips
(ii) 6 strips

(c) Compare the percentage error for these two approximations.

Solution:

(a) $I = \int_1^4 x \ln x \, dx$

$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \quad \Rightarrow \quad v = \frac{1}{2}x^2$

$\therefore I = \left[\frac{1}{2}x^2 \ln x \right]_1^4 - \int_1^4 \frac{1}{2}x^2 \times \frac{1}{x} dx$

$= 8 \ln 4 - \left[\frac{x^2}{4} \right]_1^4$

$= 8 \ln 4 - \left(4 - \frac{1}{4} \right)$

$= 8 \ln 4 - \frac{15}{4}$

(b) (i)

x	1	2	3	4
$x \ln x$	0	1.386	3.296	5.545

$I \approx \frac{1}{2} \times 1 \left[5.545 + 2 \left(1.386 + 3.296 \right) \right]$

$= \frac{1}{2} \left(14.909 \right) = 7.4545 = 7.45$ (3 s.f.)

(ii)

x	1	1.5	2	2.5	3	3.5	4
$x\ln x$	0	0.608	1.386	2.291	3.296	4.385	5.545

$$I \approx \frac{1}{2} \times 0.5 \left[5.545 + 2 \left(0.608 + 1.386 + 2.291 + 3.296 + 4.385 \right) \right]$$

$$= \frac{1}{4} \left[29.477 \right] = 7.36925 = 7.37 \text{ (3 s.f.)}$$

(c) % error using 3 strips: $\frac{[7.4545 - (8\ln 4 - 3.75)] \times 100}{8\ln 4 - 3.75} = 1.6\ \%\ \ 1\text{ d.p.}$

% error using 6 strips: $\frac{[7.376925 - (8\ln 4 - 3.75)] \times 100}{8\ln 4 - 3.75} = 0.4\ \%\ \ 1\text{ d.p.}$

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Integration
Exercise H, Question 3

Question:

(a) Find an approximate value for $I = \int_0^1 e^x \tan x \, dx$ using

(i) 2 strips
(ii) 4 strips
(ii) 8 strips.

(b) Suggest a possible value for I.

Solution:

(a) (i)

x	0	0.5	1
$e^x \tan x$	0	0.901	4.233

$$I \simeq \frac{1}{2} \times 0.5 \left(0 + 4.233 + 2 \times 0.901 \right) = \frac{1}{4} \left(6.035 \right) = 1.509$$

(ii)

x	0	0.25	0.5	0.75	1
$e^x \tan x$	0	0.328	0.901	1.972	4.233

$$I \simeq \frac{1}{2} \times 0.25 \left[4.233 + 2 \left(0.328 + 0.901 + 1.972 \right) \right]$$

$$= \frac{1}{8} \left(10.635 \right) = 1.329$$

(iii)

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$e^x \tan x$	0	0.142	0.328	0.573	0.901	1.348	1.972	2.872	4.233

$$I \simeq \frac{1}{2} \times \frac{1}{8} \left[4.233 + 2 \left(0.142 + 0.328 + 0.573 + 0.901 + 1.348 + 1.972 + 2.872 \right) \right]$$

$$= \frac{1}{16}\left(20.505\right) = 1.282$$

(b) Halving *h* reduces differences by about $\frac{1}{3}$:

	1.5098	→	1.329	→	1.282	→ ?
Differences:		0.18		0.05		0.01/2

So an answer in the range 1.25 – 1.27 seems likely.
(**Note:** Calculator gives 1.265)

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Integration
Exercise H, Question 4

Question:

(a) Find the exact value of $I = \int_0^2 x\sqrt{(2-x)}\,dx$.

(b) Find an approximate value for I using the trapezium rule with
(i) 4 and
(ii) 6 strips.

(c) Compare the percentage error for these two approximations.

Solution:

(a) $u^2 = 2 - x \quad \Rightarrow \quad 2u\,du = -dx$ and $x = 2 - u^2$
Also $u = 0$ when $x = 2$
and $u = \sqrt{2}$ when $x = 0$.

$\therefore I = \int_{\sqrt{2}}^0 (2 - u^2)\, u \times (-2u)\, du$

$= \int_0^{\sqrt{2}} (2 - u^2)\, 2u^2\, du$

$= \int_0^{\sqrt{2}} (4u^2 - 2u^4)\, du$

$= \left[\frac{4u^3}{3} - \frac{2u^5}{5} \right]_0^{\sqrt{2}}$

$= \left(\frac{4 \times 2\sqrt{2}}{3} - \frac{2 \times 4\sqrt{2}}{5} \right) - \left(0 \right)$

$= \frac{16\sqrt{2}}{15}$

(b) (i)

x	0	0.5	1	1.5	2
$x\sqrt{2-x}$	0	0.612	1	1.061	0

$I \simeq \frac{1}{2} \times 0.5 \left[0 + 2\left(0.612 + 1 + 1.061 \right) \right]$

$= \frac{1}{4}\left(5.346 \right) = 1.3365 = 1.34$ (2 d.p.)

(ii)

x	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
$x\sqrt{2-x}$	0	0.430	0.770	1	1.089	0.962	0

$$I \simeq \frac{1}{2} \times \frac{1}{3} \left[0 + 2 \left(0.430 + 0.770 + 1 + 1.089 + 0.962 \right) \right]$$

$$= \frac{1}{6} \left(8.502 \right) = 1.417 = 1.42 \text{ (2 d.p.)}$$

(c) (i) % error with 4 strips $= \dfrac{\frac{16}{15}\sqrt{2} - 1.3365}{\frac{16}{15}\sqrt{2}} \times 100 = 11.4\ \%$

(ii) % error with 6 strips $= \dfrac{\frac{16\sqrt{2}}{15} - 1.417}{\frac{16}{15}\sqrt{2}} \times 100 = 6.1\ \%$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 1

Question:

The region R is bounded by the curve with equation $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. In each of the following cases find the exact value of:
(i) the area of R,
(ii) the volume of the solid of revolution formed by rotating R through 2π radians about the x-axis.

(a) $f(x) = \frac{2}{1+x}$; $a = 0$, $b = 1$

(b) $f(x) = \sec x$; $a = 0$, $b = \frac{\pi}{3}$

(c) $f(x) = \ln x$; $a = 1$, $b = 2$

(d) $f(x) = \sec x \tan x$; $a = 0$, $b = \frac{\pi}{4}$

(e) $f(x) = x\sqrt{4 - x^2}$; $a = 0$, $b = 2$

Solution:

(a) (i) Area $= \int_0^1 \frac{2}{1+x} dx = \left[2\ln|1+x| \right]_0^1 = \left(2\ln 2 \right) - \left(2\ln 1 \right)$

$\therefore$ Area $= 2\ln 2$

(ii) Volume $= \pi \int_0^1 \left(\frac{2}{1+x} \right)^2 dx$

$= \pi \int_0^1 \frac{4}{(1+x)^2} dx$

$= \pi \left[4 \frac{(1+x)^{-1}}{-1} \right]_0^1$

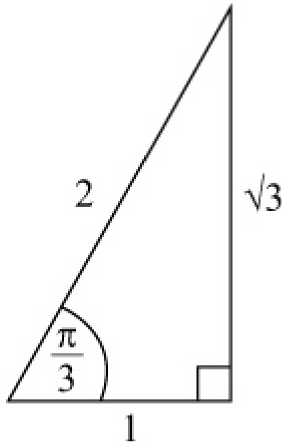
$= \pi \left[-\frac{4}{1+x} \right]_0^1$

$= \pi \left[\left(- \frac{4}{2} \right) - \left(- \frac{4}{1} \right) \right]$

$= 2\pi$

(b) (i) Area $= \int_0^{\frac{\pi}{3}} \sec x \, dx$

$= \left[\ln | \sec x + \tan x | \right]_0^{\frac{\pi}{3}}$



$= \left[\ln \left(2 + \sqrt{3} \right) \right] - \left[\ln \left(1 \right) \right]$

$\therefore$ Area $= \ln \left(2 + \sqrt{3} \right)$

(ii) Volume $= \pi \int_0^{\frac{\pi}{3}} \sec^2 x \, dx$

$= \pi \left[\tan x \right]_0^{\frac{\pi}{3}}$

$= \pi \left[\left(\sqrt{3} \right) - \left(0 \right) \right]$

$= \sqrt{3}\pi$

(c) (i) Area $= \int_1^2 \ln x \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1 \Rightarrow v = x$

$\therefore$ Area $= \left[x \ln x \right]_1^2 - \int_1^2 x \times \frac{1}{x} dx$

$= \left(2 \ln 2 \right) - \left(0 \right) - \left[x \right]_1^2$

$= 2 \ln 2 - 1$

(ii) Volume $= \pi \int_1^2 \left(\ln x \right)^2 dx$

$u = (\ln x)^2 \quad \Rightarrow \quad \frac{du}{dx} = 2\ln x \times \frac{1}{x}$

$\frac{dv}{dx} = 1 \quad \Rightarrow \quad v = x$

$\therefore V = \pi \left\{ \left[x(\ln x)^2 \right]_1^2 - 2\int_1^2 x \times \ln x \times \frac{1}{x} dx \right\}$

$= \pi \{ [2(\ln 2)^2] - (0) \{ -2\pi \int_1^2 \ln x \, dx$

But $\int_1^2 \ln x \, dx = 2\ln 2 - 1$ from (i)

$\therefore V = 2\pi(\ln 2)^2 - 2\pi(2\ln 2 - 1)$

(d) (i) Area $= \int_0^{\frac{\pi}{4}} \sec x \tan x \, dx$

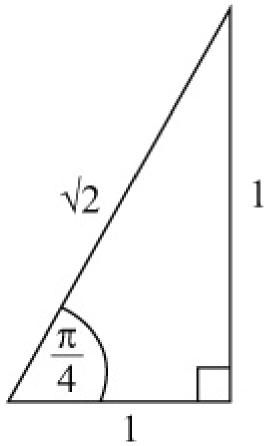
$= \left[\sec x \right]_0^{\frac{\pi}{4}}$

$= (\sqrt{2}) - (1)$

$\therefore$ Area $= \sqrt{2} - 1$

(ii) Volume $= \pi \int_0^{\frac{\pi}{4}} \sec^2 x \tan^2 x \, dx$

$= \pi \left[\frac{\tan^3 x}{3} \right]_0^{\frac{\pi}{4}}$



$= \pi \left[\left(\frac{1^3}{3} \right) - \left(0 \right) \right]$

$= \frac{\pi}{3}$

(e) (i) Area $= \int_0^2 x\sqrt{4 - x^2} \, dx$

Let $y = \left(4 - x^2\right)^{\frac{3}{2}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}\left(4 - x^2\right)^{\frac{1}{2}} \times \left(-2x\right) = -3x\left(4 - x^2\right)^{\frac{1}{2}}$

$\therefore$ Area $= \left[-\frac{1}{3}\left(4 - x^2\right)^{\frac{3}{2}}\right]_0^2 = \left(0\right) - \left(-\frac{1}{3} \times 2^3\right) = \frac{8}{3}$

(ii) Volume $= \pi \int_0^2 x^2\left(4 - x^2\right) dx$

$= \pi \int_0^2 \left(4x^2 - x^4\right) dx$

$= \pi \left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2$

$= \pi \left[\left(\frac{32}{3} - \frac{32}{5}\right) - \left(0\right)\right]$

$= \frac{64\pi}{15}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 2

Question:

Find the exact area between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ where:

(a) $f(x) = \frac{4x+3}{(x+2)(2x-1)}$; $a = 1$, $b = 2$

(b) $f(x) = \frac{x}{(x+1)^2}$; $a = 0$, $b = 2$

(c) $f(x) = x \sin x$; $a = 0$, $b = \frac{\pi}{2}$

(d) $f(x) = \cos x \sqrt{2 \sin x + 1}$; $a = 0$, $b = \frac{\pi}{6}$

(e) $f(x) = xe^{-x}$; $a = 0$, $b = \ln 2$

Solution:

(a) $\frac{4x+3}{(x+2)(2x-1)} \equiv \frac{A}{x+2} + \frac{B}{2x-1}$

$\Rightarrow \quad 4x + 3 \equiv A(2x-1) + B(x+2)$

$x = \frac{1}{2} \quad \Rightarrow \quad 5 = \frac{5}{2}B \quad \Rightarrow \quad B = 2$

$x = -2 \quad \Rightarrow \quad -5 = -5A \quad \Rightarrow \quad A = 1$

$\therefore$ area $= \int_1^2 \frac{4x+3}{(x+2)(2x-1)} dx$

$= \int_1^2 \left(\frac{1}{x+2} + \frac{2}{2x-1} \right) dx$

$= [\ln|x+2| + \ln|2x-1|]_1^2$

$= (\ln 4 + \ln 3) - (\ln 3 + \ln 1)$

$= \ln 4 \quad$ or $\quad 2 \ln 2$

(b) $\frac{x}{(x+1)^2} \equiv \frac{A}{(x+1)^2} + \frac{B}{x+1}$

$\Rightarrow \quad x \equiv A + B(x+1)$

Compare coefficient of x: $1 = B \quad \Rightarrow \quad B = 1$

Compare constants: $0 = A + B \quad \Rightarrow \quad A = -1$

$\therefore$ area $= \int_0^2 \frac{x}{(x+1)^2} dx$

$= \int_0^2 \left(\frac{1}{x+1} - \frac{1}{(x+1)^2} \right) dx$

$= \left[\ln|x+1| + \frac{1}{x+1} \right]_0^2$

$= \left(\ln 3 + \frac{1}{3} \right) - \left(\ln 1 + 1 \right)$

$= \ln 3 - \frac{2}{3}$

(c) Area $= \int_0^{\frac{\pi}{2}} x \sin x \, dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \quad \Rightarrow \quad v = -\cos x$

$\therefore$ area $= \left[-x\cos x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(-\cos x \right) dx$

$= \left(-\frac{\pi}{2} \cos \frac{\pi}{2} \right) - \left(0 \right) + \int_0^{\frac{\pi}{2}} \cos x \, dx$

$= 0 + \left[\sin x \right]_0^{\frac{\pi}{2}}$

$= \left(\sin \frac{\pi}{2} - 0 \right)$

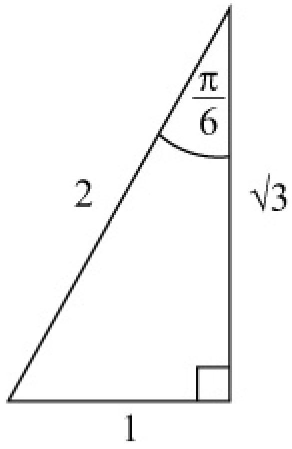
$= 1$

(d) Area $= \int_0^{\frac{\pi}{6}} \cos x \sqrt{2\sin x + 1} \, dx$

Let $y = (2\sin x + 1)^{\frac{3}{2}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{2}(2\sin x + 1)^{\frac{1}{2}} \times 2\cos x = 3\cos x (2\sin x + 1)^{\frac{1}{2}}$

$\therefore$ area $= \left[\frac{1}{3}(2\sin x + 1)^{\frac{3}{2}} \right]_0^{\frac{\pi}{6}}$



$= \left(\frac{1}{3}2^{\frac{3}{2}}\right) - \left(\frac{1}{3}1^{\frac{3}{2}}\right)$

$= \frac{2\sqrt{2}}{3} - \frac{1}{3}$

$= \frac{2\sqrt{2}-1}{3}$

(e) Area $= \int_0^{\ln 2} x e^{-x} dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$

$\therefore$ area $= [-xe^{-x}]_0^{\ln 2} - \int_0^{\ln 2}(-e^{-x})\ dx$

$= (-\ln 2 \times e^{-\ln 2}) - (0) + \int_0^{\ln 2} e^{-x} dx$

$= -\ln 2 \times \frac{1}{2} + \left[-e^{-x}\right]_0^{\ln 2}$

$= -\frac{1}{2}\ln 2 + \left(-e^{-\ln 2}\right) - \left(-e^{-0}\right)$

$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$= \frac{1}{2}\left(1 - \ln 2\right)$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 3

Question:

The region R is bounded by the curve C, the x-axis and the lines $x = -8$ and $x = +8$. The parametric equations for C are $x = t^3$ and $y = t^2$. Find:

(a) the area of R,

(b) the volume of the solid of revolution formed when R is rotated through 2π radians about the x-axis.

