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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

Answer this question by using calculus.

Find the moment of inertia of a thin uniform rod of mass m and length l about an axis through one end perpendicular to its length.

Solution:

Divide the rod into small pieces of length δx at a distance x from the axis.

The mass per unit length of the rod $= \frac{m}{l}$.

So the mass of a small piece $= \frac{m}{l}\delta x$.

For the whole rod

$$I = \sum_{i=1}^{n} m_i r^2$$

$$= \sum_{x=0}^{x=l} \frac{mx^2}{l}\delta x$$

As $\delta x \to 0$ summations become integrals and

$$I = \int_0^l \frac{mx^2}{l}\,\mathrm{d}x$$

$$= \left[\frac{1}{3}\frac{m}{l}x^3\right]_0^l$$

$$= \frac{1}{3}ml^2$$

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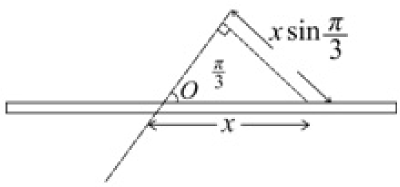
Exercise A, Question 2

Question:

Answer this question by using calculus.

Find the moment of inertia of a thin uniform rod of mass m and length l about an axis through its centre and inclined at an angle of $\frac{\pi}{3}$ to its length.

Solution:



Divide the rod into small pieces. As the mass per unit length of the rod is $\frac{m}{l}$, the mass of small piece of length δx is $\frac{m}{l}\delta x$.

If the piece is at a distance x along the rod from O, the centre of the rod, then its distance from the axis is $x\sin\frac{\pi}{3}$.

For the whole rod $I = \sum_{i=1}^{n} m_i r^2$ where $r = x\sin\frac{\pi}{3}$.

As $\delta x \to 0$, summations become integrals and

$I = \int_{\frac{-l}{2}}^{\frac{l}{2}} \frac{m}{l}x^2\sin^2\frac{\pi}{3}\,dx$, where $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\begin{aligned}\therefore\ I &= \frac{m}{l}\times\frac{3}{4}\left[\frac{1}{3}x^3\right]_{\frac{-l}{2}}^{\frac{l}{2}}\\ &= \frac{3m}{4l}\left[\frac{l^3}{24}+\frac{l^3}{24}\right]\\ &= \frac{ml^2}{16}\end{aligned}$$

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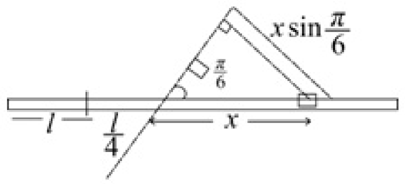
Exercise A, Question 3

Question:

Answer this question by using calculus.

Find the moment of inertia of a thin uniform rod of mass m and length $2l$ about an axis through a point at a distance $\frac{l}{4}$ from its centre and inclined at an angle of $\frac{\pi}{6}$ to its length.

Solution:



The mass per unit length of the rod $= \frac{m}{2l}$

Divide the rod into small pieces of length δx at a distance x along the rod from where the axis meets the rod.

Then a small piece is at distance $r = x\sin\frac{\pi}{6}$ from axis and the mass $m_i = \frac{m}{2l}\delta x$.

For the whole rod $I = \sum_{i=1}^{n} m_i r^2$ where $r^2 = x^2\sin^2\frac{\pi}{6} = \frac{x^2}{4}$

As $\delta x \to 0$, summations become integrals and

$$I = \int_{-\frac{5l}{4}}^{\frac{3l}{4}} \frac{m}{2l} \times \frac{x^2}{4}\,dx$$

$$= \frac{m}{8l}\left[\frac{1}{3}x^3\right]_{-\frac{5l}{4}}^{\frac{3l}{4}}$$

$$= \frac{m}{8l}\left[\frac{9l^3}{64} + \frac{125l^3}{192}\right]$$

$$= \frac{m}{8l} \times \frac{152l^3}{192}$$

$$= \frac{19ml^2}{192}$$

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Exercise A, Question 4

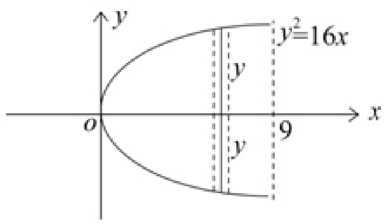
Question:

Answer this question by using calculus.

A uniform lamina, of mass M, is bounded by the curve with equation $y^2 = 16x$ and the line with equation $x = 9$.

Using calculus, find its moment of inertia about the x axis.

Solution:



Let ρ be mass per unit area of the lamina.

Then $m = \rho \int_0^9 2y\,dx$

$= 2\rho \int_0^9 4x^{\frac{1}{2}}\,dx$

$= 2\rho \left[\frac{2}{3} \times 4x^{\frac{3}{2}}\right]_0^9$

$= 2\rho \times \frac{2 \times 4 \times 27}{3}$

$= 144\rho$

$\therefore \rho = \frac{M}{144}$

The moment of inertia of a strip about axis through centre, perpendicular to strip $= \frac{1}{3}\delta m y^2 = \frac{1}{3}\rho 2y \cdot \delta x \cdot y^2$

So $I = \int_0^9 \frac{2}{3}\rho y^3\,dx$

$= \frac{2}{3} \times \frac{M}{144} \int_0^9 64x^{\frac{3}{2}}\,dx$

$= \frac{2}{3} \times \frac{M}{144} \times 64 \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^9$

$= \frac{8}{27} M \times \frac{2}{5} \times 243$

$= \frac{144}{5} M$

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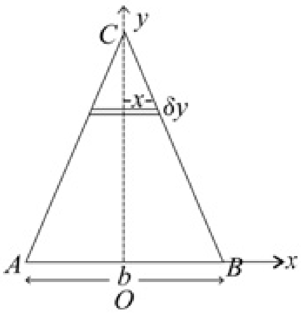
Exercise A, Question 5

Question:

Answer this question by using calculus.
Find the moment of inertia of a uniform triangular lamina of mass m which is isosceles with base b and height h about its axis of symmetry.

Solution:



Let the mass per unit area be ρ.

Then $m = \frac{1}{2} \times b \times h \times \rho$

So $\rho = \frac{2m}{bh}$

Divide the triangle into strips.
The one shown has mass $2x\delta y \times \rho$

i.e. $\frac{2m}{bh} \cdot 2x\delta y = \delta m$

The moment of inertia of the strip about the y axis is $\frac{1}{3}\delta m\, x^2 = \frac{4m}{3bh} x^3 \delta y$.

So total M.I. as $\delta y \to 0$ is given by

$$I = \int_0^h \frac{4m}{3bh} x^3 \, \mathrm{d}y$$

The equation of the line CB is $\frac{y}{h} + \frac{2x}{b} = 1$

i.e. $x = \frac{b}{2}\left(1 - \frac{y}{h}\right)$

$$\therefore\ I = \frac{4m}{3bh} \int_0^h \frac{b^3}{8}\left(1 - \frac{y}{h}\right)^3 \mathrm{d}y$$

$$= \frac{mb^2}{6h} \int_0^h 1 - \frac{3y}{h} + \frac{3y^2}{h^2} - \frac{y^3}{h^3} \, \mathrm{d}y$$

$$= \frac{mb^2}{6h}\left[y - \frac{3y^2}{2h} + \frac{y^3}{h^2} - \frac{y^4}{4h^3}\right]_0^h$$

$$= \frac{mb^2}{6h} \times \frac{h}{4}$$

$$= \frac{mb^2}{24}$$

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Exercise A, Question 6

Question:

Answer this question by using calculus.

A uniform lamina, of mass M, is bounded by the positive x and y axes and the portion of the curve $y = 2\cos x$ for which $0 \le x \in \frac{\pi}{2}$.

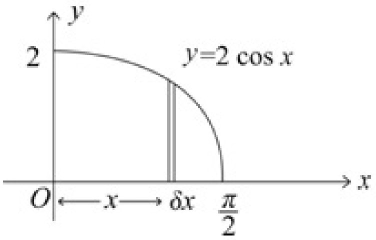
Using calculus, find its moment of inertia

a about the x axis

b about the y axis.

Solution:

a



For the strip shown

$\delta m = \rho y \delta x$

M.I. of strip about $Ox = \frac{1}{3}\delta m\, y^2$

$= \frac{1}{3}\rho y^3 \delta x$

$$\text{So total } I = \frac{1}{3}\rho \int_0^{\frac{\pi}{2}} y^3 \, dx$$

$$= \frac{1}{3}\rho \int_0^{\frac{\pi}{2}} 2^3 \cos^3 x \, dx$$

$$= \frac{8}{3}\rho \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)\cos x \, dx$$

$$= \frac{8}{3}\rho \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3}\rho \left[1 - \frac{1}{3}\right]$$

$$= \frac{16}{9}\rho$$

$$\text{But } M = \rho \int_0^{\frac{\pi}{2}} 2\cos x \, dx$$

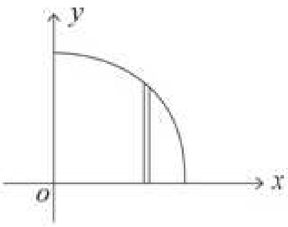
$$= \rho [2\sin x]_0^{\frac{\pi}{2}}$$

$$= 2\rho$$

$$\therefore \rho = \frac{M}{2} \text{ and so } I = \frac{16}{9} \times \frac{M}{2}$$

$$\text{i.e. } I = \frac{8}{9}M$$

b



M.I. of strip about the y-axis $= \delta m x^2$

$$= \rho y x^2 \delta x$$

$$= \frac{M}{2} y x^2 \delta x$$

$$\text{So } I = \frac{M}{2}\int_0^{\frac{\pi}{2}} x^2\, 2\cos x \, dx$$

$$= M\int_0^{\frac{\pi}{2}} x^2 \cos x \, dx$$

$$= M\left[x^2 \sin x - \int 2x \sin x \, dx\right]_0^{\frac{\pi}{2}}$$

$$= M\left[\frac{\pi^2}{4} - \left\{-2x\cos x + \int 2\cos x \, dx\right\}_0^{\frac{\pi}{2}}\right]$$

$$= M\left[\frac{\pi^2}{4} + \left[2x\cos x\right]_0^{\frac{\pi}{2}} - \left[2\sin x\right]_0^{\frac{\pi}{2}}\right]$$

$$= M\left[\frac{\pi^2}{4} - 2\right]$$

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Exercise A, Question 7

Question:

Answer this question by using additive rule and quoting known results.
A uniform ring of radius r and mass m has a particle of mass m attached to it. Find the moment of inertia of the composite body about an axis through the centre of the ring and perpendicular to the plane of the ring.

