Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 1

Question:

A particle P moves in a straight line. At time t seconds its displacement from a fixed point O on the line is x metres. The motion of P is modelled by the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 12\cos 2t - 6\sin 2t$.

When $t = 0, P$ is at rest at O.

a Find, in terms of t, the displacement of P from O.

b Show that P comes to instantaneous rest when $t = \frac{\pi}{4}$.

c Find, in metres to 3 significant figures, the displacement of P from O when $t = \frac{\pi}{4}$.

d Find the approximate period of the motion for large values of t. **[E]**

Solution:

a $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 12\cos 2t - 6\sin 2t$

Auxiliary equation: $m^2 + 2m + 2 = 0$

> An expression for x is needed. Refer to book FP2 Chapter 5 for the method of solving these equations.

$m = \frac{-2 \pm \sqrt{(4-8)}}{2} = -1 \pm i$

Complementary function is

$x = e^{-t}(A\cos t + B\sin t)$

Let $x = p\cos 2t + q\sin 2t$

> Try this for a particular integrat.

$\dot{x} = -2p\sin 2t + 2q\cos 2t$

$\ddot{x} = -4p\cos 2t - 4q\sin 2t$

$\therefore -4p\cos 2t - 4q\sin 2t$

$+ 2(-2p\sin 2t + 2q\cos 2t)$

$+ 2(p\cos 2t + q\sin 2t)$

$= 12\cos 2t - 6\sin 2t$

> Substitute the expressions for $x, \dot{x}$ and $\ddot{x}$ into the differential equation.

$\cos 2t(-4p + 4q + 2p)$

$+ \sin 2t(-4q - 4p + 2q)$

$= 12\cos 2t - 6\sin 2t$

$-4p + 4q + 2p = 12$

> Equate coefficients of cos $2t$...

$-2p + 4q = 12$ ①

$-4q - 4p + 2q = -6$

$-4p - 2q = -6$

> ... and of sin $2t$.

$-2p - q = -3$ ②

$5q = 15$

$q = 3, p = 0$

> Solve ① and ②.

$\therefore x = e^{-t}(A\cos t + B\sin t) + 3\sin 2t$

$t = 0, x = 0 \;\therefore 0 = A$

> Use the initial conditions given in the question to obtain values for A and B.

$\dot{x} = -e^{-t}B\sin t + e^{-t}B\cos t + 6\cos 2t$

$t = 0, \dot{x} = 0 \;\therefore 0 = B + 6$

$B = -6$

$\therefore x = 3\sin 2t - 6e^{-t}\sin t$

b $\dot{x} = 6e^{-t}\sin t - 6e^{-t}\cos t + 6\cos 2t$

$t = \frac{\pi}{4}, \dot{x} = 6\left[e^{-\frac{\pi}{4}}\sin\frac{\pi}{4} - e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} + \cos\frac{\pi}{2}\right]$

$= 0$

> $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} \quad \cos\frac{\pi}{2} = 0$

$\therefore$ P comes to instantaneous test when $t = \frac{\pi}{4}$

c $x = 3\sin 2t - 6e^{-t}\sin t$

$t = \frac{\pi}{4}$ $x = 3\sin\frac{\pi}{2} - 6e^{-\frac{\pi}{4}}\sin\frac{\pi}{4}$

$= 3 - 6e^{-\frac{\pi}{4}} \times \frac{1}{\sqrt{2}}$

$= 1.07$ (3 s.f.)

d $t \to \infty$ | Large values of t needed, so let $t \to \infty$.

$x \approx 3\sin 2t$ | $e^{-t} \to 0$ as $t \to \infty$

$\therefore$ approximate period is π

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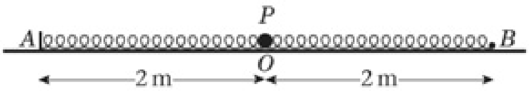
Review Exercise 2
Exercise A, Question 2

Question:



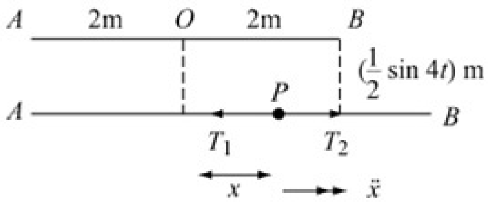
A particle P of mass 2 kg is attached to the mid-point of a light elastic spring of natural length 2 m and modulus of elasticity 4 N. One end A of the elastic spring is attached to a fixed point on a smooth horizontal table. The spring is then stretched until its length is 4 m and its other end B is held at a point on the table where $AB = 4$ m . At time $t = 0, P$ is at rest on the table at the point O where $AO = 2$ m, as shown. The end B is now moved on the table in such a way that AOB remains a straight line. At time t seconds, $AB = (4 + \frac{1}{2}\sin 4t)$ m and $AP = (2 + x)$ m.

a Show that $\frac{d^2x}{dt^2} + 4x = \sin 4t$.

