Solutionbank C3
Edexcel AS and A Level Modular Mathematics

Review Exercise
Exercise A, Question 1

Question:

Simplify

i $\dfrac{2x^2-7x-15}{x^2-25}$

ii $\dfrac{x^3+1}{x+1}$.

Solution:

a

$$\frac{2x^2-7x-15}{x^2-25} = \frac{(2x+3)\cancel{(x-5)}}{(x+5)\cancel{(x-5)}}$$

$$= \frac{2x+3}{x+5}$$

Take care with signs.

Difference of two squares.

b

$$x^3+1 = (x+1)(x^2-x+1)$$

$$\Rightarrow \frac{x^3+1}{x+1} = x^2-x+1$$

Use the factor theorem to find the factor $x+1$.

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Review Exercise
Exercise A, Question 2

Question:

Express $\frac{4x}{x^2-2x-3}+\frac{1}{x^2+x}$ as a single fraction, giving your answer in its simplest form.

Solution:

$$x^2-2x-3 = (x-3)(x+1)$$
$$x^2+x = x(x+1)$$

Factorise both denominators.

$$\Rightarrow \frac{4x}{x^3-2x-3} + \frac{1}{x^2+x}$$
$$= \frac{4x}{(x-3)(x+1)}+\frac{1}{x(x+1)}$$

The L.C.M. of the denominators is $x(x+1)(x-3)$.

$$= \frac{4x^2+(x-3)}{x(x+1)(x-3)}$$
$$= \frac{(4x-3)\,\cancel{(x+1)}}{x\,\cancel{(x+1)}\,(x-3)}$$

Factorise quadratic numerator.

$$= \frac{4x-3}{x(x-3)}$$

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Review Exercise

Exercise A, Question 3

Question:

Express $\frac{2x^2+3x}{(2x+3)(x-2)}-\frac{6}{x^2-x-2}$ as a single fraction in its simplest form. *E*

Solution:

$$\frac{2x^2+3x}{(2x+3)(x-2)}-\frac{6}{x^2-x-2}$$

$$=\frac{x\cancel{(2x+3)}}{\cancel{(2x+3)}(x-2)}-\frac{6}{(x-2)(x+1)}$$

Factorise all the quadratic expressions.

$$=\frac{x}{x-2}-\frac{6}{(x-2)(x+1)}$$

$$=\frac{x(x+1)-6}{(x-2)(x+1)}$$

$$=\frac{x^2+x-6}{(x-2)(x+1)}$$

Check the quadratic numerator to see if it factorises.

$$=\frac{(x+3)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)}$$

$$=\frac{x+3}{x+1}$$

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Review Exercise
Exercise A, Question 4

Question:

a Given that
$16x^3 - 36x^2 - 12x + 5 \equiv (2x+1)(8x^2 + ax + b)$,
find the value of a and the value of b.

b Hence, or otherwise, simplify

$$\frac{16x^3 - 36x^2 - 12x + 5}{4x - 1}$$

Solution:

a

$$16x^3 - 36x^2 - 12x + 5 \equiv (2x+1)(8x^2 + ax + b)$$
$$b = 5$$
$$\text{and } -36 = 8 + 2a$$
$$\Rightarrow 2a = -44$$
$$a = -22$$

Compare the constant term on both sides.

Compare the coefficient of x^2 on both sides.

So

$$16x^3 - 36x^2 - 12x + 5 \equiv (2x+1)(8x^2 - 22x + 5)$$
$$= (2x+1)(2x-5)(4x-1)$$

You can check your values by comparing the coefficient of x on both sides.

b Using the result in **a**

$$\frac{16x^3 - 36x^2 - 12x + 5}{4x - 1} = (2x+1)(2x-5)$$

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Review Exercise
Exercise A, Question 5

Question:

$$f(x) = 1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}, x \neq -2$$

a Show that $f(x) = \frac{x^2 + x + 1}{(x+2)^2}, x \neq -2$

b Show that $x^2 + x + 1 > 0$ for all values of x.

c Show that $f(x) > 0$ for all values of x, $x \neq 2$. *E*

Solution:

a

$$1 - \frac{3}{x+2} + \frac{3}{(x+2)^2}$$

$$= \frac{(x+2)^2 - 3(x+2) + 3}{(x+2)^2}$$

$$= \frac{x^2 + 4x + 4 - 3x - 6 + 3}{(x+2)^2}$$

$$= \frac{x^2 + x + 1}{(x+2)^2}$$

b

$$x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$$

$$\geq \frac{3}{4}$$

$$> 0 \text{ for all values of } x$$

Use the method of completing the square.

As $(x + \frac{1}{2})^2 \geq 0$.

c $\frac{x^2 + x + 1}{(x+2)^2} > 0$

as $x^2 + x + 1 > 0$ from **b**

and $(x+2)^2 > 0$ as $x \neq -2$

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Review Exercise
Exercise A, Question 6

Question:

a Show that

$$\frac{4}{(x+1)^2}-\frac{1}{(x+1)}-\frac{1}{2}=\frac{5-4x-x^2}{2(x+1)^2}.$$

b Hence solve

$$\frac{4}{(x+1)^2}<\frac{1}{(x+1)}+\frac{1}{2}, x\neq -1$$

Solution:

a

$$\frac{4}{(x+1)^2}-\frac{1}{(x+1)}-\frac{1}{2}$$

$$= \frac{4(2)-2(x+1)-(x+1)^2}{2(x+1)^2}$$

$$= \frac{8-2x-2-x^2-2x-1}{2(x+1)^2}$$

Be careful with signs.

$$= \frac{5-4x-x^2}{2(x+1)^2}$$

b

$$= \frac{4}{(x+1)^2}<\frac{1}{(x+1)}+\frac{1}{2}$$

$$\Rightarrow \frac{4}{(x+1)^2}-\frac{1}{x+1}-\frac{1}{2}<0$$

You can see the relationship with **a**.

$$\Rightarrow \frac{5-4x-x^2}{2(x+1)^2}<0$$

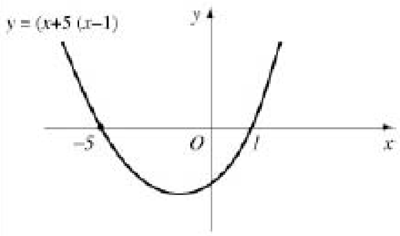
$$\Rightarrow 5-4x-x^2<0$$

If $x \neq -1, (x+1)^2>0$.

$$\Rightarrow x^2+4x-5>0$$

$$\Rightarrow (x+5)(x-1)>0$$

$$\Rightarrow x<-5, x>1$$



The curve is above the x-axis where $x<-5, x>1$.