Solution:

Moment of inertia of ring $= mr^2$
Moment of inertia of particle $= mr^2$
By additive rule, moment of inertia of composite body $= 2mr^2$

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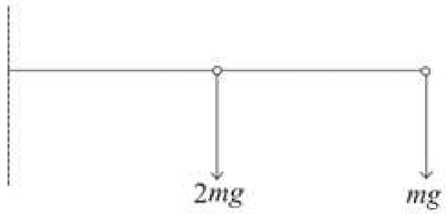
Exercise A, Question 8

Question:

Answer this question by using additive rule and quoting known results.
A uniform rod of mass $2m$ and length l has a particle of mass m fixed to one end.
Find the moment of inertia of the system about an axis through the other end of the rod and perpendicular to the rod.

Solution:



$$\text{Moment of inertia of rod } = \frac{4}{3} \cdot 2m \cdot \left(\frac{l}{2}\right)^2$$
$$= \frac{2ml^2}{3}$$

Moment of inertia of mass $m = ml^2$

$$\therefore \text{ Total M.I. } = \frac{2ml^2}{3} + ml^2 \text{(by additive rule)}$$
$$= \frac{5ml^2}{3}$$

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Exercise A, Question 9

Question:

Answer this question by using additive rule and quoting known results.
A uniform rod of mass M and length l is attached at one of its ends to the centre of a uniform disc of radius r, which is perpendicular to the rod. Find the moment of inertia of the system about an axis along the rod.

Solution:

Moment of inertia of rod $= 0$

Moment of inertia of disc $= \frac{1}{2}mr^2$

$\therefore$ Total M. I $= \frac{1}{2}mr^2$

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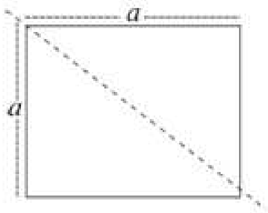
Exercise A, Question 10

Question:

Answer this question by using additive rule and quoting known results.
Four uniform rods each of mass M are rigidly jointed to form a square of side a. Find the moment of inertia of this structure about a diagonal.

Solution:



For each rod use the formula $I = \frac{4}{3}ml^2 \sin^2\theta$,
found in Example 3**c**.

So $I = \frac{4}{3} \times M \times \left(\frac{a}{2}\right)^2 \sin^2 45°$

$= \frac{1}{6}Ma^2$

But there are four rods.

So total $I = 4 \times \frac{1}{6}Ma^2$ (additive rule)

$= \frac{2}{3}Ma^2$

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Exercise A, Question 11

Question:

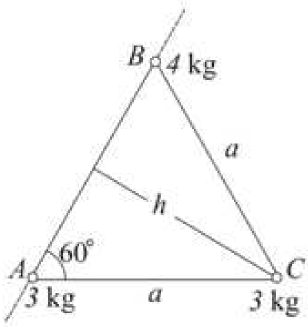
Answer this question by using additive rule and quoting known results.
Particles A, B and C of mass 3 kg, 4 kg and 3 kg respectively, are rigidly jointed by light rods to form an equilateral triangle with sides of length a.
Find the moment of inertia of the composite body about an axis

a along AB,

b through C along the axis of symmetry of the triangle which is perpendicular to AB.

Solution:

a

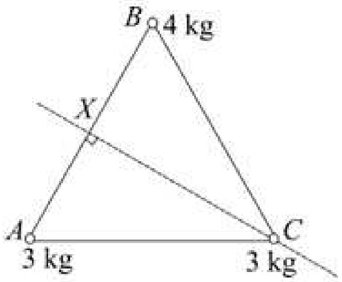


Total M.I. about $AB = 3 \times 0 + 4 \times 0 + 3 \times h^2$ where $h = a \sin 60° = \frac{a\sqrt{3}}{2}$

$\therefore \text{M.I.}_{AB} = 3 \times \frac{3a^2}{4}$

$= \frac{9a^2}{4}$

b



Let X be mid-point of AB

Total M.I. about $CX = 3 \times \left(\frac{a}{2}\right)^2 + 4 \times \left(\frac{a}{2}\right)^2 + 3 \times 0$

$= \frac{3a^2}{4} + \frac{4a^2}{4}$

$\text{M.I.}_{CX} = \frac{7a^2}{4}$

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Exercise A, Question 12

Question:

Answer this question by using additive rule and quoting known results.

In a similar configuration to that described in question **11**, particles A, B and C of mass 3 kg, 4 kg and 3 kg respectively, are rigidly joined by **heavy** rods, each of mass 2 kg, to form an equilateral triangle with sides of length a.

Find the moment of inertia of this composite body about an axis

a along AB,

b through C along the axis of symmetry of the triangle which is perpendicular to AB.

Solution:

a M.I. of rod AB about $AB = 0$

M.I. of rod BC about $AB = \frac{4m}{3}l^2\sin^2\theta$ (from Example **3c**)

$$= \frac{4\times 2}{3}\left(\frac{a}{2}\right)^2\sin^2 60°$$

$$= \frac{2}{3}a^2\times\frac{3}{4}$$

$$= \frac{a^2}{2}$$

Similarly M.I. of rod AC about $AB = \frac{a^2}{2}$

Total M.I. of particles about $AB = \frac{9a^2}{4}$ (from Question **11**)

$\therefore$ Total M.I. about $AB = \frac{9a^2}{4} + \frac{a^2}{2} + \frac{a^2}{2} = \frac{13a^2}{4}$

b M.I. of rod AB about $CX = \frac{1}{3}\times 2\times\left(\frac{a}{2}\right)^2 = \frac{a^2}{6}$

M.I. of rod BC about $CX = \frac{4\times 2}{3}\left(\frac{a}{2}\right)^2\sin^2 30° = \frac{a^2}{6}$

Similarly M.I. of rod AC about $CX = \frac{a^2}{6}$

Total M.I. of particles about $CX = \frac{7a^2}{4}$ (from Question **11**)

$\therefore$ Total M.I. about $CX = \frac{7a^2}{4} + \frac{a^2}{6} + \frac{a^2}{6} + \frac{a^2}{6}$

$$= \frac{9a^2}{4}$$

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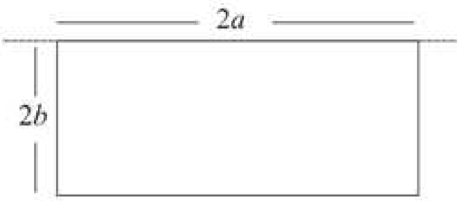
Exercise B, Question 1

Question:

Find the moment of inertia of a uniform rectangular lamina of mass m with length $2a$ and width $2b$ about an axis along the side of length $2a$.

Solution:



From the standard result $I = \frac{4}{3}mb^2$

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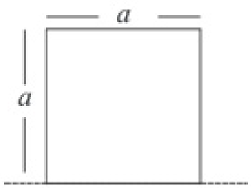
Exercise B, Question 2

Question:

Find the moment of inertia of a square lamina of mass m with sides of length a about an axis along one of the sides.

Solution:



From the standard result

$$I = \frac{4}{3}m\left(\frac{a}{2}\right)^2$$

$$= \frac{1}{3}ma^2$$

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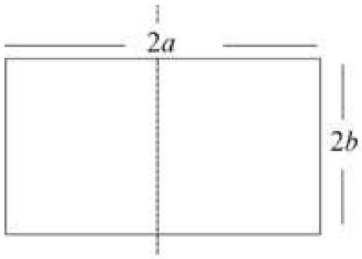
Exercise B, Question 3

Question:

Find the moment of inertia of a uniform rectangular lamina of mass m with length $2a$ and width $2b$ about an axis in the plane of the lamina, parallel to the sides of length $2b$ and bisecting the sides of length $2a$ at right angles.

Solution:



From the standard result $I = \frac{1}{3}ma^2$

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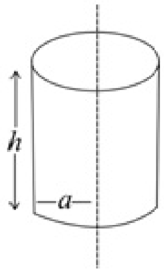
Exercise B, Question 4

Question:

Find the moment of inertia of a uniform circular solid cylinder of mass m, length h and base radius a, about its axis of symmetry.

Solution:



From standard results $I = \frac{1}{2}ma^2$

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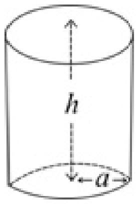
Exercise B, Question 5

Question:

Find the radius of gyration of a uniform circular hollow cylinder with height h and with a circular base of radius a of the same material, about its axis of symmetry. The total mass of the cylinder with its base is m.

Solution:



Let the mass per unit area be ρ. From standard results, the moment of inertia of the hollow cylinder is m_1a^2, where m_1 is its mass and $m_1 = \rho \cdot 2\pi ah$

i.e. $I_1 = \rho \cdot 2\pi a^3 h$.

The moment of inertia of the circular base is $\frac{m_2a^2}{2}$, where m_2 is its mass and

$m_2 = \rho \cdot \pi a^2$

i.e. $I_2 = \rho \frac{\pi a^4}{2}$

$\therefore$ Total M.I. can be obtained from additive rule and

$I = 2\pi\rho a^3 h + \frac{\pi}{2}\rho a^4 = \pi a^3 \rho (2h + \frac{a}{2})$ *

But $m = m_1 + m_2 = \rho \cdot 2\pi ah + \rho \cdot \pi a^2$

i.e. $m = \pi a \rho (2h + a) \Rightarrow \rho = \frac{m}{\pi a (2h + a)}$

Substituting into * gives

$$I = \frac{ma^2(2h + \frac{a}{2})}{2h + a}$$

i.e. $I = \frac{ma^2(4h + a)}{2(2h + a)}$

Radius of gyration, $R = \sqrt{\frac{I}{m}}$

$$= a\sqrt{\frac{4h + a}{2(2h + a)}}$$

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Exercise B, Question 6

Question:

Find the moment of inertia, about its axis of symmetry, of a uniform circular hollow cylinder of height h and base radius a, which has a circular base and circular top of twice the density of the material which forms the curved surface. The total mass of the cylinder with its base and top is m.