Solution:

(a) Area $= \int_{x=-8}^{x=8} y \, dx$

$x = t^3 \Rightarrow dx = 3t^2 \, dt$

Also $t = 2$ when $x = 8$

and $t = -2$ when $x = -8$.

$\therefore$ area $= \int_{-2}^{2} t^2 \times 3t^2 \, dt$

$= \int_{-2}^{2} 3t^4 \, dt$

$= \left[\frac{3t^5}{5} \right]_{-2}^{2}$

$= \left(\frac{96}{5} \right) - \left(-\frac{96}{5} \right)$

$= \frac{192}{5}$

(b) $V = \pi \int_{x=-8}^{x=8} y^2 \, dx$

$= \pi \int_{-2}^{2} t^4 \times 3t^2 \, dt$

$= \pi \int_{-2}^{2} 3t^6 \, dt$

$= \pi \left[\frac{3t^7}{7} \right]_{-2}^{2}$

$= \pi \left[\left(\frac{3 \times 128}{7} \right) - \left(\frac{-3 \times 128}{7} \right) \right]$

$= \frac{768}{7} \pi$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise I, Question 4

Question:

The curve C has parametric equations $x = \sin t$, $y = \sin 2t$, $0 \leq t \leq \frac{\pi}{2}$.

(a) Find the area of the region bounded by C and the x-axis.
If this region is revolved through 2π radians about the x-axis,

(b) find the volume of the solid formed.

Solution:

(a) Area $= \int_{t=0}^{t=\frac{\pi}{2}} y \, dx$

$x = \sin t \Rightarrow dx = \cos t \, dt$

$\therefore$ area $= \int_0^{\frac{\pi}{2}} \sin 2t \times \cos t \, dt$

$= \int_0^{\frac{\pi}{2}} 2 \cos^2 t \sin t \, dt$

$= \left[-\frac{2}{3} \cos^3 t \right]_0^{\frac{\pi}{2}}$

$= \left(0 \right) - \left(-\frac{2}{3} \right)$

$= \frac{2}{3}$

(b) $V = \pi \int_{t=0}^{t=\frac{\pi}{2}} y^2 \, dx$

$= \int_0^{\frac{\pi}{2}} \sin^2 2t \cos t \, dt$

$= \pi \int_0^{\frac{\pi}{2}} 4 \cos^3 t \sin t \times \sin t \, dt$

$u = \sin t \Rightarrow \frac{du}{dt} = \cos t$

$\frac{dv}{dt} = 4 \cos^3 t \sin t \Rightarrow v = -\cos^4 t$

$$\therefore V = \pi \left\{ \left[-\sin t \cos^4 t \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos^5 t \, dt \right\}$$

$$= \pi \int_0^{\frac{\pi}{2}} \cos^5 t \, dt$$

$$= \pi \int_0^{\frac{\pi}{2}} (\cos^2 t)^2 \times \cos t \, dt \qquad \text{Let } y = \sin t \quad \Rightarrow \quad dy = \cos t \, dt$$

$$= \pi \int_0^1 (1 - y^2)^2 dy$$

$$= \pi \int_0^1 (1 - 2y^2 + y^4) \, dy$$

$$= \pi \left[y - \frac{2}{3}y^3 + \frac{y^5}{5} \right]_0^1$$

$$= \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(0 \right)$$

$$= \frac{8\pi}{15}$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise J, Question 1

Question:

Find general solutions of the following differential equations. Leave your answer in the form $y = f(x)$.

(a) $\frac{dy}{dx} = \left(1 + y \right) \left(1 - 2x \right)$

(b) $\frac{dy}{dx} = y \tan x$

(c) $\cos^2 x \frac{dy}{dx} = y^2 \sin^2 x$

(d) $\frac{dy}{dx} = 2e^{x - y}$

(e) $x^2 \frac{dy}{dx} = y + xy$

Solution:

(a) $\frac{dy}{dx} = \left(1 + y \right) \left(1 - 2x \right)$

$\Rightarrow \int \frac{1}{1 + y} dy = \int \left(1 - 2x \right) dx$

$\Rightarrow \ln | 1 + y | = x - x^2 + C$

$\Rightarrow 1 + y = e^{(x - x^2 + C)}$

$\Rightarrow 1 + y = A e^{x - x^2}$, $(A = e^C)$

$\Rightarrow y = A e^{x - x^2} - 1$

(b) $\frac{dy}{dx} = y \tan x$

$\Rightarrow \int \frac{1}{y} dy = \int \tan x \, dx$

$\Rightarrow \ln | y | = \ln | \sec x | + C$

$\Rightarrow \quad \ln|y| = \ln|k\sec x|, \qquad (C = \ln k)$

$\Rightarrow \quad y = k\sec x$

(c) $\cos^2 x \frac{dy}{dx} = y^2 \sin^2 x$

$\Rightarrow \quad \int \frac{1}{y^2} dy = \int \frac{\sin^2 x}{\cos^2 x} dx$

$\Rightarrow \quad \int \frac{1}{y^2} dy = \int \tan^2 x \, dx = \int \left(\sec^2 x - 1 \right) dx$

$\Rightarrow \quad -\frac{1}{y} = \tan x - x + C$

$\Rightarrow \quad y = \frac{-1}{\tan x - x + C}$

(d) $\frac{dy}{dx} = 2e^{x-y} = 2e^x e^{-y}$

$\Rightarrow \quad \int \frac{1}{e^{-y}} dy = \int 2e^x dx$

i.e. $\Rightarrow \quad \int e^y dy = \int 2e^x dx$

$\Rightarrow \quad e^y = 2e^x + C$

$\Rightarrow \quad y = \ln\left(2e^x + C\right)$

(e) $x^2 \frac{dy}{dx} = y + xy = y\left(1 + x\right)$

$\Rightarrow \quad \int \frac{1}{y} dy = \int x^{-2} + \frac{1}{x} dx$

$\Rightarrow \quad \ln|y| = -x^{-1} + \ln|x| + C$

$\Rightarrow \quad \ln|y| - \ln|x| = C - \frac{1}{x}$

$\Rightarrow \quad \ln\left|\frac{y}{x}\right| = C - \frac{1}{x}$

$\Rightarrow \quad \frac{y}{x} = e^{C - \frac{1}{x}}$

$\Rightarrow \quad \frac{y}{x} = Ae^{-\frac{1}{x}}, \qquad \left(e^C = A\right)$

$\Rightarrow \quad y = Axe^{-\frac{1}{x}}$

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Integration
Exercise J, Question 2

Question:

Find a general solution of the following differential equations. (You do not need to write the answers in the form $y = \mathrm{f}(x)$.)

(a) $\frac{dy}{dx} = \tan y \tan x$

(b) $\sin y \cos x \frac{dy}{dx} = \frac{x \cos y}{\cos x}$

(c) $\left(1 + x^2\right) \frac{dy}{dx} = x \left(1 - y^2\right)$

(d) $\cos y \sin 2x \frac{dy}{dx} = \cot x \operatorname{cosec} y$

(e) $e^{x+y} \frac{dy}{dx} = x \left(2 + e^y\right)$

Solution:

(a) $\frac{dy}{dx} = \tan y \tan x$

$$\Rightarrow \int \frac{1}{\tan y} dy = \int \tan x \, dx$$

$$\Rightarrow \int \cot y \, dy = \int \tan x \, dx$$

$$\Rightarrow \ln|\sin y| = \ln|\sec x| + C = \ln|k \sec x| \qquad (\ln k = C)$$

$$\Rightarrow \sin y = k \sec x$$

(b) $\sin y \cos x \frac{dy}{dx} = \frac{x \cos y}{\cos x}$

$$\Rightarrow \int \frac{\sin y}{\cos y} dy = \int \frac{x}{\cos^2 x} dx$$

$$\Rightarrow \int \tan y \, dy = \int x \sec^2 x \, dx$$

$$\Rightarrow \ln|\sec y| = \int x \sec^2 x \, dx$$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \quad \Rightarrow \quad v = \tan x$

$\Rightarrow \quad \ln|\sec y| = x\tan x - \int \tan x \, dx$

$\Rightarrow \quad \ln|\sec y| = x\tan x - \ln|\sec x| + C$

(c) $\left(1 + x^2\right)\frac{dy}{dx} = x\left(1 - y^2\right)$

$\Rightarrow \quad \int \frac{1}{1 - y^2} dy = \int \frac{x}{1 + x^2} dx$

$\frac{1}{1 - y^2} \equiv \frac{A}{1 - y} + \frac{B}{1 + y}$

$\Rightarrow \quad 1 \equiv A(1 + y) + B(1 - y)$

$y = 1 \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

$y = -1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

$\therefore \int \left(\frac{\left(\frac{1}{2}\right)}{1 - y} + \frac{\left(\frac{1}{2}\right)}{1 + y} \right) dy = \int \frac{x}{1 + x^2} dx$

$\Rightarrow \quad \frac{1}{2}\ln|1 + y| - \frac{1}{2}\ln|1 - y| = \frac{1}{2}\ln|1 + x^2| + C$

(using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$)

$\Rightarrow \quad \ln\left|\frac{1 + y}{1 - y}\right| = \ln|1 + x^2| + 2C$

$\Rightarrow \quad \left|\frac{1 + y}{1 - y}\right| = k\left(1 + x^2\right) \qquad \left(\ln k = 2C\right)$

(d) $\cos y \sin 2x \frac{dy}{dx} = \cot x \operatorname{cosec} y$

$\Rightarrow \quad \int \frac{\cos y}{\operatorname{cosec} y} dy = \int \frac{\cot x}{\sin 2x} dx$

$\Rightarrow \quad \int \sin y \cos y \, dy = \int \frac{\cos x}{\sin x \quad 2\sin x \cos x} dx$

$$\Rightarrow \quad \int \frac{1}{2}\sin 2y\,\mathrm{d}y = \int \frac{1}{2}\operatorname{cosec}^2 x\,\mathrm{d}x$$

$$\Rightarrow \quad -\frac{1}{4}\cos 2y = -\frac{1}{2}\cot x + C$$

or $\cos 2y = 2\cot x + k$

(e) $e^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(2 + e^y\right)$

$$\Rightarrow \quad e^x e^y \frac{\mathrm{d}y}{\mathrm{d}x} = x\left(2 + e^y\right)$$

$$\Rightarrow \quad \int \frac{e^y}{2 + e^y}\,\mathrm{d}y = \int xe^{-x}\,\mathrm{d}x$$

$$u = x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x} \quad \Rightarrow \quad v = -e^{-x}$$

$$\therefore \ln\left|2 + e^y\right| = -xe^{-x} + \int e^{-x}\,\mathrm{d}x$$

$$\Rightarrow \quad \ln\left|2 + e^y\right| = -xe^{-x} - e^{-x} + C$$

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Integration
Exercise J, Question 3

Question:

Find general solutions of the following differential equations:

(a) $\frac{dy}{dx} = ye^{x}$

(b) $\frac{dy}{dx} = xe^{y}$

(c) $\frac{dy}{dx} = y \cos x$

(d) $\frac{dy}{dx} = x \cos y$

(e) $\frac{dy}{dx} = \left(1 + \cos 2x \right) \cos y$

(f) $\frac{dy}{dx} = \left(1 + \cos 2y \right) \cos x$

Solution:

(a) $\frac{dy}{dx} = ye^{x}$

$\Rightarrow \quad \int \frac{1}{y} dy = \int e^{x} dx$

$\Rightarrow \quad \ln | y | = e^{x} + C$

(b) $\frac{dy}{dx} = xe^{y}$

$\Rightarrow \quad \int \frac{1}{e^{y}} dy = \int x\, dx$

$\Rightarrow \quad \int e^{-y} dy = \int x\, dx$

$\Rightarrow \quad -e^{-y} = \frac{1}{2}x^{2} + C$

(c) $\frac{dy}{dx} = y\cos x$

$\Rightarrow \int \frac{1}{y} dy = \int \cos x\, dx$

$\Rightarrow \ln|y| = \sin x + C$

or $y = Ae^{\sin x}$

(d) $\frac{dy}{dx} = x\cos y$

$\Rightarrow \int \frac{1}{\cos y} dy = \int x\, dx$

$\Rightarrow \int \sec y\, dy = \int x\, dx$

$\Rightarrow \ln|\sec y + \tan y| = \frac{x^2}{2} + C$

(e) $\frac{dy}{dx} = \left(1 + \cos 2x\right)\cos y$

$\Rightarrow \int \frac{1}{\cos y} dy = \int \left(1 + \cos 2x\right) dx$

$\Rightarrow \int \sec y\, dy = \int (1 + \cos 2x)\, dx$

$\Rightarrow \ln|\sec y + \tan y| = x + \frac{1}{2}\sin 2x + C$

(f) $\frac{dy}{dx} = \left(1 + \cos 2y\right)\cos x$

$\Rightarrow \int \frac{1}{1 + \cos 2y} dy = \int \cos x\, dx$

$\Rightarrow \int \frac{1}{2\cos^2 y} dy = \int \cos x\, dx$

$\Rightarrow \int \frac{1}{2}\sec^2 y\, dy = \int \cos x\, dx$

$\Rightarrow \frac{1}{2}\tan y = \sin x + C$

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Integration
Exercise J, Question 4

Question:

Find particular solutions of the following differential equations using the given boundary conditions.