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Review Exercise
Exercise A, Question 7

Question:

$$f(x)=\frac{x}{x+3}-\frac{x+24}{2x^2+5x-3},\left\{x\in\mathbb{R},x>\frac{1}{2}\right\}.$$

a Show that $f(x)=\frac{2(x-4)}{2x-1}\left\{x\in\mathbb{R},x>\frac{1}{2}\right\}$.

b Find $f^{-1}(x)$.

Solution:

a

$$\begin{aligned} f(x) &= \frac{x}{x+3}-\frac{x+24}{(2x-1)(x+3)} \\ &= \frac{x(2x-1)-(x+24)}{(2x-1)(x+3)} \\ &= \frac{2x^2-2x-24}{(2x-1)(x+3)} \\ &= \frac{2(x^2-x-12)}{(2x-1)(x+3)} \\ &= \frac{2(x-4)\cancel{(x+3)}}{(2x-1)\cancel{(x+3)}} \\ &= \frac{2(x-4)}{2x-1} \end{aligned}$$

Be careful: always insert brackets. It is very common to see $-x+24$.

b

$$\begin{aligned} \text{Let } y &= \frac{2x-8}{2x-1} \\ \Rightarrow 2xy-y &= 2x-8 \\ 2xy-2x &= y-8 \\ 2x(y-1) &= y-8 \\ x &= \frac{y-8}{2(y-1)} \\ f^{-1}(x) &= \frac{x-8}{2(x-1)} \end{aligned}$$

$f^{-1}(x)$ is a function of x; you must express it in terms of x.

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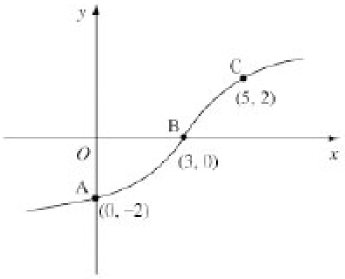
Review Exercise
Exercise A, Question 8

Question:

The graph of the increasing function f passes through the points $A(0, -2)$, B(3, 0) and C(5, 2), as shown.

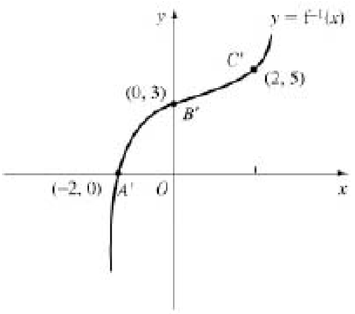


a Sketch the graph of f^{-1}, showing the images of A, B and C.
The function g is defined by
$g : x \to \sqrt{x^2+2}, x \in \mathbb{R}$

b Find **i** $fg(\sqrt{23})$, **ii** $gf(0)$.

Solution:

a



$y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.
Points with coordinates

b **i**

$$fg(\sqrt{23}) = f(\sqrt{23+2}) = f(5) = 2$$

As $f(5) = 2$, indicated by the point C on graph of f.

ii

$$gf(0) = g(-2) = \sqrt{(-2)^2+2} = \sqrt{6}$$

The point $(0, -2)$ on the graph of $y = f(x)$ implies that $f(0) = -2$.

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Review Exercise
Exercise A, Question 9

Question:

The functions f and g are defined by

$f : x \rightarrow 3x + 4, \quad x \in \mathbb{R}, x > 0,$

$g : x \rightarrow \dfrac{x}{x-2}, \quad x \in \mathbb{R}, x > 2.$

a Find the inverse function $f^{-1}(x)$, stating its domain.

b Find the exact value of $gf\left(\frac{1}{2}\right)$.

c State the range of g.

d Find $g^{-1}(x)$, stating its domain.

Solution:

a Let

$$y = 3x + 4$$
$$\Rightarrow 3x = y - 4$$
$$x = \frac{y-4}{3}$$
$$f^{-1}(x) = \frac{x-4}{3}$$

Remember to write in terms of x.

domain of $f^{-1}(x)$ is $x > 4$

The domain of $f^{-1}(x)$ is the range of $f(x)$.
As $x > 0, 3x + 4 > 4$.

b

$$gf\left(\frac{1}{2}\right) = g(5\tfrac{1}{2})$$

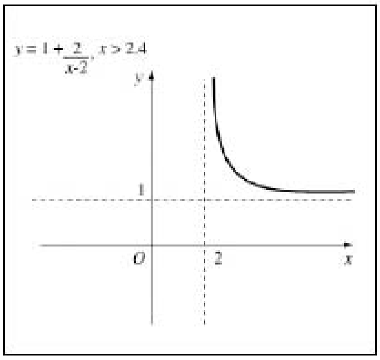
$f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) + 4$

$$= \frac{5\frac{1}{2}}{5\frac{1}{2} - 2}$$
$$= \frac{5\frac{1}{2}}{3\frac{1}{2}}$$
$$= \frac{11}{2} \times \frac{2}{7} = \frac{11}{7}$$

You must not give a decimal answer.

c

$$g(x) = \frac{x}{x-2} = \frac{x-2+2}{x-2} = 1 + \frac{2}{x-2}$$



range $g(x) > 1$

d $y = \frac{x}{x-2}$

$$\Rightarrow yx - 2y = x$$
$$\Rightarrow x(y-1) = 2y$$
$$\Rightarrow x = \frac{2y}{y-1}$$

$g^{-1}(x) = \frac{2x}{x-1}$, domain $x > 1$

The domain of $g^{-1}(x)$ is the range of $g(x)$ found in **c**.

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Review Exercise
Exercise A, Question 10

Question:

The function f is defined by

$$f : x \to \frac{5x+1}{x^2+x-2} - \frac{3}{x+2}, x > 1.$$

a Show that $f(x) = \frac{2}{x-1}, x > 1$.

b Find $f^{-1}(x)$.

The function g is defined by

$$g : x \to x^2 + 5, \quad x \in \mathbb{R}$$

c Solve $fg(x) = \frac{1}{4}$. *E*

Solution:

a

$$\frac{5x+1}{(x+2)(x-1)}-\frac{3}{x+2}$$

← Factorise the quadratic expression.

$$=\frac{5x+1-3(x-1)}{(x+2)(x-1)}$$

$$=\frac{2x+4}{(x+2)(x-1)}$$

$$=\frac{2\cancel{(x+2)}}{\cancel{(x+2)}(x-1)}$$

← $x \neq -2$ as $x > 1$

so $\mathrm{f}(x)=\frac{2}{x-1} \quad x>1$

b Let $y=\frac{2}{x-1}$

$$\Rightarrow yx-y = 2$$

$$yx = 2+y$$

$$x = \frac{2+y}{y}$$

$$\mathrm{f}^{-1}(x) = \frac{2+x}{x}\left[\text{or } 1+\frac{2}{x}\right]$$

← The domain is $x>0$.

c

$$\mathrm{fg}(x) = \mathrm{f}(x^2+5)$$

$$= \frac{2}{x^2+4}$$

$$\frac{2}{x^2+4} = \frac{1}{4} \Rightarrow x^2=4$$

$$\Rightarrow \quad x=\pm 2$$

Both answers are valid, as g is defined for $x \in \mathbb{R}$.