Solution:

Let the mass per unit area $= \rho$.
The mass of the hollow cyclinder is $2\pi ah\rho$
The mass of the circular base is $\pi a^2 \cdot 2\rho$
The mass of the circular top is $\pi a^2 \cdot 2\rho$

$\therefore$ Total mass $m = \left(2\pi ah + 2\pi a^2 + 2\pi a^2\right)\rho$ and $\rho = \dfrac{m}{2\pi ah + 4\pi a^2}$ *

The M.I. of the hollow cylinder is $(2\pi ah\rho)a^2$

M.I. of the circular base is $\left(2\pi a^2 \rho\right)\dfrac{a^2}{2}$

M.I. of the circular top is $\left(2\pi a^2 \rho\right)\dfrac{a^2}{2}$

$$\begin{aligned}\therefore \text{Total M.I.} &= 2\pi a^3 h\rho + \pi a^4 \rho + \pi a^4 \rho \\ &= \pi a^3 \rho(2h + 2a)\end{aligned}$$

Substituting $\rho = \dfrac{m}{2\pi a(h + 2a)}$ from *

$$\begin{aligned}\text{gives M.I.} &= \frac{m\pi a^3(2h + 2a)}{2\pi a(h + 2a)} \\ &= \frac{ma^2(h + a)}{h + 2a}\end{aligned}$$

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Exercise B, Question 7

Question:

Use the additive rule, and the standard result for the moment of inertia of a solid sphere, to show that the radius of gyration of a uniform solid hemisphere of mass m and radius r about a diameter of the circular base is $\sqrt{\frac{2}{5}}r$.

Solution:

Let the moment of inertia of the hemisphere about a diameter of the base be I.

Then as two hemispheres form a sphere $I + I = \frac{2}{5}mr^2$, where m is mass of sphere.

So $I = \frac{1}{2} \times \frac{2}{5}mr^2$

$= \frac{2}{5}\left(\frac{m}{2}\right)r^2$

But $\frac{m}{2} = m'$ the mass of the hemisphere

So $I = \frac{2}{5}m'r^2 = m'k^2$ where k, the radius of gyration, $= \sqrt{\frac{2}{5}}r$

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Exercise B, Question 8

Question:

Use the additive rule, and the standard result for the moment of inertia of a uniform circular disc, to find the radius of gyration of a uniform semicircular lamina of mass M and radius a about an axis perpendicular to the lamina through the mid-point of the straight edge.

Solution:

M.I. of circular disc about perpendicular axis through centre $= \frac{ma^2}{2}$

$\therefore$ moment of inertia, I, of semicircular disc about same axis $= \frac{ma^2}{4}$

(as $\mathrm{I} + \mathrm{I} = \frac{ma^2}{2}$, by additive rule)

But mass of semicircular disc $M = \frac{m}{2}$

$\therefore$ M.I. of semicircular disc $= \frac{1}{2}Ma^2$.

So radius of gyration, $k = \frac{1}{\sqrt{2}}a$.

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Exercise B, Question 9

Question:

A non-uniform solid sphere of radius R and mass M has mass kr per unit volume for all points at distance r from the centre of the sphere.

a Express k in terms of M and R.

b Use calculus to find the moment of inertia of the sphere about a diameter, giving your answer in terms of M and R.

Solution:

Divide the sphere up into concentric shells. Consider one such shell of radius r and thickness δr

its mass $\delta M \approx 4\pi r^2 \delta r \times kr$

$\therefore$ Total mass of sphere $= \sum_{r=0}^{R} 4\pi kr^3 \, \delta r$

$$\text{As } \delta r \to 0 \quad M = \int_0^R 4\pi kr^3 \, dr$$

$$= \left[\pi kr^4\right]_0^R$$

$$\therefore \ k = \frac{M}{\pi R^4}$$

The moment of inertia of the shell

$$\delta I \approx \frac{2}{3}\delta m r^2 = \frac{8}{3}k\pi r^5 \delta r$$

$$\therefore \text{ As } \delta r \to 0 \quad I = \int_0^R \frac{8}{3}k\pi r^5 \, dr$$

$$= \left[\frac{8}{18}k\pi r^6\right]_0^R$$

$$= \frac{4}{9}\pi R^6 k$$

$$= \frac{4}{9}\pi R^6 \times \frac{M}{\pi R^4}$$

$$= \frac{4}{9}MR^2$$

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Exercise B, Question 10

Question:

Using the formula for the moment of inertia of a uniform solid sphere,

a find the moment of inertia of a uniform spherical shell of inner radius r and outer radius R and mass m.

b Show that as $r \to R$ the moment of inertia reaches the value $\frac{2}{3}mr^2$.

Solution:

a By additive rule

Moment of inertia of sphere with radius r	+	Moment of inertia of shell	=	Moment of inertia of sphere with radius R

Let the sphere have mass per unit volume ρ.

Then moment of inertia of large sphere $= \frac{2}{5} \times \frac{4}{3}\pi R^3 \rho \times R^2$

$$= \frac{8}{15}\pi\rho R^5$$

Also moment of inertia of small sphere $= \frac{8}{15}\pi\rho r^5$

$\therefore$ Moment of inertia of shell $= \frac{8}{15}\pi\rho R^5 - \frac{8}{15}\pi\rho r^5$

i.e. $I = \frac{8}{15}\pi\rho\left(R^5 - r^5\right)$ ①

But mass of shell $= m = \left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho$

$$= \frac{4}{3}\pi\rho\left(R^3 - r^3\right) \quad ②$$

Dividing ① by ② gives

$$\frac{I}{m} = \frac{\frac{8}{15}\pi\rho\left(R^5 - r^5\right)}{\frac{4}{3}\pi\rho\left(R^3 - r^3\right)}$$

$$= \frac{2}{5}\frac{(R-r)\left(R^4 + R^3 r + R^2 r^2 + R r^3 + r^4\right)}{(R-r)\left(R^2 + Rr + r^2\right)}$$

$$\therefore I = m \times 2\frac{\left(R^4 + R^3 r + R^2 r^2 + R r^3 + r^4\right)}{5\left(R^2 + Rr + r^2\right)}$$

b As $r \to R$

$$I = m \times \frac{2}{5} \times \frac{5R^4}{3R^2}$$

$$= \frac{2}{3}mR^2$$

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Exercise B, Question 11

Question:

Using the formula for the moment of inertia of a uniform solid cone, (found in Example 13)

a find the moment of inertia of a conical shell, with inner radius r and inner height h' and outer radius R and outer height h and mass m. You should assume that the inner and outer cone are geometrically similar.

b Show that as $r \to R$ the moment of inertia reaches the value $\frac{1}{2}mr^2$.

c Explain how you could have deduced the value of the moment of inertia by considering a circular disc divided into a large number of concentric hoops.

Solution:

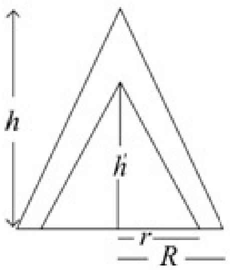
a Moment of inertia of conical shell = Moment of inertia of large cone − Moment of inertia of small cone

$$= \frac{3}{10}M_1R^2 - \frac{3}{10}M_2r^2$$

where M_1 is the mass of the large cone and M_2 is the mass of the small cone.

Let ρ be the mass per unit volume of the cones.

Then $M_1 = \frac{1}{3}\pi R^2 h\rho$ and $M_2 = \frac{1}{3}\pi r^2 h'\rho$



From similar triangles $\frac{h'}{r} = \frac{h}{R}$

So $h' = h\frac{r}{R}$

So $m = M_1 - M_2$

i.e. $m = \frac{1}{3}\pi\rho\left(R^2h - \frac{hr^3}{R}\right)$ ①

Also $I = \frac{3}{10} \times \frac{1}{3}\pi R^2 h\rho \times R^2 - \frac{3}{10} \times \frac{1}{3}\pi r^2 \frac{hr}{R}\rho \times r^2$

$$I = \frac{1}{10}\pi h\rho\left(R^4 - \frac{r^5}{R}\right) \quad ②$$

Divide equation ② by equation ① to give

$$\frac{I}{m} = \frac{\frac{1}{10}\pi h\rho\left(R^4 - \frac{r^5}{R}\right)}{\frac{1}{3}\pi\rho h\left(R^2 - \frac{r^3}{R}\right)}$$

$$= \frac{3}{10}\frac{\left(R^5 - r^5\right)}{\left(R^3 - r^3\right)}$$

i.e. $I = \dfrac{3m}{10}\dfrac{\left(R^4 + R^3r + R^2r^2 + Rr^3 + r^4\right)}{\left(R^2 + Rr + r^2\right)}$

b As $r \to R$ $\quad I = \dfrac{3}{10}m \times \dfrac{5R^4}{3R^2} = \dfrac{1}{2}mR^2$

c Consider the cone divided up into a large number of thin hoops centred on its axis of symmetry.

This is similar to a disc of the same radius divided up into a large number of thin hoops.

They have the same mass distribution and so the same moment of inertia i.e. $\dfrac{1}{2}mr^2$.

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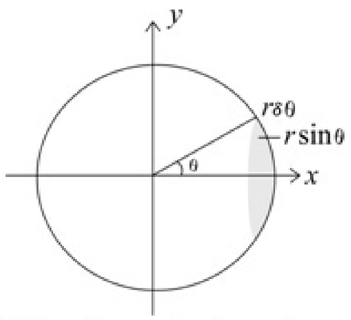
Exercise B, Question 12

Question:

Find, by integration, the moment of inertia of a uniform hollow sphere of mass m and radius r about an axis through the centre of the sphere.

Divide the sphere into composite hoops of surface area $2\pi r \sin\theta \times r\delta\theta$, where θ is the angle between the axis and the radius which joins a point on the outer circular boundary of the hoop to the centre of the sphere.

Solution:



Divide the sphere into hoops one of which is shown.
The surface area of the hoop $= 2\pi r \sin\theta \times r\delta\theta$

The mass per unit area of the sphere $= \frac{m}{4\pi r^2}$

$\therefore$ The mass of the hoop shown $= \frac{2\pi r^2 \sin\theta m\delta\theta}{4\pi r^2}$

$= \frac{1}{2} m \sin\theta\delta\theta$

M.I. of hoop about x-axis = mass × radius^2

$= \frac{1}{2} m \sin\theta \times r^2 \sin^2\theta\delta\theta$

Adding the moments of inertia of all such hoops and letting $\delta\theta \to 0$

$$\text{M.I. of sphere about } x\text{-axis} = \int_0^\pi \frac{mr^2}{2} \sin\theta \cdot \sin^2\theta \, d\theta$$

$$= \frac{mr^2}{2} \int_0^\pi \left(1 - \cos^2\theta\right) \sin\theta \, d\theta$$

$$= \frac{mr^2}{2} \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^\pi$$

$$= \frac{mr^2}{2} \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right]$$

$$= \frac{mr^2}{2} \times \frac{4}{3}$$

$$= \frac{2mr^2}{3}$$

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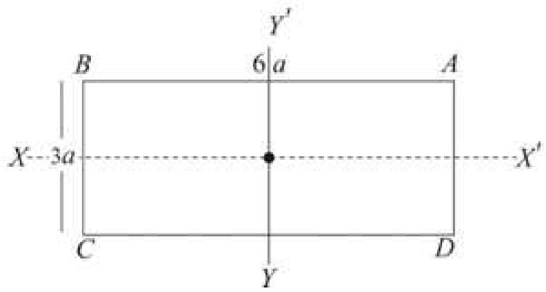
Exercise C, Question 1

Question:

A uniform lamina of mass m is in the shape of a rectangle $ABCD$ where $AB = 6a$ and $BC = 3a$. Find the moment of inertia of the lamina about an axis perpendicular to the lamina, acting through the centre of the lamina.