(a) $\frac{dy}{dx} = \sin x \cos^2 x$; $y = 0$, $x = \frac{\pi}{3}$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y$; $y = 0$, $x = \frac{\pi}{4}$

(c) $\frac{dy}{dx} = 2\cos^2 y \cos^2 x$; $y = \frac{\pi}{4}$, $x = 0$

(d) $\left(1 - x^2 \right) \frac{dy}{dx} = xy + y$; $x = 0.5$, $y = 6$

(e) $2 \left(1 + x \right) \frac{dy}{dx} = 1 - y^2$; $x = 5$, $y = \frac{1}{2}$

Solution:

(a) $\frac{dy}{dx} = \sin x \cos^2 x$

$\Rightarrow \quad \int dy = \int \sin x \cos^2 x \, dx$

$\Rightarrow \quad y = - \frac{\cos^3 x}{3} + C$

$y = 0, x = \frac{\pi}{3} \quad \Rightarrow \quad 0 = - \frac{\left(\frac{1}{8} \right)}{3} + C \quad \Rightarrow \quad C = \frac{1}{24}$

$\therefore y = \frac{1}{24} - \frac{1}{3} \cos^3 x$

(b) $\frac{dy}{dx} = \sec^2 x \sec^2 y$

$\Rightarrow \int \frac{1}{\sec^2 y} \, dy = \int \sec^2 x \, dx$

$\Rightarrow \int \cos^2 y \, dy = \int \sec^2 x \, dx$

$\Rightarrow \int \left(\frac{1}{2} + \frac{1}{2} \cos 2y \right) dy = \int \sec^2 x \, dx$

$\Rightarrow \frac{1}{2} y + \frac{1}{4} \sin 2y = \tan x + C$

or $\sin 2y + 2y = 4 \tan x + k$

$y = 0, x = \frac{\pi}{4} \Rightarrow 0 = 4 + k \Rightarrow k = -4$

$\therefore \sin 2y + 2y = 4 \tan x - 4$

(c) $\frac{dy}{dx} = 2 \cos^2 y \cos^2 x$

$\Rightarrow \int \frac{1}{\cos^2 y} \, dy = \int 2 \cos^2 x \, dx$

$\Rightarrow \int \sec^2 y \, dy = \int (1 + \cos 2x) \, dx$

$\Rightarrow \tan y = x + \frac{1}{2} \sin 2x + C$

$x = 0, \quad y = \frac{\pi}{4} \Rightarrow 1 = 0 + C$

$\therefore \tan y = x + \frac{1}{2} \sin 2x + 1$

(d) $\left(1 - x^2 \right) \frac{dy}{dx} = xy + y$

$\Rightarrow \left(1 - x^2 \right) \frac{dy}{dx} = \left(x + 1 \right) y$

$\Rightarrow \int \frac{1}{y} \, dy = \int \frac{1 + x}{1 - x^2} \, dx$

$\Rightarrow \int \frac{1}{y} \, dy = \int \frac{1 + x}{(1 - x)(1 + x)} \, dx$

$\Rightarrow \int \frac{1}{y} \, dy = \int \frac{1}{1 - x} \, dx$

$\Rightarrow \ln |y| = -\ln |1 - x| + C$

$x = 0.5, \quad y = 6 \Rightarrow \ln 6 = -\ln \frac{1}{2} + C \Rightarrow C = \ln 3$

$\therefore \ln |y| = \ln 3 - \ln |1 - x|$

or $y = \dfrac{3}{1-x}$

(e) $2\left(1+x\right)\dfrac{dy}{dx} = 1 - y^2$

$\Rightarrow \int \dfrac{2}{1-y^2}\,dy = \int \dfrac{1}{1+x}\,dx$

$\dfrac{2}{1-y^2} \equiv \dfrac{A}{1-y} + \dfrac{B}{1+y}$

$\Rightarrow \quad 2 \equiv A(1+y) + B(1-y)$

$y = 1 \quad \Rightarrow \quad 2 = 2A \quad \Rightarrow \quad A = 1$

$y = -1 \quad \Rightarrow \quad 2 = 2B \quad \Rightarrow \quad B = 1$

$\therefore \int \left(\dfrac{1}{1+y} + \dfrac{1}{1-y}\right) dy = \int \dfrac{1}{1+x}\,dx$

$\Rightarrow \quad \ln|1+y| - \ln|1-y| = \ln|1+x| + C$

$\Rightarrow \quad \ln\left|\dfrac{1+y}{1-y}\right| = \ln|k(1+x)| \qquad \left(C = \ln k\right)$

$\Rightarrow \quad \dfrac{1+y}{1-y} = k\left(1+x\right)$

$x = 5, y = \dfrac{1}{2} \quad \Rightarrow \quad \dfrac{\frac{3}{2}}{\frac{1}{2}} = 6k \quad \Rightarrow \quad k = \dfrac{1}{2}$

$\therefore \dfrac{1+y}{1-y} = \dfrac{1+x}{2}$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise K, Question 1

Question:

The size of a certain population at time t is given by P. The rate of increase of P is given by $\frac{dP}{dt} = 2P$. Given that at time $t = 0$, the population was 3, find the population at time $t = 2$.

Solution:

$\frac{dP}{dt} = 2P$

$\Rightarrow \quad \int \frac{1}{P} dP = \int 2 \, dt$

$\Rightarrow \quad \ln | P | = 2t + C$

$\Rightarrow \quad P = Ae^{2t}$

$t = 0, P = 3 \quad \Rightarrow \quad 3 = Ae^0 \quad \Rightarrow \quad A = 3$

$\therefore P = 3e^{2t}$

When $t = 2$, $P = 3e^4 = 164$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise K, Question 2

Question:

The number of particles at time t of a certain radioactive substance is N. The substance is decaying in such a way that $\frac{dN}{dt} = -\frac{N}{3}$.

Given that at time $t = 0$ the number of particles is N_0, find the time when the number of particles remaining is $\frac{1}{2}N_0$.

Solution:

$$\frac{dN}{dt} = -\frac{N}{3}$$

$$\Rightarrow \quad \int \frac{1}{N} dN = \int -\frac{1}{3} dt$$

$$\Rightarrow \quad \ln|N| = -\frac{1}{3}t + C$$

$$\Rightarrow \quad N = Ae^{-\frac{1}{3}t}$$

$t = 0, N = N_0 \quad \Rightarrow \quad N_0 = Ae^0 \quad \Rightarrow \quad A = N_0$

$$\therefore N = N_0 e^{-\frac{1}{3}t}$$

$N = \frac{1}{2} N_0 \quad \Rightarrow \quad \frac{1}{2} = e^{-\frac{1}{3}t}$

$$\Rightarrow \quad -\ln 2 = -\frac{1}{3}t$$

$$\Rightarrow \quad t = 3\ln 2 \text{ or } 2.08$$

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Integration
Exercise K, Question 3

Question:

The mass M at time t of the leaves of a certain plant varies according to the differential equation $\frac{dM}{dt} = M - M^2$.

(a) Given that at time $t = 0$, $M = 0.5$, find an expression for M in terms of t.

(b) Find a value for M when $t = \ln 2$.

(c) Explain what happens to the value of M as t increases.

Solution:

$$\frac{dM}{dt} = M - M^2$$

$$\Rightarrow \int \frac{1}{M(1-M)} dM = \int 1\, dt \quad \text{but} \quad \frac{1}{M(1-M)} \equiv \frac{A}{M} + \frac{B}{1-M}$$

$$\therefore 1 \equiv A(1-M) + BM$$

$$M = 0 : 1 = 1A,\ A = 1$$

$$M = 1 : 1 = 1B,\ B = 1$$

$$\Rightarrow \int \left(\frac{1}{M} + \frac{1}{1-M} \right) dM = \int 1\, dt$$

$$\Rightarrow \ln|M| - \ln|1-M| = t + C$$

$$\Rightarrow \ln\left|\frac{M}{1-M}\right| = t + C$$

$$\Rightarrow \frac{M}{1-M} = Ae^t$$

(a) $t = 0$, $M = 0.5 \quad \Rightarrow \quad \frac{0.5}{0.5} = Ae^0 \quad \Rightarrow \quad A = 1$

$$\therefore M = e^t - e^t M \quad \Rightarrow \quad M = \frac{e^t}{1 + e^t}$$

(b) $t = \ln 2 \quad \Rightarrow \quad M = \dfrac{e^{\ln 2}}{1 + e^{\ln 2}} = \dfrac{2}{1 + 2} = \dfrac{2}{3}$

(c) $t \to \infty \quad \Rightarrow \quad M = \dfrac{1}{e^{-t} + 1} \to \dfrac{1}{1} = 1$

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Integration
Exercise K, Question 4

Question:

The volume of liquid $V\text{cm}^3$ at time t seconds satisfies

$$-15\frac{dV}{dt} = 2V - 450.$$

Given that initially the volume is $300\,\text{cm}^3$, find to the nearest cm^3 the volume after 15 seconds.

Solution:

$$-15\frac{dV}{dt} = 2V - 450$$

$$\Rightarrow \int \frac{1}{2V - 450}dV = \int -\frac{1}{15}dt$$

$$\Rightarrow \frac{1}{2}\ln|2V - 450| = -\frac{1}{15}t + C$$

$$\Rightarrow 2V - 450 = Ae^{-\frac{2}{15}t}$$

$$t = 0, V = 300 \Rightarrow 150 = Ae^0 \Rightarrow A = 150$$

$$\therefore 2V = 150e^{-\frac{2}{15}t} + 450$$

$$t = 15 \Rightarrow 2V = 150e^{-2} + 450$$

$$\Rightarrow V = \frac{150}{2}\left(e^{-2} + 3\right)$$

$$\Rightarrow V = 75(3 + e^{-2}) = 235$$

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Integration
Exercise K, Question 5

Question:

The thickness of ice x mm on a pond is increasing and $\frac{dx}{dt} = \frac{1}{20x^2}$, where t is measured in hours. Find how long it takes the thickness of ice to increase from 1 mm to 2 mm.

Solution:

$$\frac{dx}{dt} = \frac{1}{20x^2}$$

$$\Rightarrow \quad \int x^2 \, dx = \int \frac{1}{20} dt$$

$$\Rightarrow \quad \frac{1}{3}x^3 = \frac{t}{20} + C$$

$$t = 0, x = 1 \quad \Rightarrow \quad \frac{1}{3} = C$$

$$\therefore \quad \frac{20(x^3 - 1)}{3} = t$$

$$x = 2 \quad \Rightarrow \quad t = \frac{20}{3}\left(8 - 1\right)$$

$$\Rightarrow \quad t = \frac{140}{3} \text{ or } 46\frac{2}{3}$$

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Integration
Exercise K, Question 6

Question:

The depth h metres of fluid in a tank at time t minutes satisfies $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant. Find, in terms of k, how long it takes the depth to decrease from 9 m to 4 m.

Solution:

$$\frac{dh}{dt} = -k\sqrt{h}$$

$$\Rightarrow \int \frac{1}{h^{\frac{1}{2}}} dh = \int -k\,dt$$

$$\Rightarrow \int h^{-\frac{1}{2}} dh = \int -k\,dt$$

$$\Rightarrow 2h^{\frac{1}{2}} = -kt + C$$

$t = 0, h = 9 \Rightarrow 2 \times 3 = 0 + C \Rightarrow C = 6$

$$\therefore 2h^{\frac{1}{2}} - 6 = -kt$$

or $t = \frac{6 - 2\sqrt{h}}{k}$

$h = 4 \Rightarrow t = \frac{6 - 2 \times 2}{k} = \frac{2}{k}$

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Integration
Exercise K, Question 7

Question:

The rate of increase of the radius r kilometres of an oil slick is given by $\frac{dr}{dt} = \frac{k}{r^2}$, where k is a positive constant. When the slick was first observed the radius was 3 km. Two days later it was 5 km. Find, to the nearest day when the radius will be 6.

Solution:

$$\frac{dr}{dt} = \frac{k}{r^2}$$

$$\Rightarrow \quad \int r^2 \, dr = \int k \, dt$$

$$\Rightarrow \quad \frac{1}{3}r^3 = kt + C$$

$$t = 0, r = 3 \quad \Rightarrow \quad \frac{27}{3} = C \quad \Rightarrow \quad C = 9$$

$$\therefore kt = \frac{1}{3}r^3 - 9$$

$$t = 2, r = 5 \quad \Rightarrow \quad 2k = \frac{125}{3} - 9 \quad \Rightarrow \quad k = 16\frac{1}{3}$$

$$\therefore \frac{49}{3}t = \frac{1}{3}r^3 - 9$$

or $t = \frac{r^3 - 27}{49}$

$$r = 6 \quad \Rightarrow \quad t = \frac{6^3 - 27}{49} = 3.85... = 4 \text{ days}$$

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Integration
Exercise L, Question 1

Question:

It is given that $y = x^{\frac{3}{2}} + \frac{48}{x}, x > 0$.

(a) Find the value of x and the value of y when $\frac{dy}{dx} = 0$.

(b) Show that the value of y which you found is a minimum.

The finite region R is bounded by the curve with equation $y = x^{\frac{3}{2}} + \frac{48}{x}$, the lines $x = 1$, $x = 4$ and the x-axis.

(c) Find, by integration, the area of R giving your answer in the form $p + q \ln r$, where the numbers p, q and r are to be found.

E

Solution:

(a) $y = x^{\frac{3}{2}} + 48x^{-1} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 48x^{-2}$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$$

$$\Rightarrow x^{\frac{5}{2}} = \frac{2}{3} \times 48 = 32$$

$$\Rightarrow x = 4, y = 2^3 + 12 = 20$$

$$\Rightarrow x = 4 \ , \ y = 20$$

(b) $\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + 96x^{-3} > 0$ for all $x > 0$

$\therefore$ 20 is a minimum value of y

(c) Area $= \int_1^4 \left(x^{\frac{3}{2}} + \frac{48}{x} \right) dx$

$$= \left[\frac{2}{5}x^{\frac{5}{2}} + 48\ln|x| \right]_1^4$$

$$= \left(\frac{2}{5} \times 32 + 48\ln 4 \right) - \left(\frac{2}{5} + 0 \right)$$

$$= \frac{62}{5} + 48\ln 4$$

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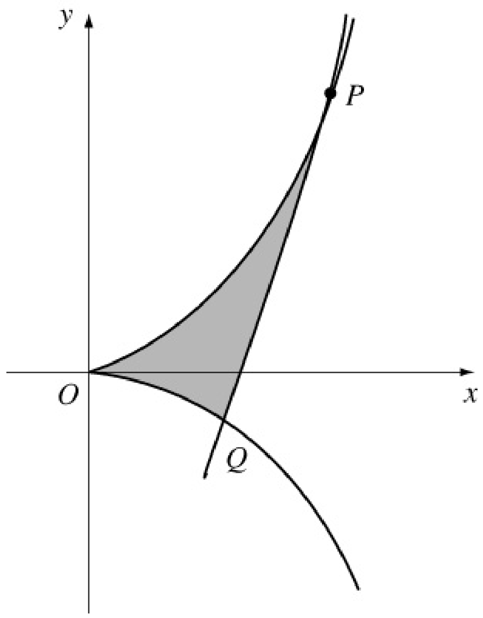
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 2

Question:

The curve C has two arcs, as shown, and the equations
$x = 3t^2$, $y = 2t^3$,
where t is a parameter.



(a) Find an equation of the tangent to C at the point P where $t = 2$.
The tangent meets the curve again at the point Q.

(b) Show that the coordinates of Q are $(3, -2)$.
The shaded region R is bounded by the arcs OP and OQ of the curve C, and the line PQ, as shown.

(c) Find the area of R.

E

Solution:

(a) $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2}{6t} = t$

P is (12, 16)
$\therefore$ tangent is $y - 16 = 2(x - 12)$ or $y = 2x - 8$

(b) Substitute $x = 3t^2$, $y = 2t^3$ into the equation for the tangent

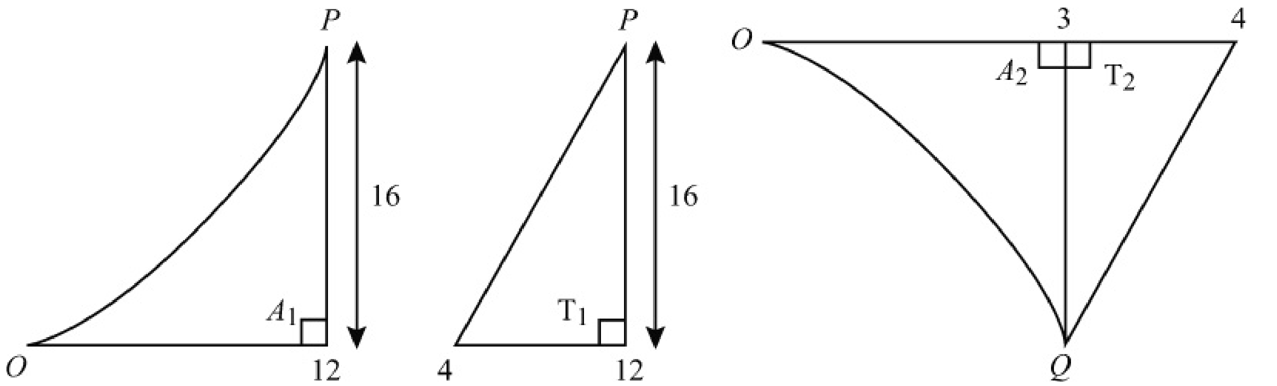
$\Rightarrow \quad 2t^3 = 6t^2 - 8$

$\Rightarrow \quad t^3 - 3t^2 + 4 = 0$

$\Rightarrow \quad (t-2)^2(t+1) = 0$

$\Rightarrow \quad t = -1$ at $Q(3, -2)$

(c)



Area of $R = A_1 - T_1 + A_2 + T_2$

$A_1 + A_2 = \int y\,dx = \int_{t=-1}^{t=2} 2t^3 \times 6t\,dt = \int_{-1}^{2} 12t^4\,dt$

$= \left[\frac{12}{5}t^5\right]_{-1}^{2} = \left(\frac{12 \times 32}{5}\right) - \left(-\frac{12}{5}\right) = 79.2$

$T_1 = \frac{1}{2} \times 16 \times 8 = 64$

$T_2 = \frac{1}{2} \times 1 \times 2 = 1$

$\therefore$ area of $R = 79.2 - 64 + 1 = 16.2$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 3

Question:

(a) Show that $(1+\sin 2x)^2 \equiv \frac{1}{2}(3+4\sin 2x-\cos 4x)$.