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Review Exercise
Exercise A, Question 11

Question:

The functions f and g are defined by

$$f : x \rightarrow (x-4)^2 - 16, \quad x \in \mathbb{R}, x > 0,$$

$$g : x \rightarrow \frac{8}{1-x}, \quad x \in \mathbb{R}, x < 1.$$

a Find the range of f.

b Explain why, with the given domain for f, $f^{-1}(x)$ does not exist.

c Show that $fg(x) = \frac{64x}{(1-x)^2}$.

d Find $g^{-1}(x)$, stating its domain.

Solution:

a $f(x) \geq -16$

Either draw a graph of $y = f(x)$ or realise that $(x-4)^2 \geq 0$ so $(x-4)^2 - 16 \geq -16$.

b For f(x) to have an inverse function it must be one-to-one. With the given domain f, is many-to-one e.g. $f(2) = -12$ and $f(6) = -12$.

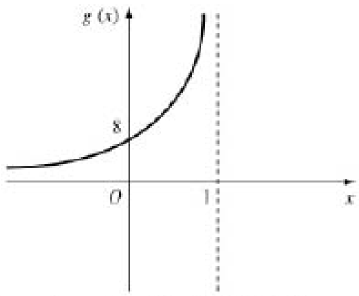
c

$$\begin{aligned} fg(x) &= f\left(\frac{8}{1-x}\right) \\ &= \left(\frac{8}{1-x} - 4\right)^2 - 16 \\ &= \left(\frac{64}{(1-x)^2} - \frac{64}{1-x} + 6\right) - 16 \\ &= \frac{64}{(1-x)^2} - \frac{64}{1-x} \\ &= \frac{64[1-(1-x)]}{(1-x)^2} \\ &= \frac{64x}{(1-x)^2} \end{aligned}$$

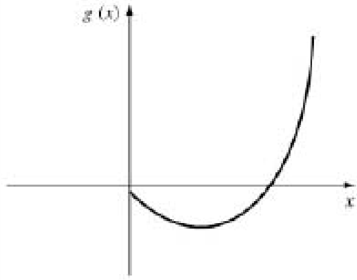
d Let $y = \frac{8}{1-x}$

$$\begin{aligned} \Rightarrow y - yx &= 8 \\ \Rightarrow xy &= y - 8 \\ x &= \frac{y-8}{y} \\ \Rightarrow g^{-1}(x) &= \frac{x-8}{x} \end{aligned}$$

domain is $x > 0$



The domain of $g^{-1}(x)$ is the range of $g(x)$ (see graph).



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Review Exercise
Exercise A, Question 12

Question:

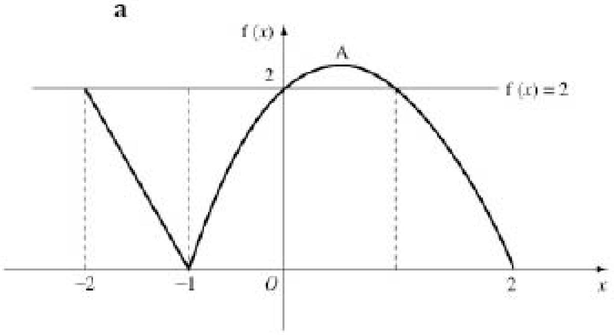
The function f(x) is defined by

$$f(x)=\begin{cases}-2(x+1) & -2\le x\le -1\\ (x+1)(2-x) & -1<x\le 2\end{cases}$$

a Sketch the graph of f(x).
b Write down the range of f.
c Find the values of x for which $f(x)<2$.

Solution:

a



$-2(x+1)$ in $[-2,-1]$ is a straight line between $(-2,2)$ and $(-1,0)$. $(x+1)(2-x)$ is a $\cap$-shaped parabola in $[-1,2]$.

b The vertex A of the parabola has x-coordinate $\frac{1}{2}$ (symmetry) so $A=(\frac{1}{2},2\frac{1}{4})$.

The greatest value of f(x) is the f(x) coordinate of thevertex of the parabola.

The range of f(x) is $0\le f(x)\le 2\frac{1}{4}$.

c The parabola crosses the line $f(x)=2$ where $x=0$ and $x=1$ (symmetry).
The line meets $f(x)=2$ at $(-2,2)$, so $f(x)<2$ where $-2<x<0,\ 1<x\le 2$

Use the graph.

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Review Exercise
Exercise A, Question 13

Question:

a Express $4x^2 - 4x - 3$ in the form $(ax - b)^2 - c$, where a, b and c are positive constants to be found.

The function f is defined by

$$f : x \rightarrow 4x^2 - 4x - 3, \quad \left\{ x \in \mathbb{R},\ x \geq \frac{1}{2} \right\}.$$

b Sketch the graph of f.

c Sketch the graph of f^{-1}.

d Find $f^{-1}(x)$, stating its domain.

Solution:

a

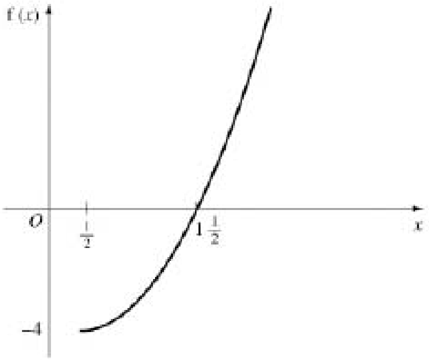
$$4x^2-4x-3 \equiv (ax-b)^2-c$$
$$\equiv a^2x^2-2abx+b^2-c$$
$$\Rightarrow a = 2$$
$$\Rightarrow 2ab = 4 \Rightarrow b=1$$
$$\Rightarrow b^2-c = -3$$
$$\Rightarrow 1-c = -3$$
$$\Rightarrow c = 4$$
$$\Rightarrow 4x^2-4x-3 \equiv (2x-1)^2-4$$

Compare coefficients of x^2.

Compare coefficients of x.