Solution:



Moment of inertia about $XX' = \frac{1}{3}m\left(\frac{3a}{2}\right)^2$

Moment of inertia about $YY' = \frac{1}{3}m(3a)^2$

$\therefore$ Moment of inertia about an axis perpendicular to the lamina acting through the centre of the lamina $= \frac{1}{3}m\left(\frac{3a}{2}\right)^2 + \frac{1}{3}m(3a)^2$ (using the perpendicular axes theorem).

i.e. required M.I. $= \frac{1}{3}m \times \frac{9a^2}{4} + \frac{1}{3}m \times 9a^2$

$$= \frac{15ma^2}{4}$$

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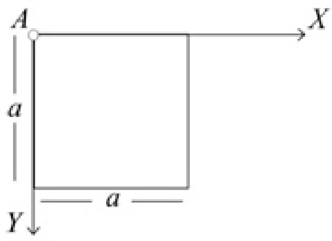
Exercise C, Question 2

Question:

Find the moment of inertia of a square lamina of mass m and side a about an axis through one corner perpendicular to the plane of the lamina.

Solution:



Choose one of the corners, A for example.

Moment of inertia of square about axis AX shown $= \frac{4}{3}m\left(\frac{a}{2}\right)^2$

$$= \frac{1}{3}ma^2$$

Moment of inertia of square about axis AY shown, $= \frac{4}{3}m\left(\frac{a}{2}\right)^2$ also

$$= \frac{1}{3}ma^2$$

$\therefore$ Moment of inertia about an axis through one corner, perpendicular to the plane is I where

$$I = \frac{1}{3}ma^2 + \frac{1}{3}ma^2 \text{(perpendicular axes theorem)}$$

i.e. $I = \frac{2}{3}ma^2$

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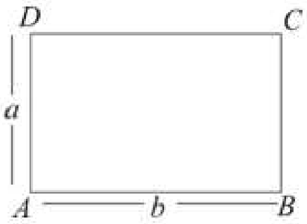
Exercise C, Question 3

Question:

Find the moment of inertia of a rectangular lamina of mass m and sides a and b about an axis through one corner perpendicular to the plane of the lamina.

Solution:



$$I_{AB} = \frac{4}{3}m\left(\frac{a}{2}\right)^2 = \frac{1}{3}ma^2$$

$$I_{AD} = \frac{4}{3}m\left(\frac{b}{2}\right)^2 = \frac{1}{3}mb^2$$

$\therefore$ Moment of inertia about an axis through A perpendicular to the

$$\text{lamina} = \frac{1}{3}ma^2 + \frac{1}{3}mb^2$$

$$= \frac{1}{3}m\left(a^2 + b^2\right)$$

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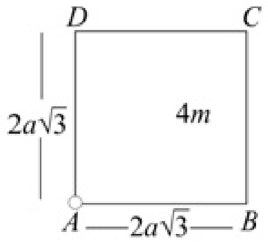
Exercise C, Question 4

Question:

A uniform square lamina $ABCD$ is of mass $4m$ and side $2a\sqrt{3}$. The axis L is a smooth fixed axis which passes through A and is perpendicular to the lamina. Show that the moment of inertia of the lamina about L is $32ma^2$.

Solution:



$$I_{AB} = \frac{4}{3} \times 4m\left(a\sqrt{3}\right)^2 = 16ma^2$$

$$I_{AD} = \frac{4}{3} \times 4m\left(a\sqrt{3}\right)^2 = 16ma^2$$

$$\therefore\ I_L = 16ma^2 + 16ma^2 \text{ (perpendicular axes theorem)}$$

$$= 32ma^2$$

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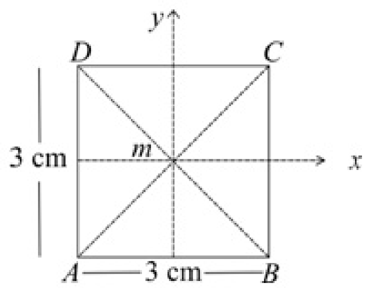
Exercise C, Question 5

Question:

A uniform lamina of mass m is in the shape of a square $ABCD$ with sides of length 3 cm. Find the moment of inertia of the lamina about the diagonal AC.

Solution:



Let centre of square be O and take x, y and z axes such that Ox is parallel to AB, Oy is parallel to AD and Oz is perpendicular to the lamina.

Then

$$I_{Ox} = \frac{1}{3}m\left(\frac{3}{2}\right)^2 = \frac{3m}{4}$$

$$I_{Oy} = \frac{1}{3}m\left(\frac{3}{2}\right)^2 = \frac{3m}{4}$$

$$\therefore\ I_{Oz} = \frac{3m}{4} + \frac{3m}{4}, \text{ by perpendicular axes theorem.}$$

$$= \frac{3m}{2}$$

Let $I_{AC} = I_{BD} = I$

Then $I + I = \frac{3m}{2}$, by perpendicular axes theorem

$\therefore I = \frac{3m}{4}$

So the moment of inertia about the diagonal $AC = \frac{3m}{4}$

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Exercise C, Question 6

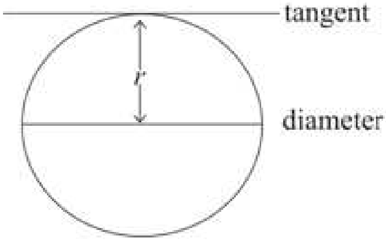
Question:

Find the radius of gyration of a uniform circular disc of radius r about a line in the plane of the disc which is tangential to the disc.

Solution:

The M.I. of the disc about an axis, through its centre, perpendicular to the disc $= \frac{mr^2}{2}$,
where m is its mass.
$\therefore$ By perpendicular axes theorem, the moment of inertia of the disc about a diameter is I where $I + I = \frac{mr^2}{2}$

$\therefore I = \frac{mr^2}{4}$



Let the moment of inertia of the disc about a tangent be I'
Then
$I' = I + mr^2$, by the parallel axes theorem

$\therefore I' = \frac{mr^2}{4} + mr^2$

$= \frac{5mr^2}{4}$

So if the radius of gyration is k

$mk^2 = 5\frac{mr^2}{4}$

$\therefore k = \frac{\sqrt{5}r}{2}$

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Exercise C, Question 7

Question:

Find the radius of gyration of a circular ring of radius r about a line in the plane of the ring which is tangential to the ring.

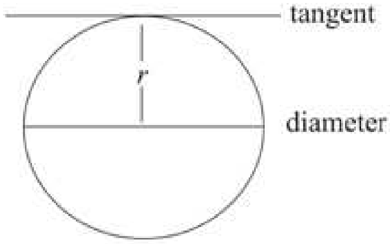
Solution:

The moment of inertia of a ring about an axis through its centre, perpendicular to the plane of the ring $= mr^2$, where m is its mass.

The moment of inertia of the ring about a diameter is I where

$I + I = mr^2$ (by perpendicular axes theorem)

$\therefore\ I = \dfrac{mr^2}{2}$



The moment of inertia of the ring about a tangent is I' where

$I' = I + mr^2$ by the parallel axes theorem

i.e. $I' = \dfrac{mr^2}{2} + mr^2$

$= \dfrac{3mr^2}{2}$

So, if the radius of gyration is k

$mk^2 = 3\dfrac{mr^2}{2}$

i.e. $k = \sqrt{\dfrac{3}{2}}r$

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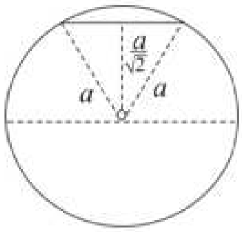
Exercise C, Question 8

Question:

Find the moment of inertia of a uniform solid sphere of radius a and mass m about a chord of the sphere which lies at a distance $\frac{a}{\sqrt{2}}$ from the centre of the sphere.

Solution:



The moment of inertia of the sphere about a diameter $= \frac{2}{5}ma^2$

This is true for any diameter and in particular for a diameter parallel to the chord.

Let I be the moment of inertia of the sphere about the chord, which is a distance $\frac{a}{\sqrt{2}}$ from O

Then

$$I = \frac{2}{5}ma^2 + m\left(\frac{a}{\sqrt{2}}\right)^2 \text{ [parallel axes theorem]}$$

$$= \frac{2}{5}ma^2 + \frac{ma^2}{2}$$

$$= \frac{9}{10}ma^2$$

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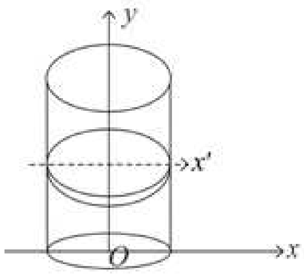
Exercise C, Question 9

Question:

Use calculus to find the moment of inertia of a thin hollow uniform right circular cylinder of mass M, radius R and height H about a diameter of an end circle. The cylinder is open at both ends.

Solution:



Let O be the centre of the circular base of the cylinder and let the x-axis be in the direction of the diameter of the base.
Let the y-axis be the axis of the cylinder.
Divide the cylinder into rings – one of which is shown. Let this ring have radius R, thickness δy and be at a distance y from the x-axis. Its mass is δm.
The moment of inertia of the ring about the y-axis is $\delta m R^2 = 2\pi\rho R\delta y \cdot R^2 = 2\pi\rho R^3\delta y$
Let the moment of inertia of the ring about a diameter perpendicular to the y-axis be $\delta I_{x'}$.
Then $\delta I_{x'} + \delta I_{x'} = 2\pi\rho R^3\delta y$ – using perpendicular axes theorem.
i.e. $\delta I_{x'} = \pi\rho R^3\delta y$
So the moment of inertia of the ring about the x-axis is $\delta I_{x'} + \delta m y^2$, using the parallel axes theorem i.e. $\pi\rho R^3\delta y + 2\pi\rho R y^2\delta y$
Adding all such rings and letting $\delta y \to 0$

$$I_x = \int_0^H \pi\rho R\left(R^2 + 2y^2\right)\mathrm{d}y$$

$$= \pi\rho R\left[R^2 y + \frac{2}{3}y^3\right]_0^H$$

$$= \pi\rho R\left[R^2 H + \frac{2}{3}H^3\right]$$

But the cylinder has mass M. So $2\pi RH\rho = M$

$$\therefore\ I_x = \frac{M}{2H}\left[R^2 H + \frac{2}{3}H^3\right]$$

$$= \frac{M}{6}\left[3R^2 + 2H^2\right]$$

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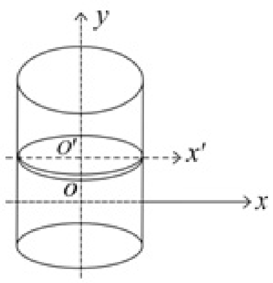
Exercise C, Question 10

Question:

Find the moment of inertia of a solid uniform right circular cylinder of mass M, radius R and height H about an axis through the centre of gravity perpendicular to the axis of the cylinder.