(b) The finite region bounded by the curve with equation $y=1+\sin 2x$, the x-axis, the y-axis and the line with equation $x=\frac{\pi}{2}$ is rotated through 2π about the x-axis.
Using calculus, calculate the volume of the solid generated, giving your answer in terms of π.

E

Solution:

(a) $(1+\sin 2x)^2 = 1+2\sin 2x+\sin^2 2x$

$= 1+2\sin 2x+\frac{1}{2}(1-\cos 4x)$

$= \frac{3}{2}+2\sin 2x-\frac{1}{2}\cos 4x$

$= \frac{1}{2}(3+4\sin 2x-\cos 4x)$

(b) $V=\pi\int y^2\,dx=\pi\int_0^{\frac{\pi}{2}}(1+\sin 2x)^2\,dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}(3+4\sin 2x-\cos 4x)\,dx$

$= \frac{\pi}{2}\left[3x-2\cos 2x-\frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}\left[\left(\frac{3\pi}{2}-2\cos\pi-\frac{1}{4}\sin 2\pi\right)-(0-2-0)\right]$

$= \frac{\pi}{2}\left(\frac{3\pi}{2}+2+2\right)$

$= \frac{\pi}{4}(3\pi+8)$

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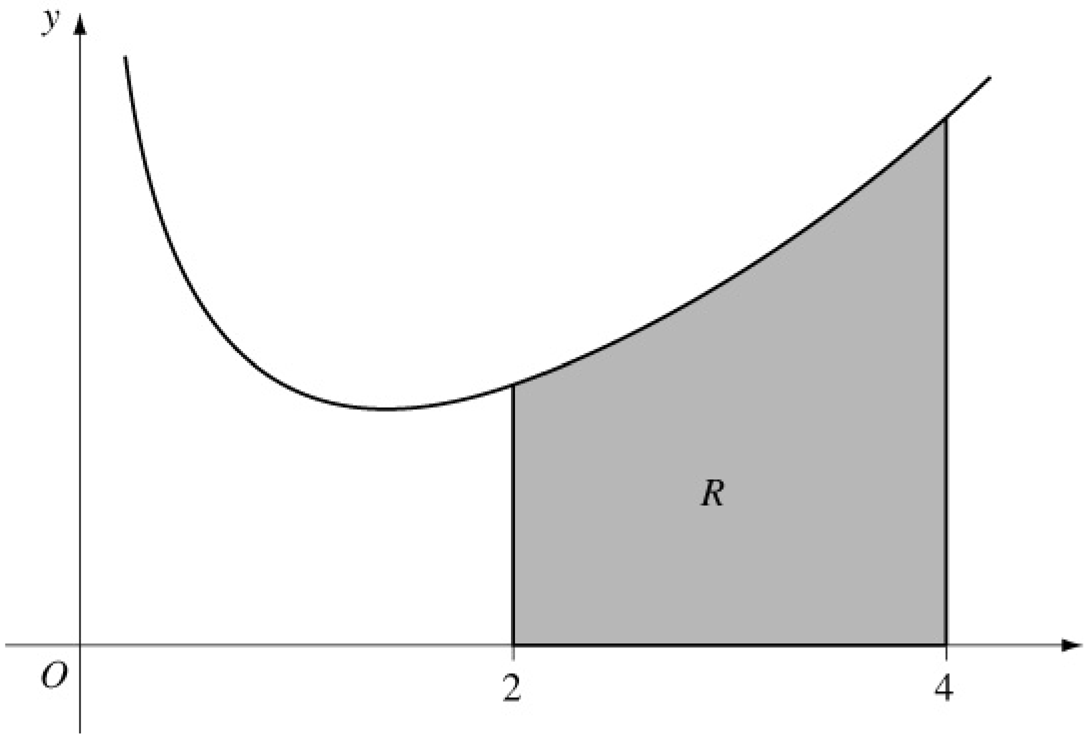
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 4

Question:

This graph shows part of the curve with equation $y = \mathrm{f}(x)$ where $\mathrm{f}(x) \equiv e^{0.5x} + \frac{1}{x}$, $x > 0$.



The curve has a stationary point at $x = \alpha$.

(a) Find $\mathrm{f}'(x)$.

(b) Hence calculate $\mathrm{f}'(1.05)$ and $\mathrm{f}'(1.10)$ and deduce that $1.05 < \alpha < 1.10$.

(c) Find $\int \mathrm{f}(x)\,\mathrm{d}x$.
The shaded region R is bounded by the curve, the x-axis and the lines $x = 2$ and $x = 4$.

(d) Find, to 2 decimal places, the area of R.

E

Solution:

(a) $f'(x) = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{x^2}$

(b) $f'(1.05) = -0.061... < 0$

$f'(1.10) = +0.040... > 0$

Change of sign $\therefore$ root α in interval (1.05, 1.10)

(c) $\int \left(e^{0.5x} + \frac{1}{x} \right) dx = 2e^{0.5x} + \ln|x| + C$

(d) Area $= \int_2^4 y\,dx$

$= \left[2e^{0.5x} + \ln|x| \right]_2^4$

$= (2e^2 + \ln 4) - (2e^1 + \ln 2)$

$= 2e^2 - 2e^1 + \ln 2$

$= 10.03$ (2 d.p.)

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Integration
Exercise L, Question 5

Question:

(a) Find $\int xe^{-x} dx$.

(b) Given that $y = \frac{\pi}{4}$ at $x = 0$, solve the differential equation

$$e^{x} \frac{dy}{dx} = \frac{x}{\sin 2y}$$

Solution:

(a) $I = \int xe^{-x} dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = -xe^{-x} - \int (-e^{-x}) dx$

i.e. $I = -xe^{-x} - e^{-x} + C$

(b) $e^{x} \frac{dy}{dx} = \frac{x}{\sin 2y}$

$\Rightarrow \int \sin 2y \, dy = \int xe^{-x} dx$

$\Rightarrow -\frac{1}{2} \cos 2y = -xe^{-x} - e^{-x} + C$

$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + C \Rightarrow C = 1$

$\therefore \frac{1}{2} \cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

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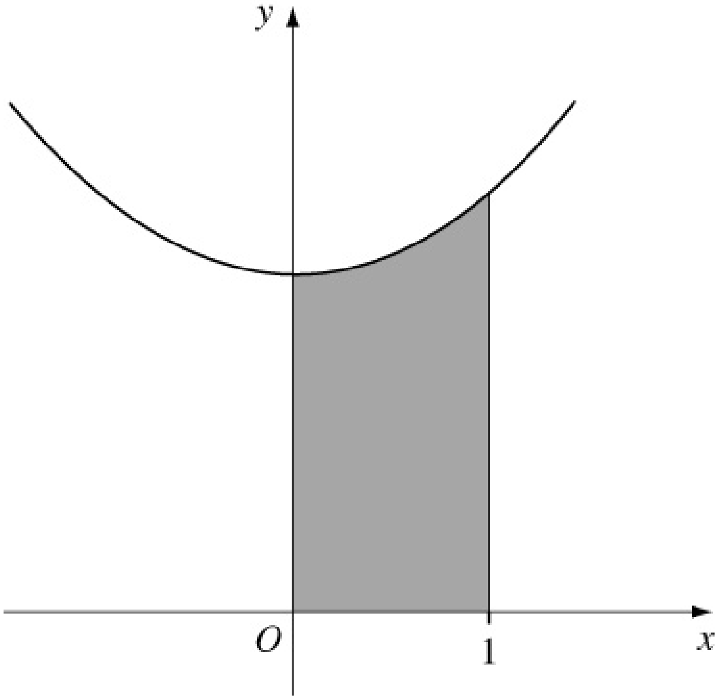
Integration
Exercise L, Question 6

Question:

The diagram shows the finite shaded region bounded by the curve with equation $y = x^2 + 3$, the lines $x = 1$, $x = 0$ and the x-axis. This region is rotated through 360° about the x-axis.
Find the volume generated.



Solution:

$$V = \pi \int_0^1 y^2 \, dx = \pi \int_0^1 \left(x^2 + 3 \right)^2 dx$$

$$= \pi \int_0^1 \left(x^4 + 6x^2 + 9 \right) dx$$

$$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} + 2 + 9 \right) - \left(0 \right) \right]$$

$$= \frac{56\pi}{5}$$

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Integration
Exercise L, Question 7

Question:

(a) Find $\int \frac{1}{x(x+1)} dx$

(b) Using the substitution $u = e^x$ and the answer to **a**, or otherwise, find $\int \frac{1}{1+e^x} dx$.

(c) Use integration by parts to find $\int x^2 \sin x \, dx$.

Solution:

(a) $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

$\therefore \int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$

$= \ln|x| - \ln|x+1| + C$

$= \ln \left| \frac{x}{x+1} \right| + C$

(b) $I = \int \frac{1}{1+e^x} dx \qquad u = e^x \Rightarrow du = e^x dx$

$\therefore I = \int \frac{1}{(1+u)} \times \frac{1}{u} du = \ln \left| \frac{u}{1+u} \right| + C$ or $\ln \left| \frac{e^x}{1+e^x} \right| + C$

(c) $I = \int x^2 \sin x \, dx$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$

$\therefore I = -x^2 \cos x - \int (-\cos x) \times 2x \, dx$

$= -x^2 \cos x + \int 2x \cos x \, dx$

Let $J = \int 2x \cos x \, dx$

$u = 2x \quad \Rightarrow \quad \frac{du}{dx} = 2$

$\frac{dv}{dx} = \cos x \quad \Rightarrow \quad v = \sin x$

$\therefore J = 2x \sin x - \int 2 \sin x \, dx$

$= 2x \sin x + 2 \cos x + C$

$\therefore I = -x^2 \cos x + 2x \sin x + 2 \cos x + k$

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Integration
Exercise L, Question 8

Question:

(a) Find $\int x \sin 2x \, dx$.

(b) Given that $y = 0$ at $x = \frac{\pi}{4}$, solve the differential equation $\frac{dy}{dx} = x \sin 2x \cos^2 y$.

E

Solution:

(a) $I = \int x \sin 2x \, dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 2x \quad \Rightarrow \quad v = \frac{-1}{2} \cos 2x$

$\therefore I = -\frac{1}{2} x \cos 2x - \int \frac{-1}{2} \cos 2x \, dx$

$= -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$

(b) $\frac{dy}{dx} = x \sin 2x \cos^2 y$

$\Rightarrow \quad \int \sec^2 y \, dy = \int x \sin 2x \, dx$

$\Rightarrow \quad \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x + C$

$y = 0, x = \frac{\pi}{4} \quad \Rightarrow \quad 0 = 0 + \frac{1}{4} + C \quad \Rightarrow \quad C = -\frac{1}{4}$

$\therefore \tan y = -\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x - \frac{1}{4}$

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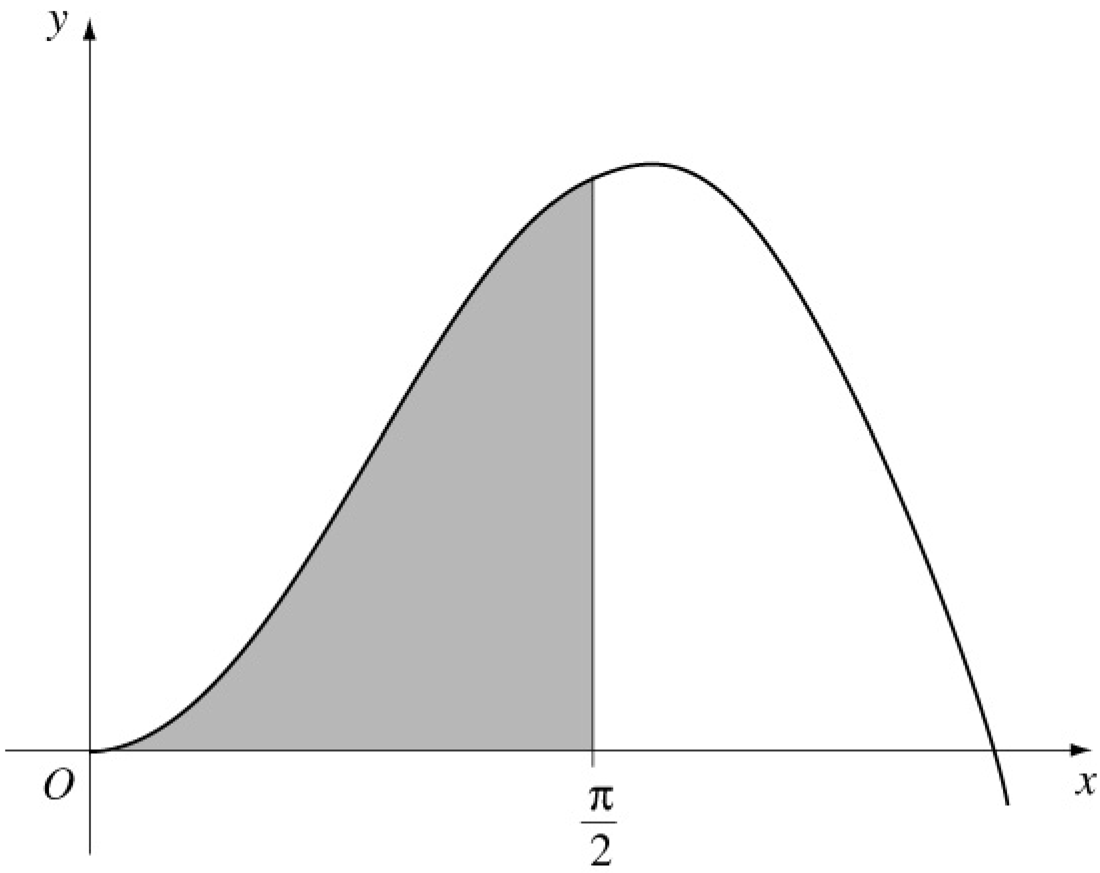
Integration
Exercise L, Question 9

Question:

(a) Find $\int x \cos 2x \, dx$.

(b) This diagram shows part of the curve with equation $y = 2x^{\frac{1}{2}} \sin x$. The shaded region in the diagram is bounded by the curve, the x-axis and the line with equation $x = \frac{\pi}{2}$. This shaded region is rotated through 2π radians about the x-axis to form a solid of revolution. Using calculus, calculate the volume of the solid of revolution formed, giving your answer in terms of π.