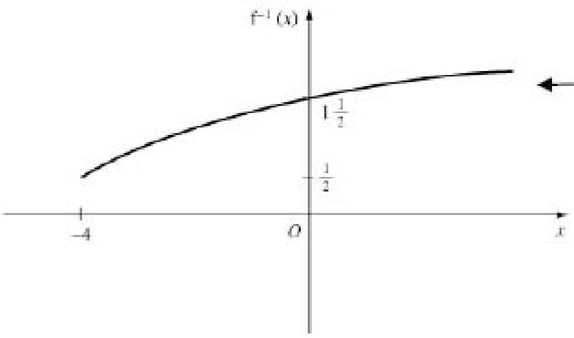
Compare coefficients of constant term.

b



As $\left(\frac{1}{2},-4\right)$ is the vertex of the parabola, f(x) is a one-to-one function for $x \geq \frac{1}{2}$.

c



Reflect graph from **b** in $f(x)=(x)$.

$(\frac{1}{2},-4) \rightarrow (-4,\frac{1}{2})$

$(1\frac{1}{2},0) \rightarrow (0,1\frac{1}{2})$

d

Let

$$y = (2x-1)^2-4$$
$$\Rightarrow (2x-1)^2 = y+4$$
$$2x-1 = \sqrt{y+4}$$
$$x = \frac{1}{2}\left(1+\sqrt{y+4}\right)$$
$$\Rightarrow f^{-1}(x) = \frac{1}{2}\left(1+\sqrt{x+4}\right),\ x \geq -4$$

$-\sqrt{y+4}$ not appropriate.

The graph in *c* gives the domain.

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Review Exercise
Exercise A, Question 14

Question:

The functions f and g are defined by

$f : x \rightarrow \frac{x+2}{x}, x \in \mathbb{R}, x \neq 0.$

$g : x \rightarrow \ln(2x-5), x \in \mathbb{R}, x > 2\frac{1}{2}.$

a Sketch the graph of f.

b Show that $f^2(x) = \frac{3x+2}{x+2}$.

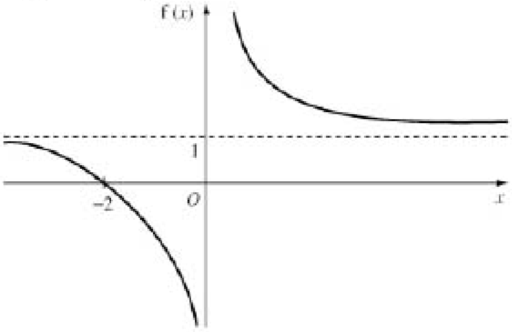
[$f^2(x)$ means ff(x)]

c Find the exact value of $gf\left(\frac{1}{4}\right)$.

d Find $g^{-1}(x)$, stating its domain.

Solution:

a $\frac{x+2}{x} = 1 + \frac{2}{x}$



b

$$\begin{aligned} f^2(x) &= f\left(\frac{x+2}{x}\right) \\ &= \frac{\frac{x+2}{x}+2}{\frac{x+2}{x}} \\ &= \frac{(3x+2)}{x} \times \frac{x}{(x+2)} \\ &= \frac{3x+2}{x+2} \end{aligned}$$

← $\frac{\frac{x+2+2x}{x}}{\frac{x+2}{x}}$

c

$$\begin{aligned} gf\left(\frac{1}{4}\right) &= g\left(\frac{2\frac{1}{4}}{\frac{1}{4}}\right) = g(9) \\ &= \ln(18-5) \\ &= \ln 13 \end{aligned}$$

d Let $y = \ln(2x-5)$

$$\begin{aligned} e^y &= 2x-5 \\ \Rightarrow x &= \frac{e^y+5}{2} \\ g^{-1}(x) &= \frac{e^x+5}{2} \quad x \in \mathbb{R} \end{aligned}$$

← The range of $g(x)$ is $x \in \mathbb{R}$ so the domain of $g^{-1}(x)$ is $x \in \mathbb{R}$.

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Review Exercise
Exercise A, Question 15

Question:

Solve the following equations, giving your answers to 3 significant figures.

a $3e^{(2x+5)} = 4$

b $3^x = 5^{1-x}$

c $2\ln(2x-1) = 1 + \ln 7$

Solution:

a

$$3\,e^{(2x+5)} = 4$$

$$\Rightarrow e^{(2x+5)} = \frac{4}{3}$$ ← Divide by 3 before taking logs.

$$\Rightarrow 2x+5 = \ln\frac{4}{3}$$ ← If $e^a = b \Rightarrow a = \ln b$

$$x = \frac{1}{2}(\ln\frac{4}{3} - 5)$$

$$= -2.36 \text{ (3 s.f.)}$$

b $3^x = 5^{1-x}$

$$x\ln 3 = (1-x)\ln 5$$ ← Take logs of both sides and use $\log a^n = n\log a$.

$$x\ln 3 + x\ln 5 = \ln 5$$

$$x(\ln 3 + \ln 5) = \ln 5$$

$$x = \frac{\ln 5}{(\ln 3 + \ln 5)}$$

$$= 0.594 \text{ (3 s.f.)}$$

c

$$2\ln(2x-1) = 1 + \ln 7$$

$$\ln(2x-1)^2 - \ln 7 = 1$$

$$\ln\frac{(2x-1)^2}{7} = 1$$

$$\frac{(2x-1)^2}{7} = e$$

$$(2x-1)^2 = 7e$$

$$2x - 1 = \pm\sqrt{7e}$$

$$x = \frac{1}{2}(1 + \sqrt{7e})$$

$$= 2.68$$

Note: $\ln(2x-1)^2 = \ln e + \ln 7 = \ln 7e$

As $x > \frac{1}{2}$, $\frac{1}{2}\left(1 - \sqrt{7e}\right)$ is not applicable.

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Review Exercise
Exercise A, Question 16

Question:

Find the exact solutions to the equations

a $\ln x + \ln 3 = \ln 6$

b $e^x + 3e^{-x} = 4$ *E*

Solution:

a $\ln x + \ln 3 = \ln 6$

so $\ln 3x = \ln 6$

$\Rightarrow 3x = 6$

$x = 2$

← Do not make the error of 'removing' the ln to give $x + 3 = 6$. You need to write $\ln a = \ln b$.

b

$e^x + 3e^{-x} = 4$

$\Rightarrow e^x + \frac{3}{e^x} = 4$

← Do not take logs of both sides. You need $e^{ax} = b$ before this strategy is valid. $\ln(e^x + 3e^{-x}) = \ln 4$ cannot be reduced any further.

$\Rightarrow (e^x)^2 - 4e^x + 3 = 0$

← Quadratic in e^x.

$\Rightarrow e^x = 3$ or $e^x = 1$

i.e. $x = \ln 3$ or $x = \ln 1 = 0$

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Review Exercise
Exercise A, Question 17

Question:

The function f is defined by

$f : x \to 3 - \ln(x + 2), \quad x \in \mathbb{R}, x > -2,$

The graph of $y = f(x)$ crosses the x-axis at the point A and crosses the y-axis at the point B.

a Find the exact coordinates of A and B.