Solution:



Take the y-axis as the axis of the cylinder and the x-axis passes through the centre of gravity as shown.
Divide the cylinder into discs. The disc shown has radius R, thickness δy and is at height y above the x-axis.

The mass per unit volume of the cylinder $= \dfrac{M}{\pi R^2 H}$

$\therefore$ mass of disc $= \dfrac{M}{\pi R^2 H} \cdot \pi R^2 \delta y = \dfrac{M \delta y}{H}$

For the disc $I_y = \left(\dfrac{M\delta y}{H}\right) \times \dfrac{R^2}{2}$

$\therefore$ The M.I. of disc about its diameter $O'x'$, parallel to the x-axis is

$I_{x'} = \left(\dfrac{M\delta y}{H}\right) \times \dfrac{R^2}{4}$ (from the perpendicular axis theorem)

$\therefore$ M.I. of disc about Ox is I_x where

$I_x = I_{x'} + \left(\dfrac{M\delta y}{H}\right) \times y^2$ (from the parallel axes theorem)

$$= \frac{M\delta y}{H}\left[\frac{R^2}{4} + y^2\right]$$

The moment of inertia for the cylinder is obtained by adding the moments of inertia for all such discs and letting $\delta y \to 0$

$$\therefore I = \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{M}{H}\left(\frac{R^2}{4} + y^2\right) dy$$

$$= \frac{M}{H}\left[\frac{R^2}{4}y + \frac{y^3}{3}\right]_{-\frac{H}{2}}^{\frac{H}{2}}$$

$$= \frac{2M}{H}\left[\frac{R^2 H}{8} + \frac{H^3}{24}\right]$$

$$= \frac{MR^2}{4} + \frac{MH^2}{12}$$

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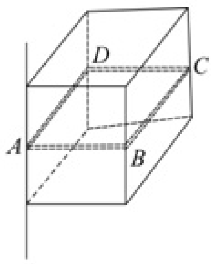
Exercise C, Question 11

Question:

Find the moment of inertia of a uniform cube of mass M and edge a about an axis along one edge.

Solution:



Consider a square cross section $ABCD$ of the cube.
Let its mass be δm.

Its M.I. about $AD = \frac{4}{3}\delta m\left(\frac{a}{2}\right)^2 = \frac{1}{3}\delta ma^2$

Also its M.I. about $AB = \frac{1}{3}\delta ma^2$

$\therefore$ By perpendicular axis theorem, its moment of inertia about an axis through A perpendicular to $ABCD$ is δI where

$$\delta I = \frac{1}{3}\delta ma^2 + \frac{1}{3}\delta ma^2 = \frac{2}{3}\delta ma^2$$

The M.I. of the cube about the edge through A is obtained by adding all such square cross sections

$$\begin{aligned}\therefore I &= \sum \frac{2}{3}\delta ma^2 \\ &= \frac{2}{3}a^2\sum \delta m \\ &= \frac{2}{3}Ma^2\end{aligned}$$

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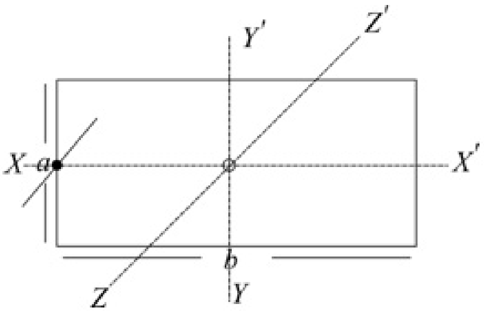
Exercise C, Question 12

Question:

Find the moment of inertia of a uniform rectangular lamina of mass M and sides a and b about an axis, perpendicular to the lamina, through the mid-point of a side of length a.

Solution:



Take XX', YY' and ZZ' as three axes meeting at O, the centre of the rectangle. XX' and YY' are parallel to sides of the rectangle and ZZ' is perpendicular to the rectangle.
Let L be the axis about which you need to find the moment of inertia. L is parallel to ZZ'

$I_{ZZ'} = I_{XX'} + I_{YY'}$ (perpendicular axis theorem)

$$= \frac{1}{3}m\left(\frac{a}{2}\right)^2 + \frac{1}{3}m\left(\frac{b}{2}\right)^2$$

$$= \frac{1}{12}m\left(a^2 + b^2\right)$$

Then $I_L = I_{ZZ'} + m\left(\frac{b}{2}\right)^2$ (parallel axes theorem)

$$= \frac{1}{12}m\left(a^2 + b^2\right) + \frac{mb^2}{4}$$

$$= \frac{1}{12}ma^2 + \frac{1}{3}mb^2$$

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Exercise C, Question 13

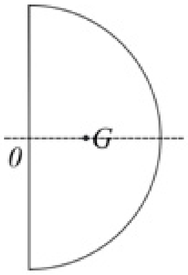
Question:

A uniform semi circular lamina has mass m and radius r.

a State the position of its centre of mass.

b Find the moment of inertia of the lamina about an axis through its centre of mass, perpendicular to the lamina.

Solution:



a On its axis of symmetry at a distance $\frac{4r}{3\pi}$ from O, the mid-point of its straight edge.

b Let the moment of inertia of the semi-circular lamina about an axis perpendicular to the lamina through O be I_O.

Then, as two such laminas make a disc of mass $2m$

$$I_O + I_O = \frac{2mr^2}{2} \text{ – by the additive rule.}$$

$$\therefore\ I_O = \frac{mr^2}{2}$$

The required moment of inertia I_G' may be obtained by using the parallel axes theorem.

As $I_O = I_G' + m\left(\frac{4r}{3\pi}\right)^2$

$$I_G' = \frac{mr^2}{2} - \frac{16mr^2}{9\pi^2}$$

$$= \frac{mr^2}{18\pi^2}\left(9\pi^2 - 32\right)$$

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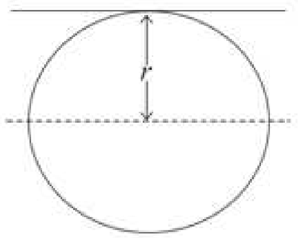
Exercise C, Question 14

Question:

Find the moment of inertia of a uniform solid sphere of mass m and radius r about a tangent at any point on the surface.

Solution:



Moment of inertia about a diameter $= \frac{2}{5}mr^2$

Then using parallel axes theorem:

$$\text{Moment of inertia about tangent} = \frac{2}{5}mr^2 + mr^2$$
$$= \frac{7}{5}mr^2$$

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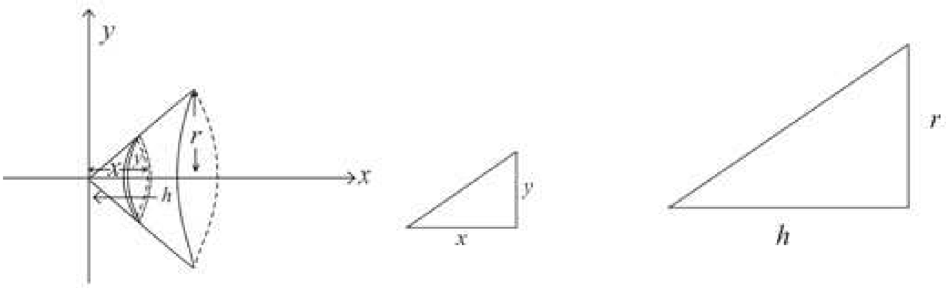
Exercise C, Question 15

Question:

Find, by integration, the moment of inertia of a uniform solid cone of mass m, base radius r and height h about a diameter of the base.

Solution:



Divide the cone into thin discs – one of which is shown.
Its mass is δm, its thickness is δx, its radius is y and it is at a distance x from the y-axis.

The moment of inertia of the disc about the x-axis is $\frac{\delta m y^2}{2}$.

Its M.I. about its diameter $= \frac{\delta m y^2}{4}$ (perpendicular axes theorem)

Its M.I. about a diameter of the base $= \frac{\delta m y^2}{4} + \delta m(h-x)^2$ (parallel axes theorem) *

The mass per unit volume of the cone $= \frac{m}{\frac{1}{3}\pi r^2 h} = \frac{3m}{\pi r^2 h}$

$$\therefore \text{ The mass } \delta m = \frac{3m}{\pi r^2 h} \times \pi y^2 \delta x$$
$$= \frac{3my^2}{r^2 h}\delta x \qquad ①$$

Also by similar triangles: $\frac{y}{x} = \frac{r}{h} \Rightarrow y = \frac{r}{h}x \qquad ②$

Substituting ① and ② into *,

$$\delta I = \frac{3m}{r^2 h}\left(\frac{y^4}{4} + y^2(h-x)^2\right)\delta x$$

i.e. $\delta I = \frac{3m}{r^2 h}\left(\frac{r^4 x^4}{4h^4} + \frac{r^2 x^2}{h^2}(h-x)^2\right)\delta x$

Let $\delta x \to 0$ and find the total moment of inertia of the cone by integration.

$$\text{So } I = \frac{3m}{r^2 h}\int_0^h \frac{r^4}{4h^4}x^4 + \frac{r^2}{h^2}\left(h^2x^2 - 2hx^3 + x^4\right)\mathrm{d}x$$
$$= \frac{3m}{r^2 h}\left[\frac{r^4 x^5}{20h^4} + \frac{r^2 x^3}{3} - \frac{2r^2 x^4}{4h} + \frac{r^2}{h^2}\frac{x^5}{5}\right]_0^h$$
$$= \frac{3m}{r^2 h}\left[\frac{r^4 h}{20} + \frac{r^2 h^3}{3} - \frac{r^2 h^3}{2} + \frac{r^2 h^3}{5}\right]$$
$$= \frac{3mr^2}{20} + \frac{mh^2}{10}$$

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Exercise D, Question 1

Question:

You may assume that the moment of inertia of a uniform circular disc, of mass m *and radius* a, *about an axis through its centre and perpendicular to its plane is* $\frac{1}{2}\text{ma}^2$.

A cartwheel is modelled as a uniform circular disc, of mass m and radius a, to which is attached a thin metal circular rim, also of mass m and radius a. The cartwheel rotates about the axis through its centre and perpendicular to its plane.