E



Solution:

(a) $I = \int x \cos 2x \, dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$$\frac{dv}{dx} = \cos 2x \quad \Rightarrow \quad v = \frac{1}{2}\sin 2x$$

$$\therefore I = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(b) $V = \pi \int_0^{\frac{\pi}{2}} y^2\,dx = \pi \int_0^{\frac{\pi}{2}} 4x\sin^2 x\,dx$

$$\cos 2A = 1 - 2\sin^2 A \quad \Rightarrow \quad 2\sin^2 x = 1 - \cos 2x$$

$$\therefore V = \pi \int_0^{\frac{\pi}{2}} 2x\left(1 - \cos 2x\right)dx$$

$$= \pi \int_0^{\frac{\pi}{2}} 2x\,dx - 2\pi \int_0^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$= \left[\pi x^2\right]_0^{\frac{\pi}{2}} - 2\pi\left[\frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^3}{4} - 2\pi\left[\left(\frac{\pi}{4}\sin\pi + \frac{1}{4}\cos\pi\right) - \left(0 + \frac{1}{4}\right)\right]$$

$$= \frac{\pi^3}{4} + \pi$$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 10

Question:

A curve has equation $y = f(x)$ and passes through the point with coordinates $(0, -1)$. Given that $f'(x) = \frac{1}{2}e^{2x} - 6x$,

(a) use integration to obtain an expression for $f(x)$,

(b) show that there is a root α of the equation $f'(x) = 0$, such that $1.41 < \alpha < 1.43$. **E**

Solution:

(a) $f'(x) = \frac{1}{2}e^{2x} - 6x$

$\Rightarrow \quad f(x) = \frac{1}{4}e^{2x} - 3x^2 + C$

$f(0) = -1 \quad \Rightarrow \quad -1 = \frac{1}{4} - 0 + C \quad \Rightarrow \quad C = -\frac{5}{4}$

$\therefore f(x) = \frac{1}{4}e^{2x} - 3x^2 - \frac{5}{4}$

(b) $f'(1.41) = -0.07... < 0$
$f'(1.43) = +0.15... > 0$
Change of sign $\therefore$ root in interval (1.41, 1.43).

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Integration
Exercise L, Question 11

Question:

$$f\left(x\right) = 16x^{\frac{1}{2}} - \frac{2}{x}, x > 0.$$

(a) Solve the equation $f(x) = 0$.

(b) Find $\int f(x)\,dx$.

(c) Evaluate $\int_1^4 f(x)\,dx$, giving your answer in the form $p + q\ln r$, where p, q and r are rational numbers.

Solution:

(a) $f\left(x\right) = 0 \Rightarrow 16x^{\frac{1}{2}} = \frac{2}{x}$

$\Rightarrow 16x^{\frac{3}{2}} = 2$

$\Rightarrow x^{\frac{3}{2}} = \frac{1}{8}$

$\Rightarrow x = \left(\sqrt[3]{\frac{1}{8}}\right)^2 = \frac{1}{4}$

(b) $\int \left(16x^{\frac{1}{2}} - \frac{2}{x}\right) dx = \frac{16x^{\frac{3}{2}}}{\frac{3}{2}} - 2\ln|x| + C$

$= \frac{32}{3}x^{\frac{3}{2}} - 2\ln|x| + C$

(c) $\int_1^4 f\left(x\right) dx = \left[\frac{32}{3}x^{\frac{3}{2}} - 2\ln|x|\right]_1^4$

$= \left(\frac{32}{3} \times 2^3 - 2\ln 4\right) - \left(\frac{32}{3} - 0\right)$

$$= \frac{224}{3} - 2\ln 4$$

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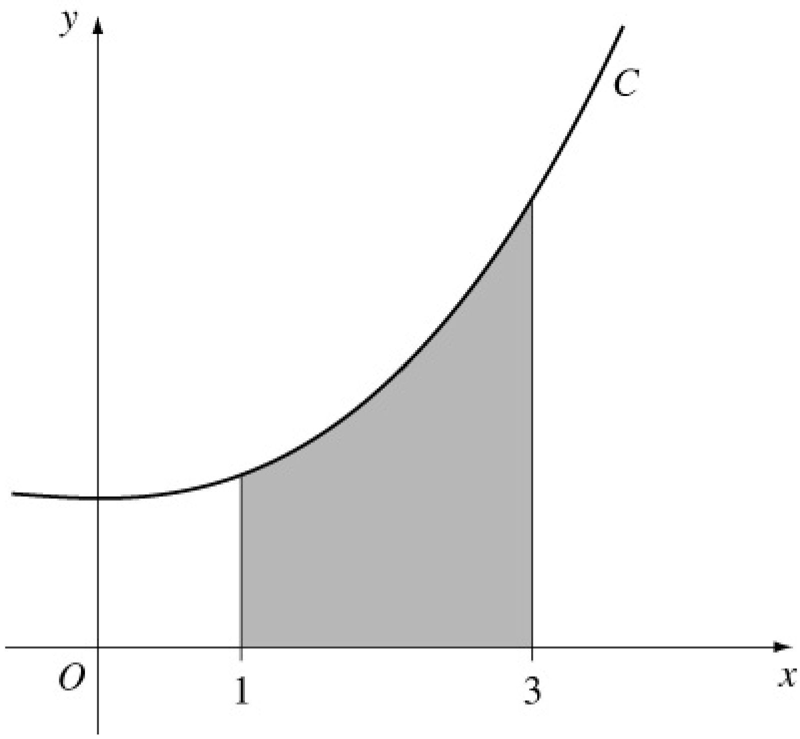
Integration
Exercise L, Question 12

Question:

Shown is part of a curve C with equation $y = x^2 + 3$. The shaded region is bounded by C, the x-axis and the lines with equations $x = 1$ and $x = 3$. The shaded region is rotated through 360° about the x-axis.
Using calculus, calculate the volume of the solid generated. Give your answer as an exact multiple of π.

E



Solution:

$V = \pi \int_1^3 y^2 \, dx = \pi \int_1^3 (x^2 + 3)^2 \, dx$

$= \pi \int_1^3 (x^4 + 6x^2 + 9) \, dx$

$= \pi \left[\frac{1}{5}x^5 + 2x^3 + 9x \right]_1^3$

$= \pi \left[\left(\frac{243}{5} + 54 + 27 \right) - \left(\frac{1}{5} + 2 + 9 \right) \right]$

$= \pi \left(\frac{242}{5} + 81 - 11 \right)$

$= 118.4\pi$

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Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 13

Question:

(a) Find $\int x (x^2 + 3)^5 dx$

(b) Show that $\int_1^e \frac{1}{x^2} \ln x \, dx = 1 - \frac{2}{e}$

(c) Given that $p > 1$, show that $\int_1^p \frac{1}{(x + 1) (2x - 1)} dx = \frac{1}{3} \ln \frac{4p - 2}{p + 1}$

E

Solution:

(a) Let $y = (x^2 + 3)^6$

$\Rightarrow \quad \frac{dy}{dx} = 6 (x^2 + 3)^5 \times 2x$

$\therefore \int x (x^2 + 3)^5 dx = \frac{1}{12} (x^2 + 3)^6 + C$

(b) $I = \int_1^e \frac{1}{x^2} \ln x \, dx$

$u = \ln x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2} \quad \Rightarrow \quad v = - \frac{1}{x}$

$\therefore I = \left[- \frac{1}{x} \ln x \right]_1^e - \int_1^e \left(- \frac{1}{x^2} \right) dx$

$= \left(- \frac{1}{e} \right) - \left(0 \right) + \left[- \frac{1}{x} \right]_1^e$

$= - \frac{1}{e} + \left(- \frac{1}{e} \right) - \left(- 1 \right)$

$= 1 - \frac{2}{e}$

(c) $\frac{1}{(x+1)(2x-1)} \equiv \frac{A}{x+1} + \frac{B}{2x-1}$

$\Rightarrow \quad 1 \equiv A(2x-1) + B(x+1)$

$x = \frac{1}{2} \quad \Rightarrow \quad 1 = \frac{3}{2}B \quad \Rightarrow \quad B = \frac{2}{3}$

$x = -1 \quad \Rightarrow \quad 1 = -3A \quad \Rightarrow \quad A = -\frac{1}{3}$

$$\therefore \int_1^p \frac{1}{(x+1)(2x-1)}\,dx = \int_1^p \left(\frac{\frac{2}{3}}{2x-1} + \frac{-\frac{1}{3}}{x+1} \right) dx$$

$$= \left[\frac{1}{3}\ln|2x-1| - \frac{1}{3}\ln|x+1| \right]_1^p$$

$$= \left[\frac{1}{3}\ln\left| \frac{2x-1}{x+1} \right| \right]_1^p$$

$$= \left[\frac{1}{3}\ln\left(\frac{2p-1}{p+1} \right) \right] - \left(\frac{1}{3}\ln\frac{1}{2} \right)$$

$$= \frac{1}{3}\ln\left(\frac{4p-2}{p+1} \right)$$

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Edexcel AS and A Level Modular Mathematics

Integration

Exercise L, Question 14

Question:

$$f\left(x\right) \equiv \frac{5x^2 - 8x + 1}{2x\left(x-1\right)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{\left(x-1\right)^2}$$

(a) Find the values of the constants A, B and C.

(b) Hence find $\int f\left(x\right)\,dx$.

(c) Hence show that $\int_4^9 f\left(x\right)\,dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$

E

Solution:

(a) $f\left(x\right) \equiv \frac{5x^2 - 8x + 1}{2x\left(x-1\right)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{\left(x-1\right)^2}$

$\Rightarrow \quad 5x^2 - 8x + 1 \equiv 2A\left(x-1\right)^2 + 2Bx\left(x-1\right) + 2Cx$

$x = 0 \quad \Rightarrow \quad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

$x = 1 \quad \Rightarrow \quad -2 = 2C \quad \Rightarrow \quad C = -1$

Coefficients of x^2: $\quad 5 = 2A + 2B \quad \Rightarrow \quad B = 2$

(b) $\int f\left(x\right)\,dx = \int\left(\frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{1}{\left(x-1\right)^2}\right)dx$

$= \frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1} + C$

(c) $\int_4^9 f\left(x\right)\,dx = \left[\frac{1}{2}\ln\left|x\right| + 2\ln\left|x-1\right| + \frac{1}{x-1}\right]_4^9$

$= \left[\ln\left|\sqrt{x}\left(x-1\right)^2\right| + \frac{1}{x-1}\right]_4^9$

$$= \left[\ln \left(3 \times 64 \right) + \frac{1}{8} \right] - \left[\ln \left(2 \times 9 \right) + \frac{1}{3} \right]$$

$$= \ln \left(\frac{3 \times 64}{2 \times 9} \right) + \frac{1}{8} - \frac{1}{3}$$

$$= \ln \frac{32}{3} - \frac{5}{24}$$

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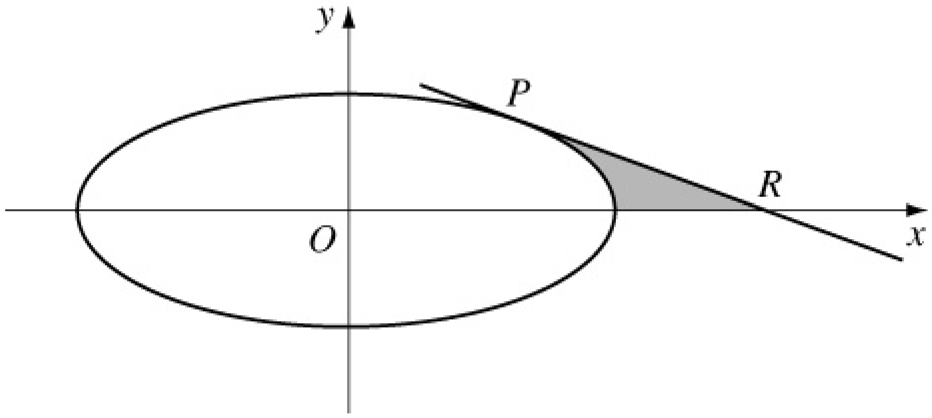
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 15

Question:

The curve shown has parametric equations

$x = 5\cos\theta,\ y = 4\sin\theta,\ 0 \leq \theta < 2\pi.$



(a) Find the gradient of the curve at the point P at which $\theta = \frac{\pi}{4}$.

(b) Find an equation of the tangent to the curve at the point P.

(c) Find the coordinates of the point R where this tangent meets the x-axis.
The shaded region is bounded by the tangent PR, the curve and the x-axis.

(d) Find the area of the shaded region, leaving your answer in terms of π.

E

Solution:

(a) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{4\cos\theta}{5\sin\theta}$

$\therefore$ gradient of tangent at $P = -\frac{4}{5}$

(b) $P = \left(\frac{5}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right)$

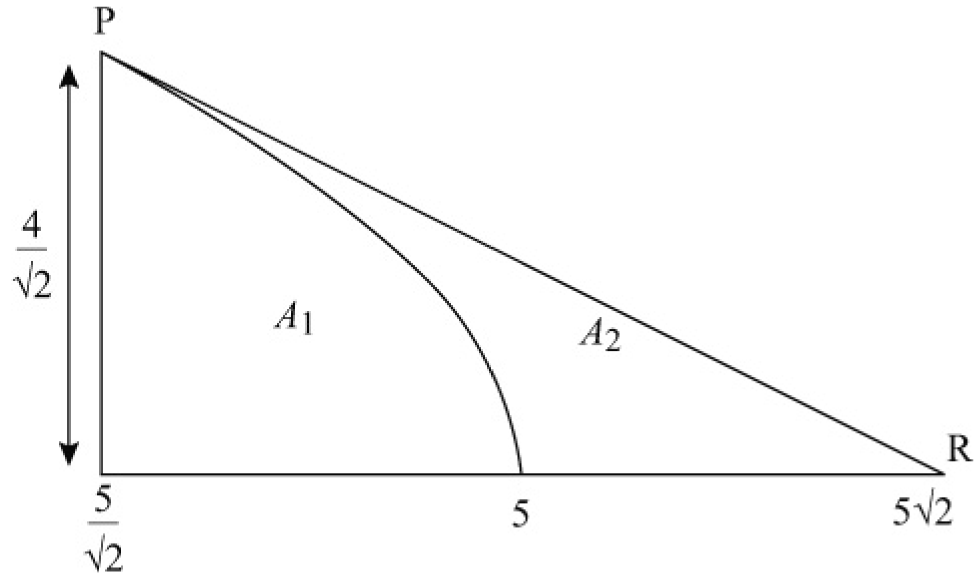
$\therefore$ equation of tangent is

$y - \frac{4}{\sqrt{2}} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$ or $y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$

(c) At R, $y = 0 \Rightarrow x = \frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2} = 5\sqrt{2}$

$\therefore R$ is $(5\sqrt{2}, 0)$

(d)



$A_1 + A_2 = \frac{1}{2} \times \left(5\sqrt{2} - \frac{5}{\sqrt{2}}\right) \times \frac{4}{\sqrt{2}} = \frac{1}{2} \times \frac{5}{\sqrt{2}} \times \frac{4}{\sqrt{2}} = 5$

$A_1 = \int y\,dx = \int_{\frac{\pi}{4}}^{0} 4\sin\theta \times \left(-5\sin\theta\right) d\theta$

$= 10\int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right) d\theta$

$= \left[10\theta - 5\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{5\pi}{2} - 5$

$\therefore A_2 = 5 - A_1 = 5 - \left(\frac{5\pi}{2} - 5\right) = 10 - 2.5\pi$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 16

Question:

(a) Obtain the general solution of the differential equation

$\frac{dy}{dx} = xy^2, y > 0.$

(b) Given also that $y = 1$ at $x = 1$, show that

$y = \frac{2}{3 - x^2}, \ -\sqrt{3} < x < \sqrt{3}$

is a particular solution of the differential equation.

The curve C has equation $y = \frac{2}{3 - x^2}, x \neq -\sqrt{3}, x \neq \sqrt{3}$

(c) Write down the gradient of C at the point (1, 1).

(d) Deduce that the line which is a tangent to C at the point (1, 1) has equation $y = x$.

(e) Find the coordinates of the point where the line $y = x$ again meets the curve C.