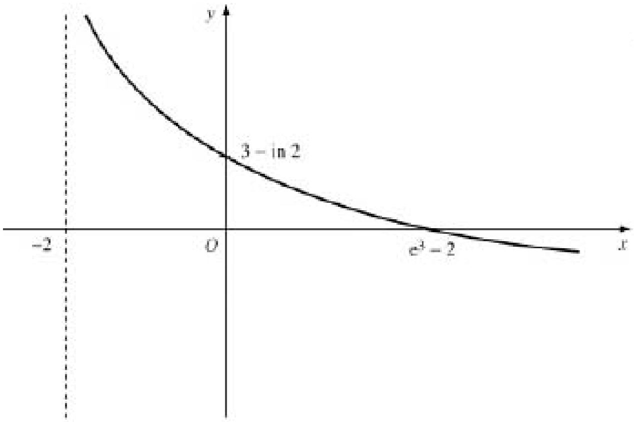
b Sketch the graph of $y = f(x), x > -2$.

Solution:

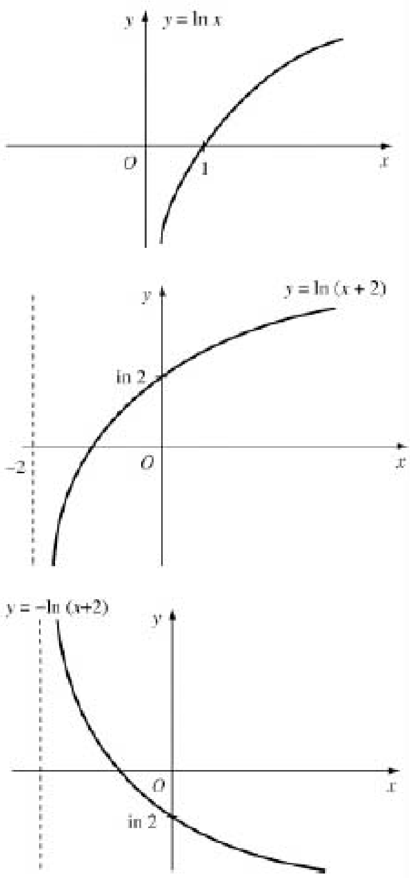
a

$$y = 3-\ln(x+2) \quad x>-2$$
$$\text{Put } y = 0$$
$$0 = 3-\ln(x+2)$$
$$\Rightarrow 3 = \ln(x+2)$$
$$e^3 = x+2$$
$$e^3-2 = x(\text{exact})$$
$$\text{So A} = (e^3-2, 0)$$
$$\text{Put } x = 0$$
$$y = 3-\ln 2$$
$$\text{So B} = (0, 3-\ln 2)$$

b



The curve is a decreasing curve.



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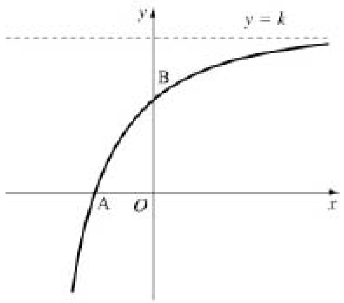
Review Exercise
Exercise A, Question 18

Question:

The graph of the function

$f(x) = 144 - 36e^{-2x}, \quad x \in \mathbb{R}$
has an asymptote $y = k$, and crosses the x and y axes at A and B respectively, as shown.



a Write down the value of k and the y-coordinate of B.
b Express the x-coordinate of A in terms of ln2.

Solution:

a The asymptote is $y = 144$, so $k = 144$. ← As $x \to \infty, e^{-2x} \to 0$ so $144 - 36e^{-2x} \to 144$.

The curve crosses the y-axis where $x = 0$,
so $y = 144 - 36e^{\circ} = 108$.
The y-coordinate of B is 108.

b The curve crosses the x-axis where $y = 0$

$$\Rightarrow 0 = 144 - 36e^{-2x}$$
$$\Rightarrow e^{-2x} = \frac{144}{36} = 4$$
$$-2x = \ln 4$$
$$x = \frac{1}{2}\ln 4$$
$$= -\ln 4^{\frac{1}{2}}$$
$$= -\ln 2$$

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Review Exercise
Exercise A, Question 19

Question:

[Part **d** requires the differentiation of e^{ax}, see Ch 8]
The functions f and g are defined by

$f : x \rightarrow 2x + \ln 2, \quad x \in \mathbb{R}$
$g : x \rightarrow e^{2x}, \quad x \in \mathbb{R}.$

a Prove that the composite function gf is

$gf : x \rightarrow 4e^{4x}, \quad x \in \mathbb{R}.$

b Sketch the curve with equation $y = gf(x)$, and show the coordinates of the point where the curve cuts the y-axis.

c Write down the range of gf.

d Find the value of x for which $\frac{d}{dx}[gf(x)] = 3$, giving your answer to 3 significant figures. *E*

Solution:

a

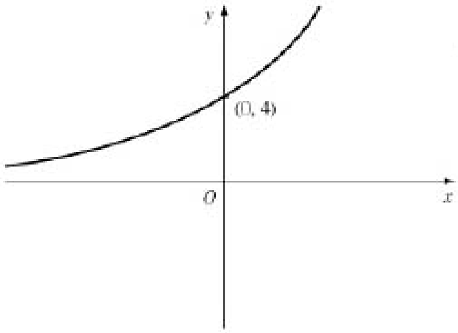
$$\begin{aligned}\text{gf}(x) &= \text{g}(2x+\ln 2)\\ &= \text{e}^{2(2x+\ln 2)}\\ &= \text{e}^{4x}\text{e}^{2\ln 2}\\ &= \text{e}^{4x}\text{e}^{\ln 2^2}\\ &= 4\text{e}^{4x}\end{aligned}$$

As $\text{e}^{a+b} = \text{e}^a\text{e}^b$

As $p\ln 2 = \ln 2^p$ and $\text{e}^{\ln 2^p} = 2^p$

$\text{gf}: x \to 4\text{e}^{4x} \quad x \in \mathbb{R}$

b



When
$x = 0$
$y = 4\text{e}^0 = 4\times 1 = 4$

c Range: $\text{gf}(x) > 0$

d

$$\begin{aligned}\frac{\text{d}}{\text{d}x}[\text{gf}(x)] &= 16\text{e}^{4x}\\ \text{if}\quad 16\text{e}^{4x} &= 3\\ \text{e}^{4x} &= \frac{3}{16}\\ \Rightarrow 4x &= \ln\frac{3}{16}\\ x &= \frac{1}{4}\ln\frac{3}{16}\\ &= -0.418\ (3\text{ s.f.})\end{aligned}$$

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Review Exercise
Exercise A, Question 20

Question:

a Show that $e^x - e^{-x} = 4$ can be rewritten in the form $e^{2x} - 4e^x - 1 = 0$

b Hence find the exact value of the real solution of $e^x - e^{-x} = 4$.

c For this value of x, find the exact value of $e^x + e^{-x}$.