Find the radius of gyration of the cartwheel about this axis. ***E***

Solution:

Moment of inertia of circular disc $= \frac{1}{2}ma^2$

Moment of inertia of circular rim $= ma^2$

$\therefore$ M.I. of cartwheel $= \frac{1}{2}ma^2 + ma^2$ (additive rule)

$= \frac{3}{2}ma^2$

But mass of cartwheel $= 2m$

$\therefore$ If its radius of gyration $= k$

$\text{M.I.} = 2mk^2 = \frac{3}{2}ma^2$

$\therefore k^2 = \frac{3}{4}a^2$

i.e. $k = \frac{\sqrt{3}a}{2}$

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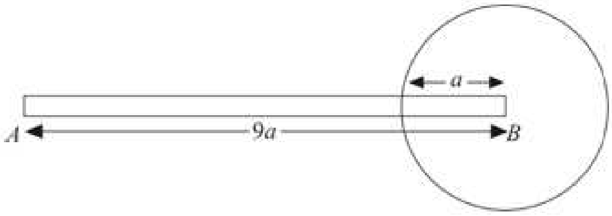
Exercise D, Question 2

Question:



A pendulum P is modelled as a uniform rod AB, of length $9a$ and mass M, rigidly fixed to a uniform circular disc of radius a and mass $2M$. The end B of the rod is attached to the centre of the disc and the rod lies in the plane of the disc as shown in the figure. The pendulum is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through end A and is perpendicular to the plane of the disc.

Show that the moment of inertia of P about L is $190Ma^2$. ***E*** (adapted)

Solution:

Moment of inertia of rod about $L = \frac{4}{3}M\left(\frac{9a}{2}\right)^2$

$= \frac{1}{3}M \times (9a)^2$

$= 27Ma^2$

Moment of inertia of disc about $L = 2M\left(\frac{a^2}{2}\right) + 2M(9a)^2$ (By parallel axes theorem)

$= Ma^2 + 162Ma^2$

$= 163Ma^2$

$\therefore$ Moment of inertia of pendulum about $L = 27Ma^2 + 163Ma^2$ (additive law)

$= 190Ma^2$

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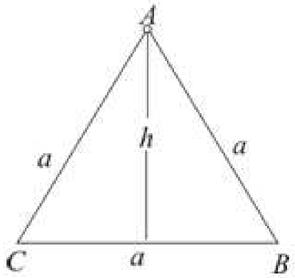
Exercise D, Question 3

Question:

A uniform wire of length $3a$ and mass $3m$ is bent into the shape of an equilateral triangle. Find the moment of inertia of the triangle about an axis through a vertex perpendicular to the plane of the lamina. ***E***

Solution:



By Pythagoras' Theorem:

$h^2 = a^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4}$ *

The diagram shows the equilateral triangle ABC. Let L be the axis through A, perpendicular to the plane of ABC.

Moment of inertia of AB about $L = \frac{4}{3}m\left(\frac{a}{2}\right)^2$

Moment of inertia AC about $L = \frac{4}{3}m\left(\frac{a}{2}\right)^2$

Moment of inertia CB about $L = \frac{m}{3}\left(\frac{a}{2}\right)^2 + mh^2$ – from parallel axes theorem

$\therefore$ By additive rule:
moment of inertia of the triangle about L

$= \frac{4}{3}m\left(\frac{a}{2}\right)^2 + \frac{4}{3}m\left(\frac{a}{2}\right)^2 + \frac{m}{3}\left(\frac{a}{2}\right)^2 + mh^2$

$= \frac{1}{3}ma^2 + \frac{1}{3}ma^2 + \frac{1}{12}ma^2 + \frac{3ma^2}{4}$ (from *)

$= \frac{18}{12}ma^2$

$= \frac{3}{2}ma^2$

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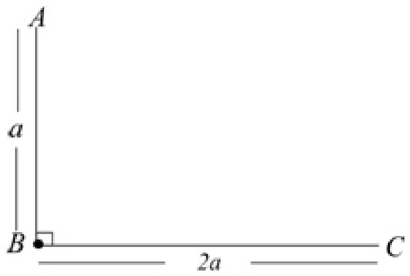
Exercise D, Question 4

Question:

A uniform piece of wire ABC, of total length $3a$ and mass m, is bent to form a right angle at B, with straight arms AB and BC of length a and $2a$ respectively. Show that the moment of inertia of the wire about the axis L through B perpendicular to the plane of the wire is ma^2. ***E***

Solution:



As the wire is uniform, mass of AB is $\frac{m}{3}$ and mass of BC is $\frac{2m}{3}$.

Moment of inertia of AB about $L = \frac{4}{3}\left(\frac{m}{3}\right)\left(\frac{a}{2}\right)^2$

Moment of inertia BC about $L = \frac{4}{3}\left(\frac{2m}{3}\right)\left(\frac{2a}{2}\right)^2$

$\therefore$ Moment of inertia of wire about $L = \frac{4}{3} \times \frac{ma^2}{12} + \frac{4}{3} \times \frac{8ma^2}{12}$

$= \frac{ma^2}{9} + \frac{8ma^2}{9}$

$= ma^2$

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Exercise D, Question 5

Question:

A thin uniform rod of mass m and length $2l$ is attached at one end to the centre of a face of a uniform solid cube of mass $8m$ and side l. The rod is perpendicular to the face to which it is attached. Find the moment of inertia of the system about an edge of the cube which is parallel to the rod. ***E***

Solution:

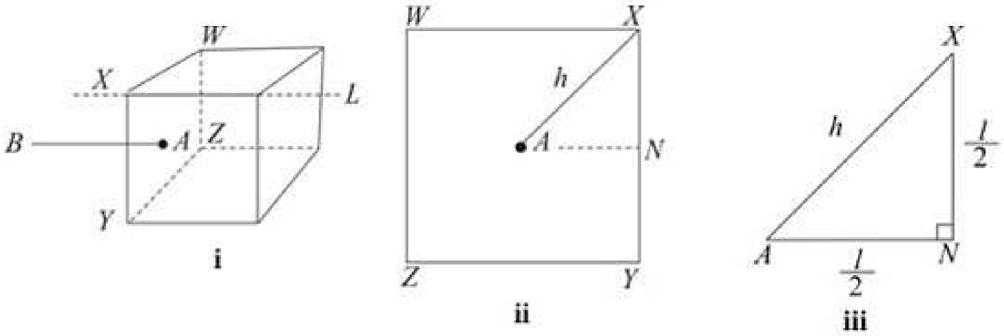

Diagram **i** shows the rod AB attached at A, the centre of the face $WXYZ$. It also shows the axis L through point X perpendicular to face $WXYZ$.
Diagram **ii** shows the face $WXYZ$ and diagram **iii** shows an enlargement of ΔANX, where N is the mid-point of the edge XY

Let $AX = h$ where $h^2 = \left(\frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2 = \frac{2l^2}{4} = \frac{l^2}{2}$ (from Pythagoras' Theorem)

Let mass of square be m'
moment of inertia of the rod AB about axis L

$= mh^2$

$= \frac{ml^2}{2}$ ①

moment of inertia of cube about $L =$ moment of inertia of square $WXYZ$ of same mass about L (stretching rule)

Moment of inertia of square about axis along $AN = \frac{1}{3}m'\left(\frac{l}{2}\right)^2$

Also M.I. of square about axis perpendicular to AN in plane of square $= \frac{1}{3}m'\left(\frac{l}{2}\right)^2$

$\therefore$ By perpendicular axes theorem M.I. of square about axis through A perpendicular to plane $= \frac{1}{3}m'\frac{l^2}{4} + \frac{1}{3}m'\frac{l^2}{4} = \frac{1}{6}m'l^2$

By parallel axes theorem M.I. of $WXYZ$ about L

$= \frac{1}{6}m'l^2 + m'h^2$

$= \frac{1}{6}m'l^2 + \frac{1}{2}m'l^2$

$= \frac{2}{3}m'l^2$

So M.I. of cube about $L = \frac{2}{3} \times 8ml^2$ ②

Using results ① and ② the M.I. of the system about L

$= \frac{ml^2}{2} + \frac{16}{3}ml^2$

$= \frac{35}{6}ml^2$

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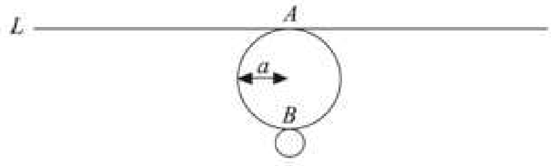
Exercise D, Question 6

Question:



A uniform disc has mass m and radius a.

a Show that the moment of inertia of the disc about a tangent L lying in the plane of the disc is $\frac{5}{4}ma^2$.

The line L is a tangent to the disc at the point A, and AB is a diameter of the disc, as shown in the figure. A particle of mass m is attached to the disc at B.

b Find the moment of inertia of the loaded disc about the tangent L. ***E***

Solution:

a M.I. of disc about axis through its centre, perpendicular to its plane $= \frac{ma^2}{2}$

This is a result you may quote.

$\therefore$ M.I. of disc about diameter $= \frac{ma^2}{4}$

From perpendicular axis theorem – as $I + I = \frac{ma^2}{2}$, then $I = \frac{ma^2}{4}$.

$\therefore$ M.I. of disc about tangent $= \frac{ma^2}{4} + ma^2$ (Parallel axes theorem)

i.e. M.I. $= \frac{5ma^2}{4}$

b $I = \frac{5ma^2}{4} + m(2a)^2$

$= \frac{21ma^2}{4}$ (Additive rule)

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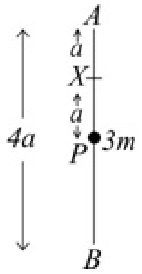
Exercise D, Question 7

Question:

A uniform rod AB of mass m and length $4a$ is free to rotate in a vertical plane about a fixed smooth horizontal axis l through the point X on the rod, where $AX = a$. The rod is hanging at rest with B below A when it is struck at its mid-point by a particle P of mass $3m$ moving horizontally with speed u in a direction perpendicular to l. Immediately after the impact P adheres to the rod. Show that after the impact, the moment of inertia about l of the rod and the particle together is $\frac{16}{3}ma^2$. ***E***

Solution:



Moment of inertia of rod about axis perpendicular to it, through mid-point $= \frac{m(2a)^2}{3} = \frac{4ma^2}{3}$

$\therefore$ M.I. of rod about axis l, through $X = \frac{4ma^2}{3} + ma^2$ (parallel axes theorem)

Moment of inertia of particle P about $l = 3ma^2$

$\therefore$ M.I. of rod and particle together $= \frac{4ma^2}{3} + ma^2 + 3ma^2$

$$= \frac{16ma^2}{3}$$

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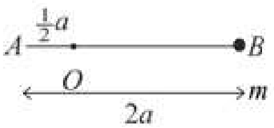
Exercise D, Question 8

Question:

A uniform rod AB has mass m and length $2a$. A particle of mass m is attached to the end B. The loaded rod is free to rotate about a fixed smooth horizontal axis L, perpendicular to the rod and passing through a point O of the rod, where $AO = \frac{1}{2}a$.