E

Solution:

(a) $\frac{dy}{dx} = xy^2$

$\Rightarrow \quad \int \frac{1}{y^2} dy = \int x \, dx$

$\Rightarrow \quad -\frac{1}{y} = \frac{x^2}{2} + C$

or $y = \frac{-2}{x^2 + k} \quad \left(k = 2C \right)$

(b) $y = 1, x = 1 \quad \Rightarrow \quad 1 = \frac{-2}{1 + k} \quad \Rightarrow \quad k = -3$

$\therefore y = \dfrac{2}{3 - x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

(c) When $x = 1$, $y = 1$ $\dfrac{dy}{dx}$ is 1

(d) Equation of tangent is $y - 1 = 1(x - 1)$, i.e. $y = x$.

(e) $x = \dfrac{2}{3 - x^2} \quad \Rightarrow \quad -x^3 + 3x = 2$ or $x^3 - 3x + 2 = 0$

$\Rightarrow \quad (x - 1)^2 (x + 2) = 0$

$\therefore y = x$ meets curve at $(-2, -2)$.

Solutionbank
Edexcel AS and A Level Modular Mathematics

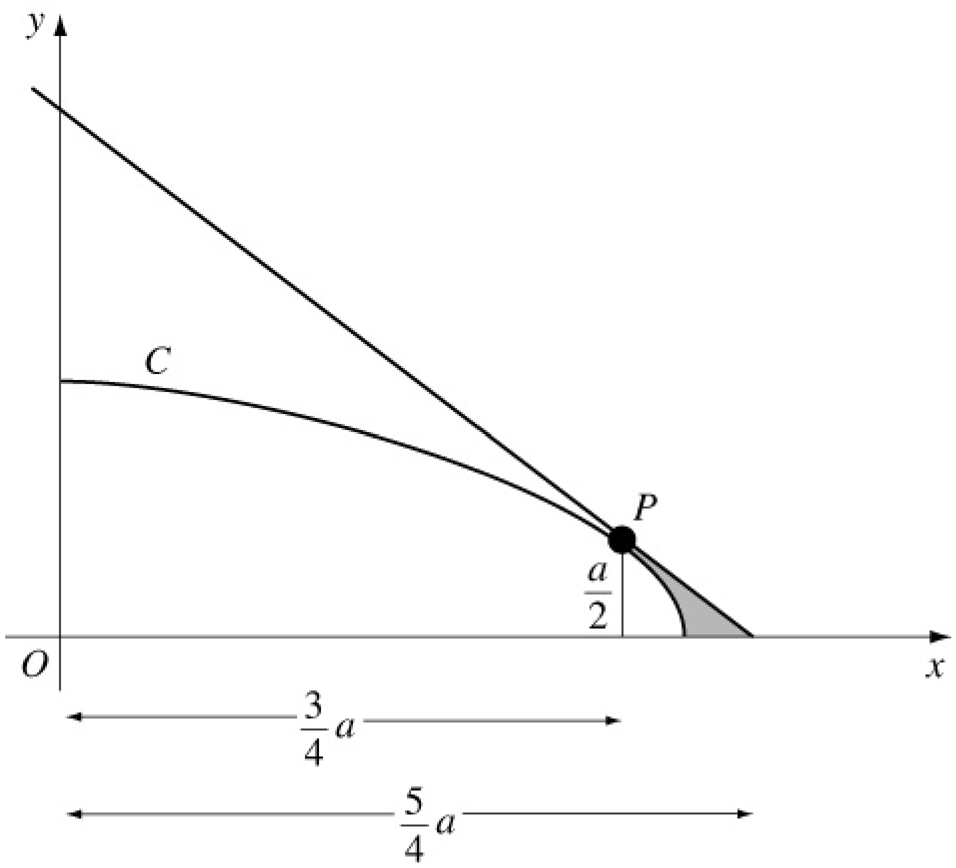
Integration
Exercise L, Question 17

Question:

The diagram shows the curve C with parametric equations

$x = a\sin^2 t,\ y = a\cos t,\ 0 \le t \le \frac{1}{2}\pi,$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a,\ \frac{1}{2}a\right)$.



(a) Find $\frac{dy}{dx}$, giving your answer in terms of t.

(b) Find an equation of the tangent to C at P.

(c) Show that a cartesian equation of C is $y^2 = a^2 - ax$.
The shaded region is bounded by C, the tangent at P and the x-axis. This shaded region is rotated through 2π radians about the x-axis to form a solid of

revolution.

(d) Use calculus to calculate the volume of the solid revolution formed, giving your answer in the form $k\pi a^3$, where k is an exact fraction. **E**

Solution:

(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = -\frac{1}{2}\sec t$

(b) P is $\left(\frac{3}{4}a, \frac{1}{2}a\right)$, so $\cos t = \frac{1}{2}$

$$\Rightarrow \quad M = -\frac{1}{2 \times \frac{1}{2}} = -1$$

$\therefore$ tangent is $y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$

or $\quad y = -x + \frac{5}{4}a$

(c) $\sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \frac{x}{a} + \frac{y^2}{a^2} = 1$

or $\quad y^2 = a^2 - ax$

(d) volume $=$ cone $- \pi \int_{\frac{3}{4}a}^{a} y^2\,dx$

cone $= \frac{1}{3}\pi\left(\frac{1}{2}a\right)^2\left(\frac{5}{4}a - \frac{3}{4}a\right) = \frac{\pi a^3}{24}$

$\pi \int_{\frac{3}{4}a}^{a} y^2\,dx = \pi\left[a^2 x - \frac{a}{2}x^2\right]_{\frac{3}{4}a}^{a}$

$= \pi\left[\left(a^3 - \frac{a^3}{2}\right) - \left(\frac{3}{4}a^3 - \frac{9}{32}a^3\right)\right] = \frac{\pi a^3}{32}$

$\therefore$ Volume $= \pi\left(\frac{a^3}{24} - \frac{a^3}{32}\right) = \frac{\pi a^3}{96}$

Solutionbank

Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 18

Question:

(a) Using the substitution $u = 1 + 2x$, or otherwise, find

$$\int \frac{4x}{(1+2x)^2} dx, \ x > -\frac{1}{2},$$

(b) Given that $y = \frac{\pi}{4}$ when $x = 0$, solve the differential equation

$$(1+2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$$

E

Solution:

(a) $I = \int \frac{4x}{(1+2x)^2} dx$

$u = 1 + 2x$

$\Rightarrow \frac{du}{2} = dx$ and $4x = 2(u-1)$

$\therefore I = \int \frac{2(u-1)}{u^2} \times \frac{du}{2}$

$= \int \left(\frac{1}{u} - u^{-2} \right) du$

$= \ln|u| + \frac{1}{u} + C$

$= \ln|1+2x| + \frac{1}{1+2x} + C$

(b) $(1+2x)^2 \frac{dy}{dx} = \frac{x}{\sin^2 y}$

$\Rightarrow \int \sin^2 y \, dy = \int \frac{x}{(1+2x)^2} dx$

$\Rightarrow \int 4\sin^2 y \, dy = \int \frac{4x}{(1+2x)^2} dx$

$\Rightarrow \int (2 - 2\cos 2y) \, dy = I$

$$\Rightarrow \quad 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + C$$

$$x = 0,\ y = \frac{\pi}{4} \quad \Rightarrow \quad \frac{\pi}{2} - 1 = \ln 1 + 1 + C$$

$$\Rightarrow \quad C = \frac{\pi}{2} - 2$$

$$\therefore\ 2y - \sin 2y = \ln|1+2x| + \frac{1}{1+2x} + \frac{\pi}{2} - 2$$

Solutionbank
Edexcel AS and A Level Modular Mathematics

Integration
Exercise L, Question 19

Question:

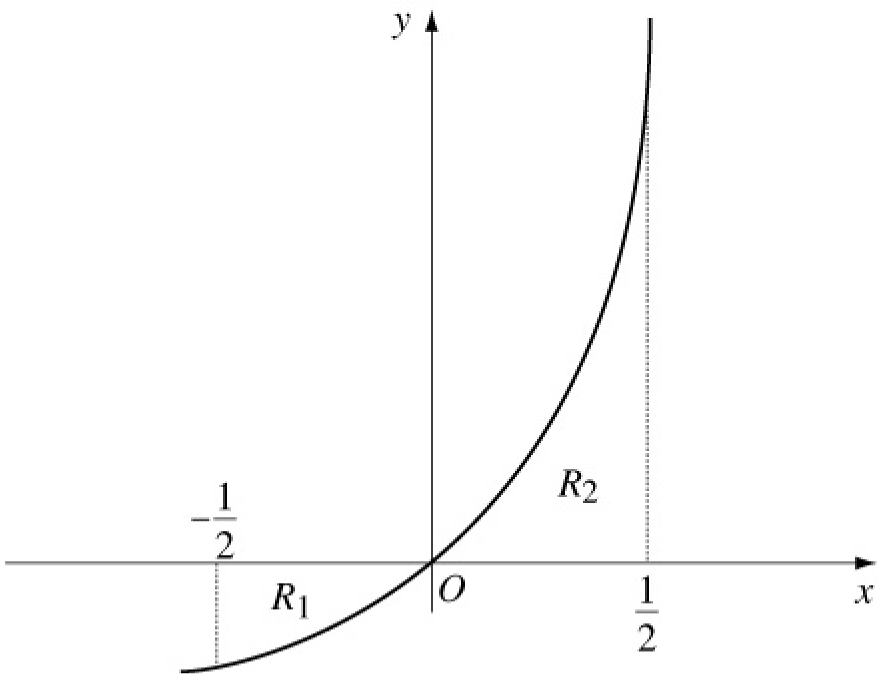
The diagram shows the curve with equation $y = xe^{2x}$, $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

The finite region R_1 bounded by the curve, the x-axis and the line $x = -\frac{1}{2}$ has area A_1.

The finite region R_2 bounded by the curve, the x-axis and the line $x = \frac{1}{2}$ has area A_2.

(a) Find the exact values of A_1 and A_2 by integration.

(b) Show that $A_1 : A_2 = (e - 2) : e$.



Solution:

(a) $\int xe^{2x} dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$$\frac{dv}{dx} = e^{2x} \quad \Rightarrow \quad v = \frac{1}{2}e^{2x}$$

$$\therefore \int xe^{2x}dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

$$A_1 = -\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \right]_{-\frac{1}{2}}^{0}$$

$$= -\left[\left(0 - \frac{1}{4} \right) - \left(-\frac{1}{4}e^{-1} - \frac{1}{4}e^{-1} \right) \right]$$

$$= \frac{1}{4}\left(1 - 2e^{-1} \right)$$

$$A_2 = \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} \right]_0^{\frac{1}{2}}$$

$$= \left(\frac{1}{4}e^{1} - \frac{1}{4}e^{1} \right) - \left(0 - \frac{1}{4} \right)$$

$$= \frac{1}{4}$$

(b) $$\frac{A_1}{A_2} = \frac{\frac{1}{4}(1 - 2e^{-1})}{\frac{1}{4}} = 1 - 2e^{-1} = \frac{e-2}{e}$$

$$\therefore A_1 : A_2 = (e - 2) : e$$

Solutionbank

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Integration
Exercise L, Question 20

Question:

Find $\int x^2 e^{-x} dx$.

Given that $y = 0$ at $x = 0$, solve the differential equation $\frac{dy}{dx} = x^2 e^{3y - x}$. (E)

Solution:

$I = \int x^2 e^{-x} dx$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = -x^2 e^{-x} - \int (-e^{-x}) \times 2x \, dx$

$= -x^2 e^{-x} + \int 2x e^{-x} dx$

$J = \int 2x e^{-x} dx$

$u = 2x \Rightarrow \frac{du}{dx} = 2$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore J = -2xe^{-x} - \int (-e^{-x}) \times 2 \, dx$

$= -2xe^{-x} - 2e^{-x} + k$

$\therefore I = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$\frac{dy}{dx} = x^2 e^{3y - x} = x^2 e^{-x} e^{3y}$

$\Rightarrow \int e^{-3y} dy = \int x^2 e^{-x} dx$

$\Rightarrow -\frac{1}{3} e^{-3y} = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$x = 0, y = 0 \Rightarrow -\frac{1}{3} = -2 + C \Rightarrow C = \frac{5}{3}$

$\therefore \frac{1}{3} e^{-3y} = x^2 e^{-x} + 2xe^{-x} + 2e^{-x} - \frac{5}{3}$

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Integration
Exercise L, Question 21

Question:

The curve with equation $y = e^{3x} + 1$ meets the line $y = 8$ at the point $(h, 8)$.

(a) Find h, giving your answer in terms of natural logarithms.

(b) Show that the area of the finite region enclosed by the curve with equation $y = e^{3x} + 1$, the x-axis, the y-axis and the line $x = h$ is $2 + \frac{1}{3}\ln 7$.

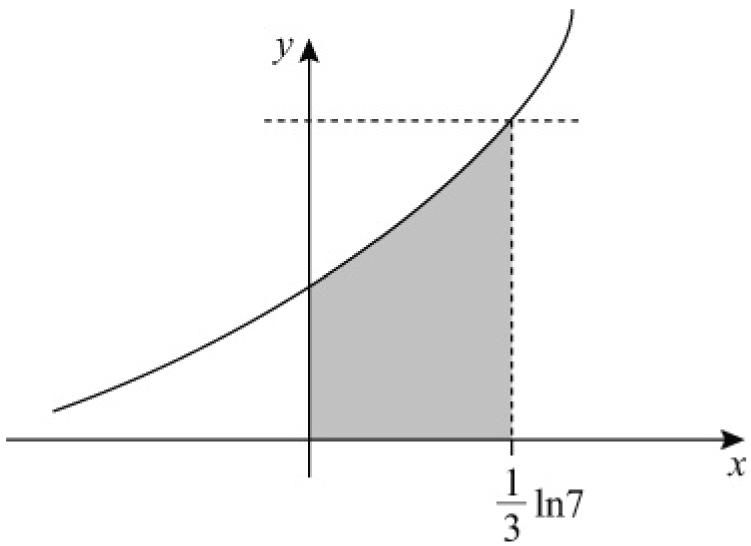
E

Solution:

(a) $8 = e^{3x} + 1 \quad \Rightarrow \quad 7 = e^{3x}$

$\therefore x = \frac{1}{3}\ln 7$, i.e. $h = \frac{1}{3}\ln 7$

(b)



$$\text{Area} = \int_0^{\frac{1}{3}\ln 7} y\,dx$$

$$= \int_0^{\frac{1}{3}\ln 7} \left(e^{3x} + 1 \right) dx$$

$$= \left[\frac{1}{3}e^{3x} + x \right]_0^{\frac{1}{3}\ln 7}$$

$$= \left(\frac{1}{3}e^{\ln 7} + \frac{1}{3}\ln 7 \right) - \left(\frac{1}{3} + 0 \right)$$

$$= \frac{1}{3}\left(7 + \ln 7 \right) - \frac{1}{3}$$

$$= \frac{1}{3}\left(6 + \ln 7 \right)$$

$$= 2 + \frac{1}{3}\ln 7$$

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Integration
Exercise L, Question 22

Question:

(a) Given that

$$\frac{x^2}{x^2-1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1},$$

find the values of the constants A, B and C.