Solution:

a

$$e^x - e^{-x} = 4$$
$$\Rightarrow e^x - \frac{1}{e^x} = 4$$
$$\Rightarrow (e^x)^2 - 1 = 4e^x$$
$$e^{2x} - 4e^x - 1 = 0$$

b

$$e^x = \frac{4 \pm \sqrt{16+4}}{2}$$
$$= \frac{4 \pm \sqrt{20}}{2}$$
$$= \frac{4 \pm 2\sqrt{5}}{2}$$
$$= 2 \pm \sqrt{5}$$
$$\text{so } e^x = 2 \pm \sqrt{5}$$
$$\Rightarrow x = \ln(2+\sqrt{5})$$

Use the formula on the quadratic $y^2 - 4y - 1 = 0$ where $y = e^x$.

As $e^x > 0$, $2-\sqrt{5}$ not applicable.

c If $e^x = 2+\sqrt{5}$

$$e^{-x} = \frac{1}{2+\sqrt{5}} = \frac{2-\sqrt{5}}{(2+\sqrt{5})(2-\sqrt{5})}$$
$$= \frac{2-\sqrt{5}}{4-5}$$
$$= \sqrt{5} - 2$$
$$\text{so } e^x + e^{-x} = (2+\sqrt{5}) + (\sqrt{5}-2)$$
$$= 2\sqrt{5}$$

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Review Exercise
Exercise A, Question 21

Question:

At time $t = 0$, a lake is stocked with k fish. The number, n, of fish in the lake at time t days can be represented by the equation
$n = 3000 + 1450e^{0.04t}$.

a State the value of k.

b Calculate the increase in the population of fish 3 weeks after stocking the lake.

c Find how many days pass, from the day the lake was stocked, before the number of fish increases to over 7000.

Solution:

$n = 3000 + 1450\,e^{0.04t}$

a At $t = 0$

$$\begin{aligned} n &= 3000 + 1450\,e^{0} \\ &= 4450 \\ \text{so } k &= 4450 \end{aligned}$$

b At $t = 21$

$$\begin{aligned} n &= 3000 + 1450\,e^{0.84} \\ &= 6\,358.7\ldots \end{aligned}$$

So increase in population is
$6358 - 4450$
$= 1908$ fish

c

$$\begin{aligned} 7000 &= 3000 + 1450\,e^{0.04t} \\ \Rightarrow 1450e^{0.04t} &= 4000 \\ e^{0.04t} &= \frac{4000}{1450} \\ 0.04t &= \ln\left[\frac{4000}{1450}\right] \\ t &= \frac{1}{0.04}\ln\left[\frac{400}{145}\right] \\ &= 25.3 \end{aligned}$$

So 26 days pass before the population reaches over 7000

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Review Exercise
Exercise A, Question 22

Question:

A heated metal ball S is dropped into a liquid. As S cools, its temperature, T°C, t minutes after it enters the liquid is given by

$T = 400e^{-0.05t} + 25, \quad t \geq 0.$

a Find the temperature of S as it enters the liquid.

b Find how long S is in the liquid before its temperature drops to 300°C. Give your answer to 3 significant figures.

c Find the rate, in °C per minute to 3 significant figures, at which the temperature of S is decreasing at the instant $t = 50$.

d With reference to the equation given above, explain why the temperature of S can never drop to 20°C *E*

Solution:

a

$$\text{Put } t = 0 \Rightarrow T = 400e^0 + 25 = 425°\text{C}$$

b $300 = 400e^{-0.05t} + 25$

$$\Rightarrow 400e^{-0.05t} = 275$$

$$e^{-0.05t} = \frac{275}{400}$$

$$\Rightarrow -0.05t = \ln\left[\frac{275}{400}\right]$$

$$t = \frac{-1}{0.05}\ln\left[\frac{275}{400}\right]$$

$$= 7.49 \text{ minutes}$$

Not using a calculator until the final step ensures that there has been no approximation in the process.

c $\frac{dT}{dt} = 400(-0.05)e^{-0.05t}$

When $t = 50$ $\frac{dT}{dt} = -20\,e^{-2.5} = -1.64$

So rate of decrease is 1.64°C/minute

d As $t \to \infty$, $400e^{-0.05t} \to 0$

so $T \to 25$

The temperature can never go below 25°C, so cannot reach 20°C

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Review Exercise
Exercise A, Question 23

Question:

A breeding programme for a particular animal is being monitored. Initially there were k breeding pairs in the survey.
A suggested model for the number of breeding pairs, n, after t years is

$$n = \frac{400}{1 + 9e^{-\frac{1}{9}t}}$$

a Find the value of k.

b Show that the above equation can be written in the form

$$t = 9\ln\left(\frac{9n}{400 - n}\right)$$

c Hence, or otherwise, calculate the number of years, according to the model, after which the number of breeding pairs will first exceed 100.

The model predicts that the number of breeding pairs cannot exceed the value A.

d Find the value of A.

Solution:

$$n = \frac{400}{1+9e^{-\frac{1}{9}t}}$$

a Initially $t = 0$, $n = k$

$$\begin{aligned} \text{so } k &= \frac{400}{1+9e^{0}} \\ &= \frac{400}{10} \\ &= 40 \end{aligned}$$

b

$$\begin{aligned} (1+9e^{-\frac{1}{9}t})n &= 400 \\ n+9ne^{-\frac{1}{9}t} &= 400 \\ 9ne^{-\frac{1}{9}t} &= 400-n \\ e^{-\frac{1}{9}t} &= \frac{400-n}{9n} \\ -\frac{1}{9}t &= \ln\left[\frac{400-n}{9n}\right] \\ \frac{1}{9}t &= -\ln\left[\frac{400-n}{9n}\right] \\ &= \ln\left[\frac{400-n}{9n}\right]^{-1} \\ t &= 9\ln\left[\frac{9n}{400-n}\right] \end{aligned}$$

c For

$$\begin{aligned} n = 100, t &= 9\ln\left[\frac{900}{300}\right] \\ &= 9.88\ldots \end{aligned}$$

so it takes 10 years

d Using the original equation with $t \to \infty$

$$n \to \frac{400}{1+0} = 400$$

$A = 400$

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Review Exercise
Exercise A, Question 24

Question:

$f(x) = x^3 - \frac{1}{x} - 2, \quad x \neq 0.$

a Show that the equation $f(x) = 0$ has a root between 1 and 2.
An approximation for this root is found using the iteration formula

$$x_{n+1} = \left(2 + \frac{1}{x_n}\right)^{\frac{1}{3}}, \text{ with } x_0 = 1.5.$$

b By calculating the values of x_1, x_2, x_3 and x_4 find an approximation to this root, giving your answer to 3 decimal places.

c By considering the change of sign of f(x) in a suitable interval, verify that your answer to part **b** is correct to 3 decimal places. *E*

Solution:

a

$f(1) = 1 - 1 - 2 = -2$

$f(2) = 8 - \frac{1}{2} - 2 = 5\frac{1}{2}$

← There is a change of sign and curve is continuous so there must be at least one root between 1 and 2.

b

$x_1 = 1.3867\ldots$
$x_2 = 1.3961\ldots$
$x_3 = 1.3953\ldots$
$x_4 = 1.3953\ldots$

← As the values oscillate and x_3 and x_4 are the same to 4 d.p., a 3 d.p. answer can be given.

approximation to the root is 1.395 (3 d.p.)

c Choosing [1.3945, 1.3955]

← You need to choose this interval or tighter, but there needs to be a sign change.