Show that the moment of inertia of the loaded rod about L is $\frac{17ma^2}{6}$. **E**

Solution:



Moment of inertia of rod about mid-point $= \frac{1}{3}ma^2$

$\therefore$ M.I. of rod about axis L, through $O = \frac{1}{3}ma^2 + m\left(\frac{1}{2}a\right)^2$ (parallel axes theorem)

M.I. of particle at B about $L = m\left(\frac{3}{2}a\right)^2$

$$\therefore \text{ M.I. of the loaded rod} = \frac{1}{3}ma^2 + \frac{1}{4}ma^2 + \frac{9}{4}ma^2$$
$$= \frac{34}{12}ma^2$$
$$= \frac{17ma^2}{6}$$

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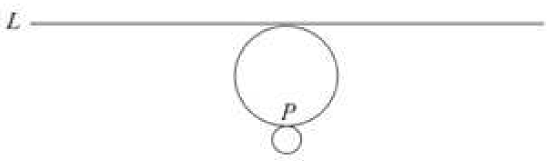
Exercise D, Question 9

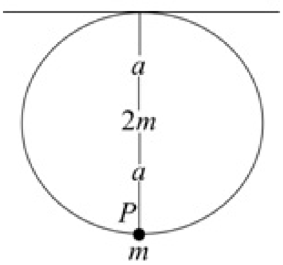
Question:



An ear-ring is modelled as a uniform solid sphere of mass $2m$ and radius a, with a particle of mass m attached to a point P on the surface of the sphere. The ear- ring is free to rotate about a fixed horizontal axis L which is tangential to the sphere and passes through a point diametrically opposite to P, as shown in the figure.

Show that the moment of inertia of the ear- ring about L is $\frac{34}{5}ma^2$. ***E***

Solution:



Moment of inertia of sphere about diameter $= \frac{2}{5}(2m)a^2$

$= \frac{4ma^2}{5}$

$\therefore$ M.I. of sphere about $L = \frac{4}{5}ma^2 + 2ma^2$ (parallel axes theorem)

$= \frac{14}{5}ma^2$

M.I. of particle at P about $L = m(2a)^2$

$= 4ma^2$

$\therefore$ Total M.I. of ear-ring about $L = \frac{14}{5}ma^2 + 4ma^2$ (additive rule)

i.e. M.I. $= \frac{34}{5}ma^2$

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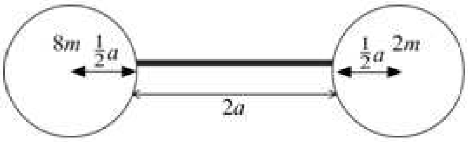
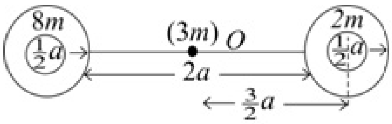
Exercise D, Question 10

Question:



A model of a timing device in a clock consists of a uniform rod, of mass $3m$ and length $2a$, the ends of which are attached to two uniform solid spheres, each of radius $\frac{1}{2}a$ as shown in the figure. One sphere has mass $8m$ and the other has mass $2m$. The device rotates freely in a vertical plane about a horizontal axis through the centre of the rod and perpendicular to it. Show that the moment of inertia of the system about this axis is $\frac{49}{2}ma^2$. ***E***

Solution:



Let O be the centre of the rod and let L be the horizontal axis through O, perpendicular to the rod.

M.I. of rod about $L = \frac{1}{3}(3m)a^2 = ma^2$

$$\text{M.I. of sphere mass } 2m \text{ about its diameter} = \frac{2}{5} \times 2m\left(\frac{1}{2}a\right)^2 = \frac{1}{5}ma^2$$

$$\therefore \text{ M.I. of sphere mass } 2m \text{ about } L = \frac{1}{5}ma^2 + 2m\left(\frac{3}{2}a\right)^2 \quad \text{(parallel axes theorem)}$$

$$= \frac{47}{10}ma^2$$

$$\text{Similarly M.I. of sphere mass } 8m \text{ about } L = \frac{2}{5} \times 8m\left(\frac{1}{2}a\right)^2 + 8m\left(\frac{3}{2}a\right)^2 = \frac{188}{10}ma^2$$

Using the additive law:

$$\therefore \text{ M.I. of whole timing device} = ma^2 + \frac{47}{10}ma^2 + \frac{188}{10}ma^2 = \frac{49ma^2}{2}$$

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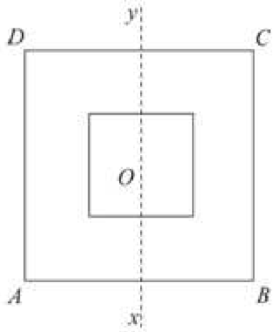
Exercise D, Question 11

Question:



A uniform lamina of mass m is formed from a square lamina $ABCD$ of side $2a$ by cutting out a square of side a. Both squares have the same centre O and their sides are parallel as shown in the figure. The points X and Y are the mid-points of AB and CD respectively.

a Find the moment of inertia of the lamina about an axis passing through X and Y.

b Hence find the radius of gyration of the lamina about an axis perpendicular to its plane passing through O. ***E***

Solution:

a Let mass per unit area be ρ.

Moment of inertia of $ABCD$ about $XY = \frac{1}{3}(\rho \times 4a^2) \times a^2$

Moment of inertia of smaller square about $XY = \frac{1}{3}(\rho \times a^2) \times \left(\frac{a}{2}\right)^2$

$\therefore$ By additive law moment of inertia of lamina $= \frac{1}{3}\rho \times 4a^4 - \frac{1}{3}\rho \times \frac{a^4}{4}$

$$= \frac{1}{3}\rho a^4\left(4 - \frac{1}{4}\right)$$

$$= \frac{5}{4}\rho a^4$$

But $m = \rho(4a^2 - a^2)$

i.e. $\rho = \frac{m}{3a^2}$

$\therefore$ M.I. $= \frac{5ma^2}{12}$

b Moment of inertia about axis perpendicular to plane

$= \frac{5ma^2}{12} + \frac{5ma^2}{12}$ (by perpendicular axes theorem)

$= \frac{5ma^2}{6}$

But M.I. $= mk^2$ where k is the radius of gyration – so $k^2 = \frac{5a^2}{6}$ and $k = \sqrt{\frac{5}{6}}a$

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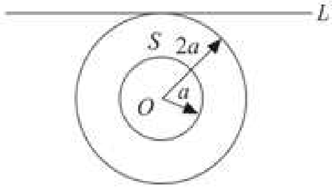
Exercise D, Question 12

Question:



A lamina S is formed from a uniform disc, centre O and radius $2a$, by removing the disc of centre O and radius a, as shown. The mass of S is M.

a Show that the moment of inertia of S about an axis through O and perpendicular to its plane is $\frac{5}{2}Ma^2$.

The lamina is free to rotate about a fixed smooth horizontal axis L. The axis L lies in the plane of S and is a tangent to its outer circumference, as shown.

b Show that the moment of inertia of S about L is $\frac{21}{4}Ma^2$. ***E*** (adapted)

Solution:

a Let the mass per unit area be ρ and let the axis through O perpendicular to the lamina be L.

The moment of inertia of the disc with radius $2a$ about L

$$= \left[\rho \cdot \pi (2a)^2\right] \times \frac{(2a)^2}{2} = 8\pi\rho a^4$$

The moment of inertia of the disc with radius a about $L = \rho\pi a^2 \times \frac{a^2}{2} = \frac{1}{2}\rho\pi a^4$

$\therefore$ Moment of inertia of lamina $S = 8\pi\rho a^4 - \frac{1}{2}\pi\rho a^4$

$$= \frac{15}{2}\pi\rho a^4 \quad ①$$

But the mass of $S = M = \pi\rho[(2a)^2 - a^2]$

$$= 3\pi\rho a^2 \quad ②$$

Substitute $\rho = \frac{M}{3\pi a^2}$ into equation ① to give M.I. $= \frac{5}{2}Ma^2$

b Let the moment of inertia of S about a diameter parallel to L be I.

The moment of inertia of S about a diameter perpendicular to L is also I.

Then $I + I = \frac{5}{2}Ma^2$ (from perpendicular axes theorem)

$$\therefore I = \frac{5}{4}Ma^2$$

The moment of inertia of S about $L = \frac{5}{4}Ma^2 + M(2a)^2$ (from the parallel axes theorem)

$\therefore$ Required moment of inertia $= \frac{5}{4}Ma^2 + 4Ma^2$

$$= \frac{21}{4}Ma^2$$

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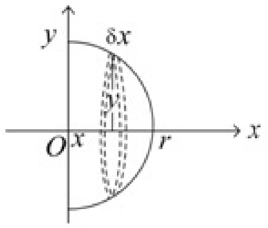
Exercise D, Question 13

Question:

Use integration to show that the radius of gyration of a uniform solid hemisphere of mass m and radius r about a diameter of the circular base is $\sqrt{\frac{2}{5}}r$. ***E***

Solution:



Let the mass per unit volume be ρ.
Divide the hemisphere up into discs of radius y, thickness δx at a distance x from the circular base.

The M.I. of the disc shown about $Ox = (\rho\pi y^2 \delta x) \times \frac{y^2}{2}$

$\therefore$ The M.I. of the disc about its diameter parallel to $Oy = (\rho\pi y^2 \delta x)\frac{y^2}{4}$

(perpendicular axes theorem)

$\therefore$ Its M.I. about $Oy = \rho\pi y^2 \delta x \frac{y^2}{4} + \rho\pi y^2 \delta x \cdot x^2$ (parallel axes theorem)

Summing all such discs and letting $\delta x \to 0$ gives I, the moment of inertia of the hemisphere.