(b) Given that $x = 2$ at $t = 1$, solve the differential equation

$$\frac{dx}{dt} = 2 - \frac{2}{x^2}, x > 1.$$

You need not simplify your final answer. (E)

Solution:

(a) $\frac{x^2}{x^2-1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1}$

$\Rightarrow \quad x^2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)$

$x = 1 \quad \Rightarrow \quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

$x = -1 \quad \Rightarrow \quad 1 = -2C \quad \Rightarrow \quad C = -\frac{1}{2}.$

Coefficients of x^2: $\quad 1 = A \quad \Rightarrow \quad A = 1$

(b) $\frac{dx}{dt} = 2\frac{(x^2-1)}{x^2}$

$\Rightarrow \quad \int \frac{x^2}{x^2-1} dx = \int 2\, dt$

$\Rightarrow \quad \int \left(1 + \frac{\left(\frac{1}{2}\right)}{x-1} - \frac{\left(\frac{1}{2}\right)}{x+1} \right) dx = 2t$

$\Rightarrow \quad x + \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| = 2t + C$

$x = 2, t = 1 \quad \Rightarrow \quad 2 + \frac{1}{2}\ln\frac{1}{3} = 2 + \quad C \quad \Rightarrow \quad C = \frac{1}{2}\ln\frac{1}{3}$

$\therefore\ x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + \frac{1}{2}\ln\frac{1}{3}$

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Integration
Exercise L, Question 23

Question:

The curve C is given by the equations

$x = 2t, y = t^2,$

where t is a parameter.

(a) Find an equation of the normal to C at the point P on C where $t = 3$.
The normal meets the y-axis at the point B. The finite region R is bounded by the part of the curve C between the origin O and P, and the lines OB and OP.

(b) Show the region R, together with its boundaries, in a sketch.
The region R is rotated through 2π about the y-axis to form a solid S.

(c) Using integration, and explaining each step in your method, find the volume of S, giving your answer in terms of π.

E

Solution:

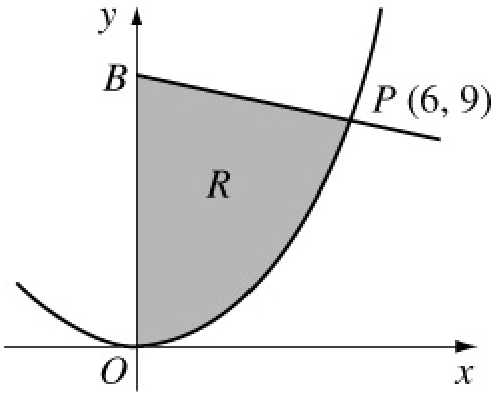
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2t}{2} = t.$

$\therefore$ at P (6, 9) gradient of normal is $-\frac{1}{3}$

$\therefore$ equation of normal is $y - 9 = -\frac{1}{3}\left(x - 6\right)$ or $y = -\frac{1}{3}x + 11$

(b) $x = 2t, y = t^2 \Rightarrow y = \frac{x^2}{4}$

B is (0, 11)



(c) volume $= \text{cone} + \pi \int_0^9 x^2 \, dy$

cone $= \frac{1}{3}\pi \times 6^2 \times 2 = 24\pi$

$\pi \int_0^9 x^2 \, dy = \pi \int_{t=0}^{t=3} 4t^2 \times 2t \, dt = \pi \int_0^3 8t^3 \, dt$

$= \pi \left[2t^4 \right]_0^3 = \pi \times 2 \times 81 = 162\pi$

$\therefore$ Volume of $S = 186\pi$

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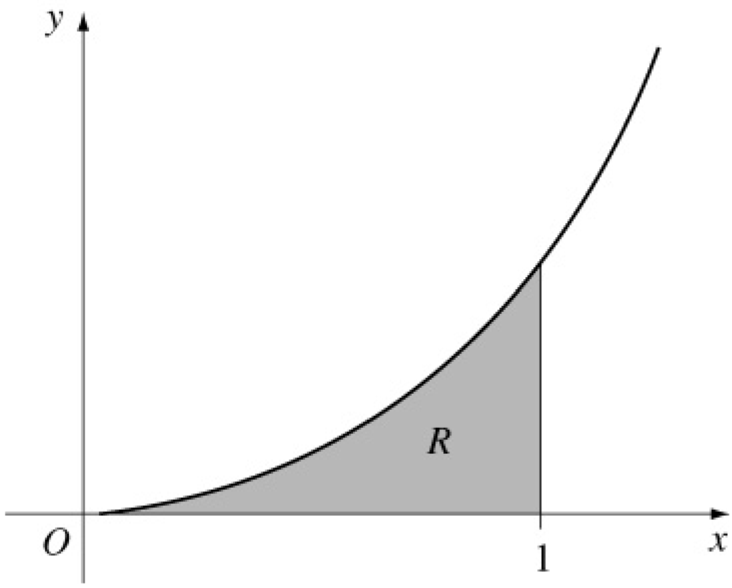
Integration
Exercise L, Question 24

Question:

Shown is part of the curve with equation $y = e^{2x} - e^{-x}$. The shaded region R is bounded by the curve, the x-axis and the line with equation $x = 1$.
Use calculus to find the area of R, giving your answer in terms of e. **E**



Solution:

$$\text{Area} = \int_0^1 \left(e^{2x} - e^{-x} \right) dx$$
$$= \left[\frac{1}{2}e^{2x} + e^{-x} \right]_0^1$$
$$= \left(\frac{1}{2}e^2 + e^{-1} \right) - \left(\frac{1}{2} + 1 \right)$$
$$= \frac{1}{2}\left(e^2 + \frac{2}{e} - 3 \right)$$

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Integration
Exercise L, Question 25

Question:

(a) Given that $2y = x - \sin x \cos x$, show that $\frac{dy}{dx} = \sin^2 x$.

(b) Hence find $\int \sin^2 x \, dx$.

(c) Hence, using integration by parts, find $\int x \sin^2 x \, dx$. (E)

Solution:

(a) $2y = x - \sin x \cos x$

$$\Rightarrow \quad 2\frac{dy}{dx} = 1 - \left[\cos^2 x + \sin x \left(-\sin x \right) \right] = 1 - \cos^2 x + \sin^2 x$$

$$\therefore \frac{dy}{dx} = \sin^2 x \qquad (\text{using } \sin^2 x = 1 - \cos^2 x)$$

(b) $\int \sin^2 x \, dx = y + C_1$

$$= \frac{x}{2} - \frac{1}{2}\sin x \cos x + C_1$$

(c) $\int x \sin^2 x \, dx$

$$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin^2 x \Rightarrow v = \left(\text{b} \right)$$

$$\therefore \int x \sin^2 x \, dx = \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \int \left(\frac{x}{2} - \frac{1}{2}\sin x \cos x \right) dx$$

$$= \frac{x^2}{2} - \frac{1}{2} x \sin x \cos x - \frac{x^2}{4} + \frac{1}{4}\int \sin 2x \, dx$$

$$= \frac{x^2}{4} - \frac{1}{2} x \sin x \cos x - \frac{1}{8}\cos 2x + C_2$$

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Integration
Exercise L, Question 26

Question:

The rate, in $\text{cm}^3 \text{s}^{-1}$, at which oil is leaking from an engine sump at any time t seconds is proportional to the volume of oil, $V \text{cm}^3$, in the sump at that instant. At time $t = 0$, $V = A$.

(a) By forming and integrating a differential equation, show that

$V = Ae^{-kt}$

where k is a positive constant.

(b) Sketch a graph to show the relation between V and t.

Given further that $V = \frac{1}{2}A$ at $t = T$,

(c) show that $kT = \ln 2$. E

Solution:

(a) $\frac{dv}{dt} = -k V$

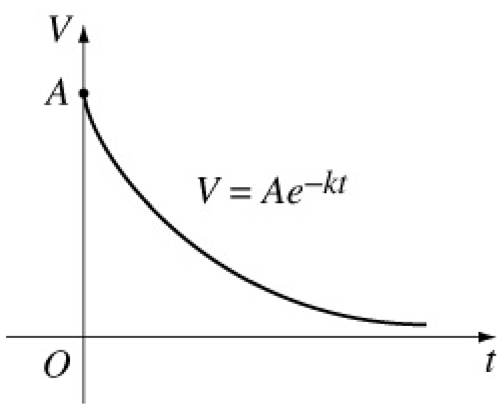
$\Rightarrow \int \frac{1}{V} dV = \int -k dt$

$\Rightarrow \ln|V| = -kt + C$

$\Rightarrow V = A_1 e^{-kt}$

$t = 0, V = A \Rightarrow V = Ae^{-kt} \quad (A_1 = A)$

(b)



(c) $t = T, V = \frac{1}{2}A \quad \Rightarrow \quad \frac{1}{2}A = Ae^{-kT}$

$\Rightarrow \quad -\ln 2 = -kT$

$\Rightarrow \quad kT = \ln 2$

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Integration
Exercise L, Question 27

Question:

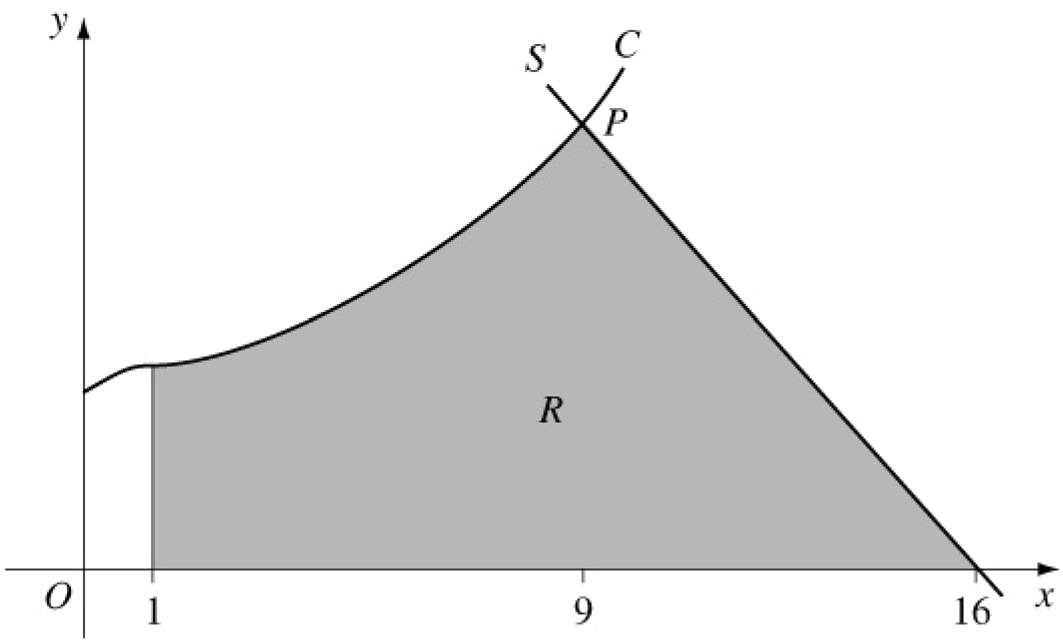
This graph shows part of the curve C with parametric equations

$$x = (t+1)^2, y = \frac{1}{2}t^3 + 3, t \geq -1.$$

P is the point on the curve where $t = 2$. The line S is the normal to C at P.

(a) Find an equation of S.
The shaded region R is bounded by C, S, the x-axis and the line with equation $x = 1$.

(b) Using integration and showing all your working, find the area of R. E



Solution:

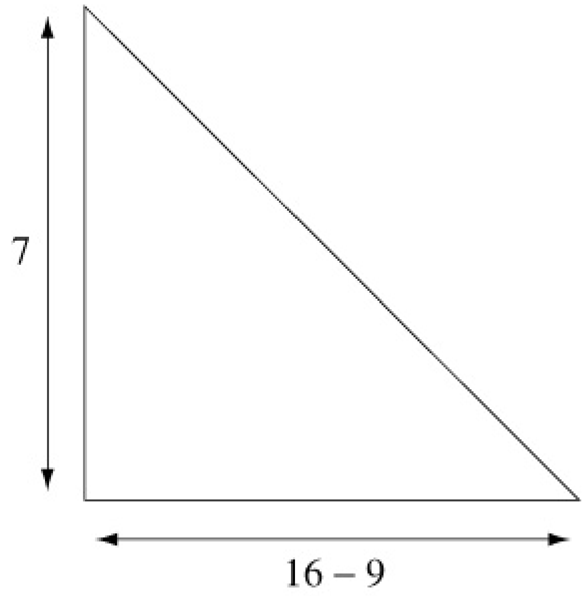
(a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{3}{2}t^2}{2(t+1)} = \frac{3t^2}{4(t+1)}$

At P (9, 7) gradient of normal is $-\frac{4 \times 3}{3 \times 2^2} = -1$

$\therefore$ equation of line S is $y - 7 = -1(x - 9)$

i.e. $y = -x + 16$ or $y + x = 16$

(b) Area $= \int_{x=1}^{x=9} y\,dx +$ area of triangle shown below



$$\int_{x=1}^{x=9} y\,dx = \int_{t=0}^{t=2} \left(\frac{1}{2}t^3 + 3 \right) 2 \left(t + 1 \right) dt$$

$$= \int_0^2 (t^4 + t^3 + 6t + 6)\,dt$$

$$= \left[\frac{1}{5}t^5 + \frac{1}{4}t^4 + \frac{6t^2}{2} + 6t \right]_0^2$$

$$= \left(\frac{32}{5} + \frac{16}{4} + 3 \times 4 + 6 \times 2 \right) - \left(0 \right)$$

$$= 34.4$$

$$\therefore \text{ Area } = 34.4 + \frac{1}{2} \times 7^2 = 58.9$$

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Integration
Exercise L, Question 28

Question:

Shown is part of the curve C with parametric equations

$x = t^2, y = \sin 2t, t \geq 0.$

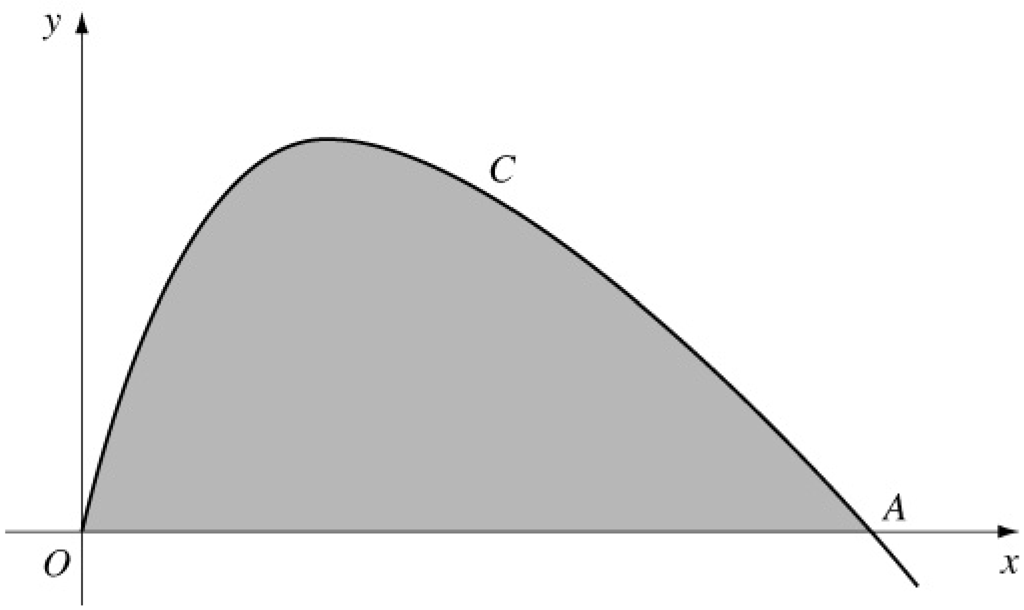
The point A is an intersection of C with the x-axis.

(a) Find, in terms of π, the x-coordinate of A.

(b) Find $\frac{dy}{dx}$ in terms of t, $t > 0$.

(c) Show that an equation of the tangent to C at A is $4x + 2\pi y = \pi^2$.
The shaded region is bounded by C and the x-axis.