$f(1.3945) = -0.0053\ldots$
$f(1.3955) = +0.0010\ldots$

← As there is a change of sign, the root lies between 1.3945 and 1.3955.

root lies between 1.3945 and 1.3955 so 1.395 is an approximation to the root to 3 d.p.

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Review Exercise
Exercise A, Question 25

Question:

a By sketching the graphs of $y = -x$ and $y = \ln x, x > 0$, on the same axes, show that the solution to the equation $x + \ln x = 0$ lies between 0 and 1.

b Show that $x + \ln x = 0$ may be written in the form $x = \frac{(2x - \ln x)}{3}$.

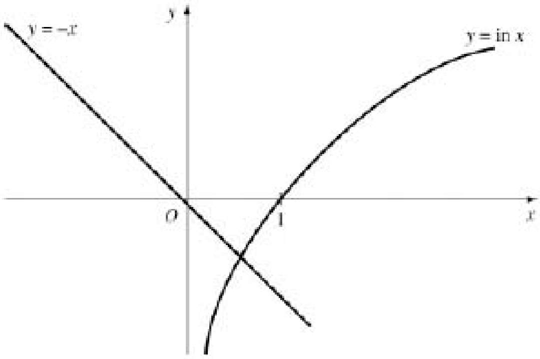
c Use the iterative formula

$$x_{n+1} = \frac{(2x_n - \ln x_n)}{3}, \quad x_0 = 1,$$

to find the solution of $x + \ln x = 0$ correct to 5 decimal places.

Solution:

a



$y=-x$ meets $y=\ln x$ where $-x=\ln x$ i.e. where $x+\ln x=0$

It is clear from the diagram that the solution lies between 0 and 1.

b

$$x = \frac{(2x-\ln x)}{3}$$
$$\Rightarrow 3x = 2x-\ln x$$
$$\Rightarrow x + \ln x = 0$$

Arrangements of $x+\ln x=0$ such as

$x = -\ln x$

$x = e^{-2x}$

do not lead to useful iterative processes.

c

$x_1 = 0.666667$ (6 d.p.)

$x_2 = 0.579599$

$x_3 = 0.568206$

$x_4 = 0.567228$

$x_5 = 0.567150$

$x_6 = 0.567144$

$x_7 = 0.567143$

$x_8 = 0.567143$

solution is 0.56714 (5 d.p.)

You can check by evaluating f(0.567135) and f(0.567145) where $f(x)=x+\ln x$.

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Review Exercise
Exercise A, Question 26

Question:

a Show that the equation $e^{2x} - 8x = 0$ has a root k between $x = 1$ and $x = 2$.
The iterative formula

$$x_n = \frac{1}{2}\ln 8x, \quad x_0 = 1.2,$$

is used to find to find an approximation for k

b Calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

c Show that, to 3 decimal places, $k = 1.077$.

d Deduce the value, to 2 decimal places, of one of the roots of $e^x = 4x$.

Solution:

a

$$f(x) = e^{2x} - 8x$$
$$f(1) = -0.6109\ldots$$
$$f(2) = 38.598\ldots$$

← Change of sign implies a root between 1 and 2.

b

$$x_1 = 1.131\ (3\text{ d.p.})$$
$$x_2 = 1.101\ (3\text{ d.p.})$$
$$x_3 = 1.088\ (3\text{ d.p.})$$

c

$$f(1.0765) = -0.0013\ldots$$
$$f(1.0775) = +0.0789\ldots$$

$1.0765 < 1.077 < 1.0775$

change of sign shows root lies between the two x values 1.0765 and 1.0775 so $k = 1.077$ (3 d.p.)

d Put $p = 2x$

then $e^{2x} - 8x = 0$ becomes $e^p - 4p = 0$

so root of $e^p = 4p$

is $p = 2x$

$= 2 \times 1.077$

$= 2.154$

$= 2.15$ (2 d.p.)

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Review Exercise
Exercise A, Question 27

Question:

The curve C has equation $y = x^5 - 1$. The tangent to C at the point $P(-1, -2)$ meets the curve again at the point Q, whose x-coordinate is k.

a Show that k is a root of the equation $x^5 - 5x - 4 = 0$.

b Show that $x^5 - 5x - 4 = 0$ can be rearranged in the form

$$x = \sqrt[4]{5 + \frac{4}{x}}.$$

The iterative formula

$$x_{n+1} = \sqrt[4]{5 + \frac{4}{x_n}}, \quad x_0 = 1.5,$$

is used to find to find an approximation for k.

c Write down the values of x_1, x_2, x_3 and x_4, giving your answers to 5 significant figures.

d Show that $k = 1.6506$ correct to 5 significant figures.

Solution:

$y = x^5 - 1$

a $\frac{dy}{dx} = 5x^4$

At the point $(-1, -2)$ the gradient of the tangent is $5(-1)^4 = 5$

Equation of tangent is $y + 2 = 5(x+1)$

i.e. $y = 5x + 3$

Use $y - y_1 = m(x - x_1)$

tangent meets the curve where

$$5x + 3 = x^5 - 1$$
$$\Rightarrow x^5 - 5x - 4 = 0$$

b

$$x^5 = 5x + 4$$
$$\Rightarrow x^4 = 5 + \frac{4}{x}$$
$$\Rightarrow x = \sqrt[4]{\left(5 + \frac{4}{x}\right)}$$

c

$x_1 = 1.6640$ (5 s.f.)

$x_2 = 1.6495$

$x_3 = 1.6507$

$x_4 = 1.6506$

d

$f(1.65055) = -0.0025\ldots$

$f(1.65065) = +0.00066\ldots$

Evaluate f(1.65055) and f(1.65065) where $f(x) = x^5 - 5x - 4$

Continued iteration is fine here as this is an oscillating set of values and convergence is rapid.