So $I = \rho\pi \int_0^r \frac{y^4}{4} \mathrm{d}x + \rho\pi \int_0^r y^2 x^2 \ \mathrm{d}x$

But $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$$\therefore I = \rho\pi \int_0^r \frac{1}{4}\left(r^4 - 2r^2x^2 + x^4\right) + \left(r^2x^2 - x^4\right)\mathrm{d}x$$

$$= \rho\pi \left[\frac{1}{4}\left(r^4x - \frac{2}{3}r^2x^3 + \frac{1}{5}x^5\right) + \frac{1}{3}r^2x^3 - \frac{1}{5}x^5\right]_0^r$$

$$= \rho\pi \left[\frac{1}{4}r^5 - \frac{1}{6}r^5 + \frac{1}{20}r^5 + \frac{1}{3}r^5 - \frac{1}{5}r^5\right]$$

$$= \frac{\rho\pi}{60}[15 - 10 + 3 + 20 - 12]r^5$$

$$= \frac{4\rho\pi r^5}{15}$$

But the mass of the hemisphere $m = \frac{2}{3}\pi\rho r^3 \Rightarrow \rho = \frac{3m}{2\pi r^3}$

$\therefore I = \frac{2}{5}mr^2$

and the radius of gyration $k = \sqrt{\frac{2}{5}}r$

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Exercise D, Question 14

Question:

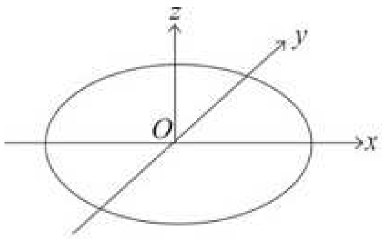
Assuming that the moment of inertia of a uniform circular disc, of mass m and radius r, about an axis through its centre and perpendicular to its plane is $\frac{1}{2}mr^2$,

a deduce that its moment of inertia about a diameter is $\frac{1}{4}mr^2$.

b Hence, using integration, show that the moment of inertia of a uniform solid circular cylinder, of mass M, radius r and height h, about a diameter of one of its plane faces is $\frac{1}{12}M(3r^2+4h^2)$. ***E***

Solution:

a



Let O be the centre of the circular disc. Take axes Ox and Oy in the plane of the disc and Oz perpendicular to the disc.

Then $I_{Oz}=\frac{1}{2}mr^2$

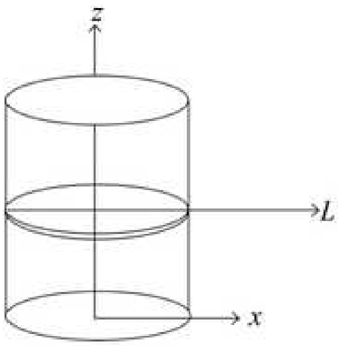
Also $I_{Ox}+I_{Oy}=I_{Oz}$ (perpendicular axes theorem)

Let $I_{Ox}=I$, then $I_{Oy}=I$ also (symmetry)

$\therefore 2I=\frac{1}{2}mr^2$

So $I=\frac{1}{4}mr^2$

b



Consider the cylinder divided up into a large number of thin discs.
Let a typical disc have radius r, thickness δz and be at a distance z from the Oxy plane.

The M.I. of this disc is $\frac{mr^2}{4}$ about its diameter in the direction L, parallel to Ox, where m is the mass of the disc.

Its M.I. about a diameter of the base of the cylinder, Ox is $\frac{mr^2}{4}+mz^2$

(by parallel axes theorem)

As the cylinder is uniform $\frac{m}{M}=\frac{\delta z}{h}$

$$\therefore m=\frac{M}{h}\delta z$$

So M.I. of cylinder about base diameter is obtained from $\sum \frac{M}{h}\left[\frac{r^2}{4}+z^2\right]\delta z$ as

$\delta z \to 0$

i.e.

$$I=\frac{M}{h}\int_0^h \frac{r^2}{4}+z^2 \,\mathrm{d}z$$

$$=\frac{M}{h}\left[\frac{r^2}{4}z+\frac{1}{3}z^3\right]_0^h=\frac{M}{h}\left[\frac{r^2h}{4}+\frac{h^3}{3}\right]$$

$$=\frac{M}{12}\left[3r^2+4h^2\right]$$

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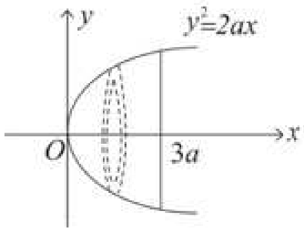
Exercise D, Question 15

Question:

You may assume, without proof, that the moment of inertia of a uniform circular disc, of mass m *and radius* r, *about an axis through its centre and perpendicular to its plane is* $\frac{1}{2}\text{mr}^2$.

A uniform solid S is generated by rotating the finite region bounded by the curve with equation $y^2 = 2ax$ and the line with equation $x = 3a$ through $180°$ about the x-axis. The volume of S is $9\pi a^3$ and its mass is M. Show, by integration, that the moment of inertia of S about its axis of symmetry is $2Ma^2$. ***E***

Solution:



Let the mass per unit volume be ρ. Divide the solid S into a large number of thin discs, perpendicular to the x-axis. A typical disc is shown. This has mass $\rho\pi y^2\delta x$ and its moment of inertia about the x-axis is $\rho\pi y^2\delta x \times \frac{y^2}{2}$

$\therefore$ By summation and letting $\delta x \to 0$ the moment of inertia of S about the axis of symmetry

$$I = \frac{\rho\pi}{2}\int_0^{3a} y^4 \, dx$$
$$= \frac{\rho\pi}{2}\int_0^{3a} (2ax)^2 \, dx$$
$$= \frac{4a^2\rho\pi}{2}\left[\frac{x^3}{3}\right]_0^{3a}$$
$$= \frac{4a^5\rho\pi \times 9}{2}$$
$$= 18a^5\rho\pi \qquad ①$$

But as the volume of S is $9\pi a^3$

$\therefore M = \rho \cdot 9\pi a^3$

$\therefore I = 2Ma^2$ by substitution into equation ①

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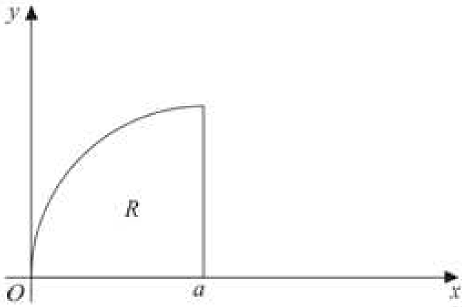
Exercise D, Question 16

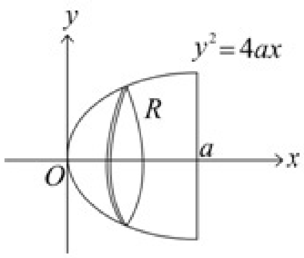
Question:

You may assume, without proof, that the moment of inertia of a uniform disc, of mass m *and radius* r*, about an axis through its centre perpendicular to its plane is* $\frac{1}{2}\text{mr}^2$.



A region R is bounded by the curve $y^2 = 4ax (y > 0)$, the x-axis and the line $x = a$ $(a > 0)$, as shown. A uniform solid S of mass M is formed by rotating R about the x-axis through $360°$. Using integration, prove that the moment of inertia of S about the x-axis is $\frac{4}{3}Ma^2$. ***E***

Solution:



Divide S into discs parallel to the circular base of the solid let the mass per unit volume be ρ.

$$\text{Then } M = \rho\int_0^a \pi y^2 \, dx$$
$$= \rho\int_0^a \pi \cdot 4ax \, dx$$
$$= \left[2\pi\rho a x^2\right]_0^a$$
$$= 2\pi\rho a^3$$
$$\therefore \rho = \frac{M}{2\pi a^3} \quad *$$

$$\text{Also } I = \rho\int_0^a \pi y^2 \times \frac{y^2}{2} \, dx$$
$$= \frac{\pi\rho}{2}\int_0^a (4ax)^2 \, dx$$
$$= 8a^2\pi\rho\left[\frac{x^3}{3}\right]_0^a$$
$$= \frac{8}{3}a^5\pi\rho$$

Substitute the value of ρ from $*$

$$\text{Then } I = \frac{4}{3}Ma^2$$

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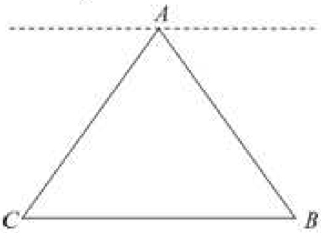
Exercise D, Question 17

Question:

a Show by integration, that the moment of inertia of a uniform rod, of length $2L$ and mass m, about an axis through the centre of the rod and inclined at an angle θ to the rod is $\frac{1}{3}mL^2\sin^2\theta$.



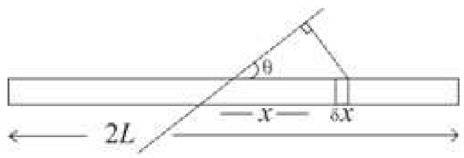
A framework in the shape of an equilateral triangle ABC is formed from three uniform rods, each of length $2L$ and mass m, as shown in the figure.

b Find the moment of inertia of the framework about an axis in the plane of the framework, parallel to BC and passing through A.

c Hence find the radius of gyration of the framework about this axis. ***E***

Solution:

a



Divide the rod into small pieces of length δx at a distance x along the rod from the middle. The perpendicular distance from the small piece shown to the axis is $x\sin\theta$, where θ is the constant given angle.

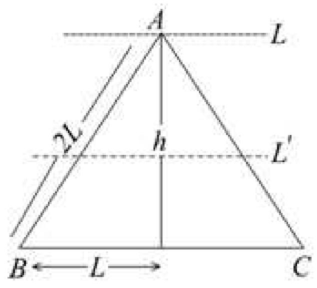
The mass of the small piece is $\frac{m}{2L}\delta x$

You obtain $I = \int_{-L}^{L}\frac{mx^2\sin^2\theta}{2L}\mathrm{d}x$

i.e. $I = \frac{m}{2L}\sin^2\theta\left[\frac{x^3}{3}\right]_{-L}^{L}$

$= \frac{mL^2\sin^2\theta}{3}$

b



Moment of inertia of AB about axis L' shown

$= \frac{mL^2 \sin^2 60^\circ}{3} = \frac{mL^2}{4}$ (from result in **a**)

Moment of inertia of AC about axis L' shown $= \frac{mL^2}{4}$ also

By parallel axis theorem, M.I. of AB about axis L = M.I. of AC about axis

$L = \frac{mL^2}{4} + m\left(\frac{h}{2}\right)^2$ *

Moment of inertia of BC about given axis $= mh^2$ where $h^2 = (2L)^2 - L^2$

(from Pythagoras' Theorem)

i.e. $h^2 = 3L^2$

So for BC moment of inertia about axis $L = 3mL^2$ and for AB and AC, each moment of inertia about L

$= \frac{mL^2}{4} + \frac{3mL^2}{4}$

$= mL^2$ (from *)

So the moment of inertia of the framework, by the additive rule,

$= mL^2 + mL^2 + 3mL^2$

$= 5mL^2$

c Let the radius of gyration of the framework be k. As its mass $= 3m$

$\therefore$ its moment of inertia $= 3mk^2 = 5mL^2$

$$\text{i.e. } k^2 = \frac{5}{3}L^2$$

$$\therefore k = \sqrt{\frac{5}{3}}L$$