(d) Use calculus to find, in terms of π, the area of the shaded region. E



Solution:

(a) At A, $y = 0 \Rightarrow \sin 2t = 0 \Rightarrow 2t = 0$ or $\pi \Rightarrow t = \frac{\pi}{2}$

$\therefore A$ is $\left(\left(\frac{\pi}{2} \right)^2, 0 \right)$ or $\left(\frac{\pi^2}{4}, 0 \right)$

(b) $\frac{dy}{dx} = \frac{2\cos 2t}{2t} = \frac{\cos 2t}{t}$

(c) Gradient of tangent at A is $\frac{\cos\pi}{\left(\frac{\pi}{2}\right)} = -\frac{1}{\left(\frac{\pi}{2}\right)} = -\frac{2}{\pi}$

$\therefore$ equation of tangent is $y - 0 = -\frac{2}{\pi}\left(x - \frac{\pi^2}{4}\right)$

$\Rightarrow \quad \pi y = -2x + \frac{2\pi^2}{4}$

or $2\pi y + 4x = \pi^2$

(d) Area $= \int y\,dx = \int_{t=0}^{t=\frac{\pi}{2}} \sin 2t \times 2t\,dt$

$u = t \quad \Rightarrow \quad \frac{du}{dt} = 1$

$\frac{dv}{dt} = 2\sin 2t \Rightarrow v = -\cos 2t$

$\therefore$ Area $= \left[-t\cos 2t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\left(-\cos 2t\right)dt$

$= \left(+\frac{\pi}{2}\right) - \left(0\right) + \left[\frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

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Integration
Exercise L, Question 29

Question:

Showing your method clearly in each case, find

(a) $\int \sin^2 x \cos x \, dx$,

(b) $\int x \ln x \, dx$.

Using the substitution $t^2 = x + 1$, where $x > -1$, $t > 0$,

(c) Find $\int \frac{x}{\sqrt{x+1}} dx$.

(d) Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$. (E)

Solution:

(a) Let $y = \sin^3 x \Rightarrow \frac{dy}{dx} = 3 \sin^2 x \cos x$

$\therefore \int \sin^2 x \cos x \, dx = \frac{1}{3} \sin^3 x + C$

(b) $\int x \ln x \, dx$

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \Rightarrow v = \frac{1}{2} x^2$

$\therefore \int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \times \frac{1}{x} dx$

$= \frac{1}{2} x^2 \ln x - \frac{x^2}{4} + C$

(c) $t^2 = x + 1 \Rightarrow 2t \, dt = dx$

$\therefore I = \int \frac{x}{\sqrt{x+1}} dx$

$= \int \frac{t^2 - 1}{t} \times 2t \, dt$

$$= \int \left(2t^2 - 2 \right) dt$$

$$= \frac{2}{3}t^3 - 2t + C$$

$$= \frac{2}{3}\left(x + 1 \right)^{\frac{3}{2}} - 2\sqrt{x + 1} + C$$

$$= \frac{2}{3}\sqrt{x + 1}\left(x - 2 \right) + C$$

(d) $\int_0^3 \frac{x}{\sqrt{x + 1}} dx = \left[\frac{2}{3}\left(x - 2 \right)\sqrt{x + 1} \right]_0^3$

$$= \left(\frac{2}{3} \times 2 \right) - \left(- \frac{4}{3} \right) = \frac{8}{3}$$

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Integration
Exercise L, Question 30

Question:

(a) Using the substitution $u = 1 + 2x^2$, find $\int x \left(1 + 2x^2 \right)^5 dx$.

(b) Given that $y = \frac{\pi}{8}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = x \left(1 + 2x^2 \right)^5 \cos^2 2y.$$

Solution:

(a) $u = 1 + 2x^2 \Rightarrow du = 4x\,dx \Rightarrow x\,dx = \frac{du}{4}$

So $\int x \left(1 + 2x^2 \right)^5 dx = \int \frac{u^5}{4} du = \frac{u^6}{24} + C_1 = \frac{\left(1 + 2x^2 \right)^6}{24} + C_1$

(b) $\frac{dy}{dx} = x \left(1 + 2x^2 \right)^5 \cos^2 2y$

$\Rightarrow \int \sec^2 2y\,dy = \int x \left(1 + 2x^2 \right)^5 dx$

$\Rightarrow \frac{1}{2} \tan 2y = \frac{\left(1 + 2x^2 \right)^6}{24} + C_2$

$y = \frac{\pi}{8}, x = 0 \Rightarrow \frac{1}{2} = \frac{1}{24} + C_2 \Rightarrow C_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{\left(1 + 2x^2 \right)^6}{12} + \frac{11}{12}$

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Integration
Exercise L, Question 31

Question:

Find $\int x^2 \ln 2x \, dx$.

Solution:

$I = \int x^2 \ln 2x \, dx$

$u = \ln 2x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \quad \Rightarrow \quad v = \frac{x^3}{3}$

$\therefore I = \frac{x^3}{3} \ln 2x - \int \frac{x^3}{3} \times \frac{1}{x} dx$

$= \frac{x^3}{3} \ln 2x - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln 2x - \frac{x^3}{9} + C$

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Integration
Exercise L, Question 32

Question:

Obtain the solution of

$$x\left(x+2\right)\frac{dy}{dx}=y\,,\ y>0,\ x>0,$$

for which $y = 2$ at $x = 2$, giving your answer in the form $y^2 = f\left(x\right)$.

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Solution:

$$x\left(x+2\right)\frac{dy}{dx}=y$$

$$\Rightarrow \quad \int\frac{1}{y}dy=\int\frac{1}{x\left(x+2\right)}dx$$

$$\frac{1}{x\left(x+2\right)}\equiv\frac{A}{x}+\frac{B}{x+2}$$

$$\Rightarrow \quad 1\equiv A\left(x+2\right)+Bx$$

$$x=0 \quad \Rightarrow \quad 1=2A \quad \Rightarrow \quad A=\frac{1}{2}$$

$$x=-2 \quad \Rightarrow \quad 1=-2B \quad \Rightarrow \quad B=-\frac{1}{2}$$

$$\text{So } \ln y=\int\left(\frac{\left(\frac{1}{2}\right)}{x}-\frac{\left(\frac{1}{2}\right)}{x+2}\right)dx$$

$$=\frac{1}{2}\ln\left|x\right|-\frac{1}{2}\ln\left|x+2\right|+C$$

$$\therefore y=\sqrt{\frac{kx}{x+2}} \qquad \left(C=\frac{1}{2}\ln k\right)$$

$$x=2,\ y=2 \quad \Rightarrow \quad 2=\sqrt{\frac{2k}{4}} \quad \Rightarrow \quad 4\times2=k$$

$$\therefore y=\sqrt{\frac{8x}{x+2}} \quad \text{or} \quad y^2=\frac{8x}{x+2}$$

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Integration
Exercise L, Question 33

Question:

(a) Use integration by parts to show that

$$\int_0^{\frac{\pi}{4}} x\sec^2 x\,dx = \frac{1}{4}\pi - \frac{1}{2}\ln 2.$$

The finite region R, bounded by the curve with equation $y = x^{\frac{1}{2}}\sec x$, the line $x = \frac{\pi}{4}$ and the x-axis is shown. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid of revolution generated.

(c) Find the gradient of the curve with equation $y = x^{\frac{1}{2}}\sec x$ at the point where $x = \frac{\pi}{4}$.

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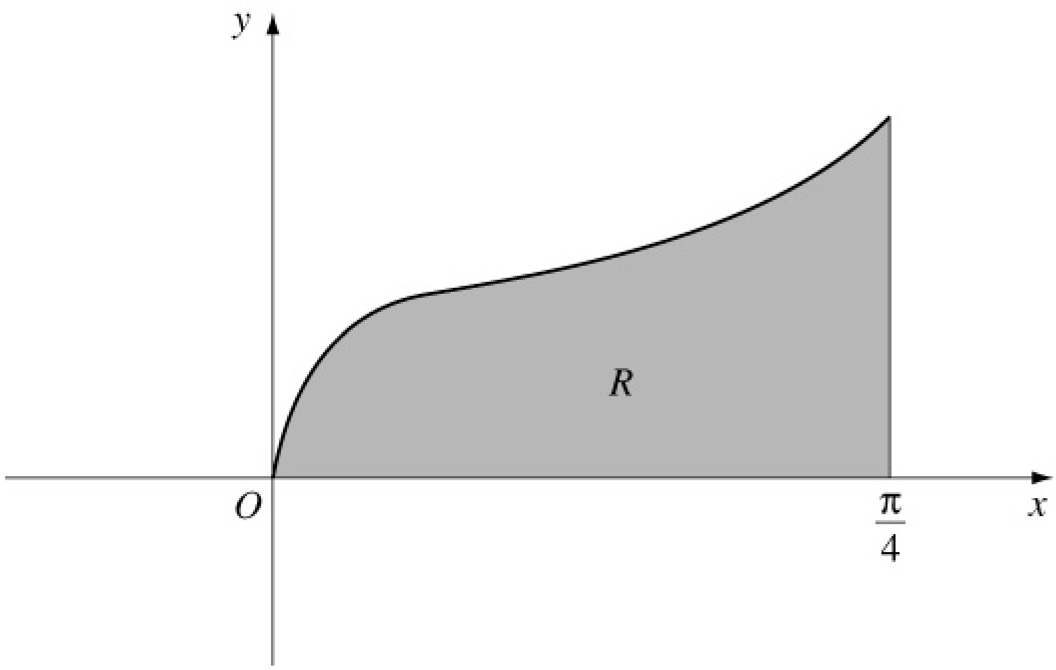

Solution:

(a) $I = \int_0^{\frac{\pi}{4}} x\sec^2 x\,dx$

$u = x \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \quad \Rightarrow \quad v = \tan x$

$\therefore I = \left[x \tan x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \left(\frac{\pi}{4} \right) - \left(0 \right) - \left[\ln | \sec x | \right]_0^{\frac{\pi}{4}}$

$= \frac{\pi}{4} - \left[\left(\ln \sqrt{2} \right) - \left(\ln 1 \right) \right]$

$= \frac{\pi}{4} - \frac{1}{2} \ln 2$

(b) $V = \pi \int_0^{\frac{\pi}{4}} y^2 \, dx = \pi \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$

Using (a) $\quad V = \frac{\pi^2}{4} - \frac{\pi}{2} \ln 2 = 1.38$ (3 s.f.)

(c) $\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \sec x + x^{\frac{1}{2}} \sec x \tan x$

At $x = \frac{\pi}{4}$ $\frac{dy}{dx} = \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \times \sqrt{2} + \frac{\sqrt{\pi}}{2} \times \sqrt{2} \times 1 = \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi}{2}} = 2.05$ (3 s.f.)

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Integration
Exercise L, Question 34

Question:

Part of the design of a stained glass window is shown. The two loops enclose an area of blue glass. The remaining area within the rectangle *ABCD* is red glass. The loops are described by the curve with parametric equations

$x = 3\cos t, y = 9\sin 2t, 0 \leq t < 2\pi.$

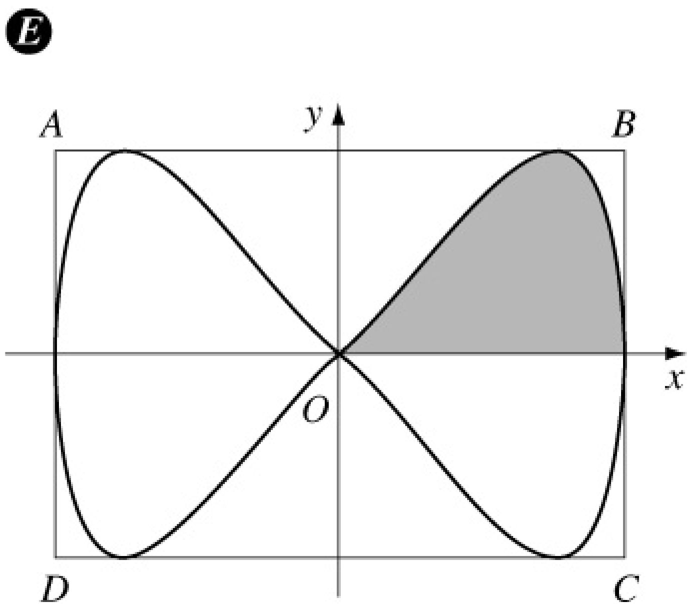
(a) Find the cartesian equation of the curve in the form $y^2 = f(x)$.

(b) Show that the shaded area enclosed by the curve and the *x*-axis, is given by

$\int_0^{\frac{\pi}{2}} A\sin 2t \sin t \, dt$, stating the value of the constant *A*.

(c) Find the value of this integral.
The sides of the rectangle *ABCD* are the tangents to the curve that are parallel to the coordinate axes. Given that 1 unit on each axis represents 1 cm,

(d) find the total area of the red glass.



Solution:

(a) $x = 3\cos t$

$y = 9\sin 2t \quad \Rightarrow \quad y = 18\cos t \sin t$

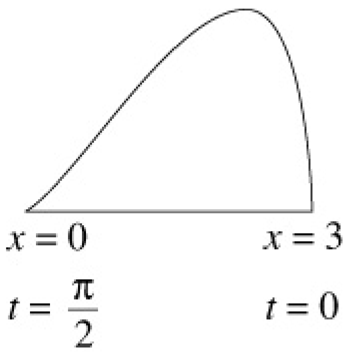
$\Rightarrow \quad y = 6x\sin t$

$\therefore \cos t = \frac{x}{3}, \sin t = \frac{y}{6x}$

$\cos^2 t + \sin^2 t = 1 \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{36x^2} = 1$

i.e. $4x^4 + y^2 = 36x^2$
or $y^2 = 4x^2 (9 - x^2)$

(b)



Area $= \int y \, dx$

$= \int_{t = \frac{\pi}{2}}^{t = 0} 9 \sin 2t \times \left(- 3 \sin t \right) dt$

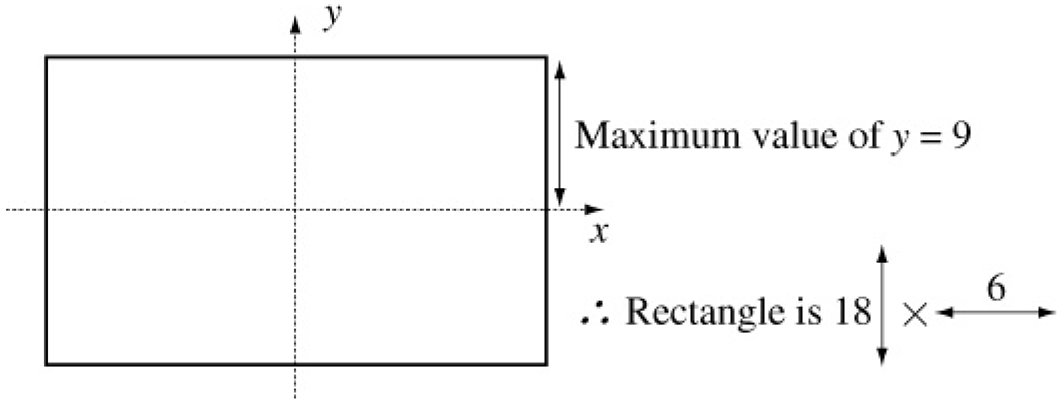
$= 27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt$

(c) $27 \int_0^{\frac{\pi}{2}} \sin 2t \sin t \, dt = 54 \int_0^{\frac{\pi}{2}} \sin^2 t \cos t \, dt$

$= \left[\frac{54 \sin^3 t}{3} \right]_0^{\frac{\pi}{2}}$

$= (18 \times 1) - (0)$
$= 18$

(d) Area of blue glass is $18 \times 4 = 72$



Area of rectangle $= 108$

$\therefore$ Area of red glass $= 108 - 72 = 36\,\text{cm}^2$