As there is a change of sign the root lies between 1.65055 and 1.65065 so $k = 1.6506$ (5 s.f.)

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Review Exercise
Exercise A, Question 28

Question:

$f(x) = 2x^3 - x - 4$.

a Show that the equation $f(x) = 0$ can be written as

$$x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}.$$

The equation $2x^3 - x - 4 = 0$ has a root between 1.35 and 1.4.

b Use the iteration formula

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)},$$

with $x_0 = 1.35$, to find, to 2 decimal places, the value of x_1, x_2 and x_3.

The only real root of $f(x) = 0$ is α.

c By choosing a suitable interval, prove that $\alpha = 1.392$, to 3 decimal places. *E*

Solution:

a

$$2x^3 - x - 4 = 0$$
$$\Rightarrow 2x^3 = x + 4$$
$$\Rightarrow x^2 = \frac{x+4}{2x} = \frac{1}{2} + \frac{2}{x}$$
$$\Rightarrow x = \sqrt{\left(\frac{2}{x} + \frac{1}{2}\right)}$$

b

$$x_{n+1} = \sqrt{\left(\frac{2}{x_n} + \frac{1}{2}\right)} \quad x_0 = 1.35$$
$$x_1 = 1.41 \text{ (2 d.p.)}$$
$$x_2 = 1.39 \text{ (2 d.p.)}$$
$$x_3 = 1.39 \text{ (2 d.p.)}$$

c Consider the interval [1.3915, 1.3925]

$$f(1.3915) = -0.00285\ldots$$
$$f(1.3925) = +0.00777\ldots$$

as there is a change of sign in the interval and $f(x)$ is continuous, the root lies in the interval so $\alpha = 1.392$ (3 d.p.)

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Review Exercise
Exercise A, Question 29

Question:

The function f is defined by

$f : x \to -5 + 4e^{2x}, \quad x \in \mathbb{R}, \quad x > 0.$

a Show that the inverse function of f is defined by

$$f^{-1} : x \to \frac{1}{2}\ln\left(\frac{x+5}{4}\right),$$

and write down the domain of f^{-1}.

b Write down the range of f^{-1}.

The graph of $y = \frac{1}{2}x$ crosses the graph of $y = f^{-1}(x)$ at $x = k$.

The iterative formula

$$x_{n+1} = \ln\left(\frac{x_n + 5}{4}\right), \quad x_0 = 0.3,$$

is used to find to find an approximation for k.

c Calculate the values of x_1 and x_2, giving your answers to 4 decimal places.

d Continue the iterative process until you have two values which are the same to 4 decimal places.

e Prove that this value does give k, correct to 4 decimal places.

Solution:

$f : x \rightarrow -5 + 4e^{2x} \quad x > 0$

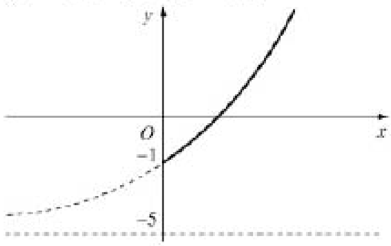
a Let

$$y = -5 + 4e^{2x}$$
$$4e^{2x} = y + 5$$
$$e^{2x} = \frac{y+5}{4}$$
$$2x = \ln\left[\frac{y+5}{4}\right]$$
$$x = \frac{1}{2}\ln\left[\frac{y+5}{4}\right]$$

so $f^{-1} : x \rightarrow \frac{1}{2}\ln\left[\frac{x+5}{4}\right]$

domain of f^{-1} is $x > -1$

The graph of $y = -5 + 4e^{2x}, x > 0$ is



This is the range of f.

b range of f^{-1} is $f^{-1}(x) > 0$

This is the domain of f.

c

$$x_1 = 0.2814$$
$$x_2 = 0.2779$$

d

$$x_3 = 0.2772$$
$$x_4 = 0.2771$$
$$x_5 = 0.2771$$

As this is a decreasing set of values, you cannot be sure that this does give k correct to 4 d.p.

e

Evaluate f(0.27705) and f(0.27715) where $f(x) = x - \ln\left(\frac{x+5}{4}\right)$

$$f(0.27705) = -2.2\ldots \times 10^{-5}$$
$$f(0.27715) = +5.8\times\ldots \times 10^{-5}$$

Change of sign indicates the root lies between 0.27705 and 0.27715.

$k = 0.2771$ (4 d.p.)

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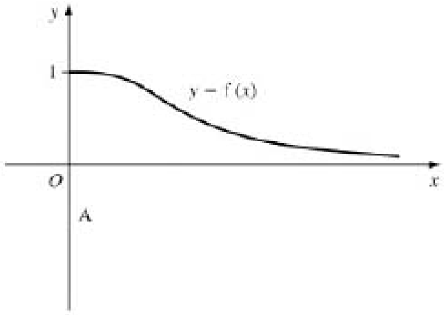
Review Exercise
Exercise A, Question 30

Question:

The graph of the function f, defined by

$$f : x \to \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \quad x \geq 0,$$

is shown.



a Copy the sketch and add to it the graph of $y = f^{-1}(x)$, showing the coordinates of the point where it meets the x-axis.

The two curves meet in the point A, with x-coordinate k.

b Explain why k is a solution of the equation $x = \frac{1}{1+x^2}$.

The iterative formula

$$x_{n+1} = \frac{1}{1+x_n^2}, \quad x_0 = 0.7$$

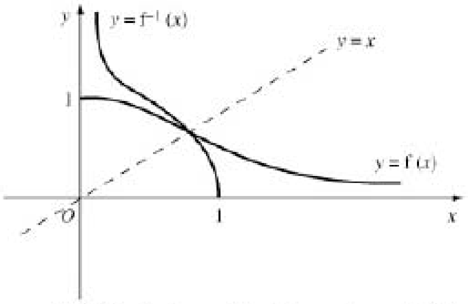
is used to find an approximation of k.

c Calculate the values of x_1, x_2, x_3 and x_4, giving your answers to 4 decimal places.

d Show that $k = 0.682$, correct to 3 decimal places.

Solution:

a



b $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in $y = x$

So the point A lies on $y = f(x)$ and $y = x$

where they meet $x = \frac{1}{1+x^2}$

It also lies on $y = f^{-1}(x)$.

c

$x_1 = 0.6711$ (4 d.p.)

$x_2 = 0.6895$

$x_3 = 0.6778$

$x_4 = 0.6852$

d

$f(0.6815) = -0.00135\ldots$

$f(0.6825) = +0.00028\ldots$

Where $f(x) = x - \frac{1}{1+x^2}$.

as change of sign root lies between 0.6815 and 0.6825

i.e. $k = 0.682$ (3 d.p.)