b Hence find the time when P first comes to instantaneous rest. **[E]**

Solution:

a



$$T_1 = 4(1+x)$$

By Hooke's law. $AP = 2 + x \Rightarrow$ extension $= 1 + x$

$$T_2 = 4\left(1+\frac{1}{2}\sin 4t - x\right)$$

By Hooke's law. $AB = \left(4+\frac{1}{2}\sin 4t\right) \Rightarrow$ extension in PB $= \left(1+\frac{1}{2}\sin 4t - x\right)$

$$T_2 - T_1 = 2\ddot{x}$$

Using $F = ma$

$$4\left(1+\frac{1}{2}\sin 4t - x\right) - 4(1+x) = 2\ddot{x}$$

Substitute for T_1 and T_2.

$$2\sin 4t + 4 - 4x - 4 - 4x = 2\ddot{x}$$

$$\ddot{x} + 4x = \sin 4t$$

b Auxiliary equation: $m^2 + 4 = 0$

$$m = \pm 2\text{i}$$

Now solve the differential equation (refer to book FP2 Chapter 5).

$\therefore$ Complementary function:

$x = A\sin 2t + B\cos 2t$

For particular integral, try

$$x = P\sin 4t$$

$$\dot{x} = 4P\cos 4t \quad \ddot{x} = -16P\sin 4t$$

$$\therefore -16P\sin 4t + 4P\sin 4t = \sin 4t$$

Substitute in the differential equation.

$$-12P = 1$$

$$P = -\frac{1}{12}$$

$$\therefore x = A\sin 2t + B\cos 2t - \frac{1}{12}\sin 4t$$

$$t = 0, x = 0 \Rightarrow B = 0$$

Use the initial conditions given in the question to obtain values for A and B.

$$\dot{x} = 2A\cos 2t - \frac{1}{3}\cos 4t$$

$$t = 0, \dot{x} = 0 \Rightarrow 0 = 2A - \frac{1}{3}$$

$$A = \frac{1}{6}$$

When $\dot{x} = 0$

$$0 = 2\times\frac{1}{6}\cos 2t - \frac{1}{3}\cos 4t$$

$$\cos 4t = \cos 2t$$

$$\therefore 4t = 2t + 2\pi \text{ or } 2\pi - 2t$$

$$t = \pi \quad \text{or} \quad t = \frac{\pi}{3}$$

$\therefore P$ first comes to rest when $t = \frac{\pi}{3}$

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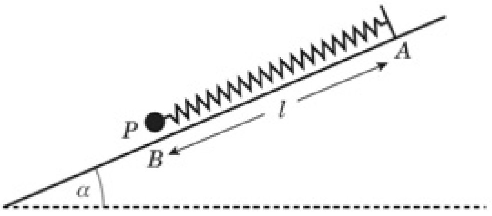
Review Exercise 2
Exercise A, Question 3

Question:



A light elastic spring has natural length l and modulus of elasticity $4mg$. One end of the spring is attached to a point A on a plane that is inclined to the horizontal at an angle α, where $\tan\alpha = \frac{3}{4}$. The other end of the spring is attached to a particle P of mass m. The plane is rough and the coefficient of friction between P and the plane is $\frac{1}{2}$. The particle P is held at a point B on the lane where B is below A and $AB = l$, with the spring lying along a line of greatest slope of the plane, as shown. At time $t = 0$, the particle is projected up the plane towards A with speed $\frac{1}{2}\sqrt{(gl)}$. At time t, the compression of the spring is x.

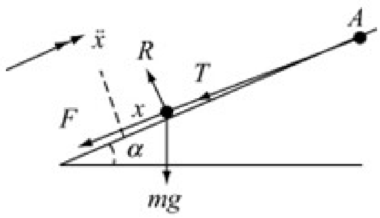
a Show that $\frac{d^2x}{dt^2} + 4\omega^2 x = -g$, where $\omega = \sqrt{\left(\frac{g}{l}\right)}$

b Find x in terms of l, ω and t.

c Find the distance that P travels up the plane before first coming to rest. **[E]**

Solution:

a



$\lambda = 4mg$

$\tan\alpha = \frac{3}{4}$

Hooke's law: $T = \frac{\lambda x}{l} = \frac{4mgx}{l}$

R($\perp$, plane): $R = mg\cos\alpha = \frac{4}{5}mg$

$F = \mu R = \frac{1}{2}\times\frac{4}{5}mg = \frac{2}{5}mg$

The magnitude of the frictional force must be obtained.

Equation of motion:

$$-F - T - mg\sin\alpha = m\ddot{x}$$

$$-\frac{2}{5}mg - \frac{4mgx}{l} - \frac{3}{5}mg = m\ddot{x}$$

$$\ddot{x} + \frac{4gx}{l} = -g$$

x is the compression in the spring. It is measured from B and increases as P travels up the plane.

Let $\frac{g}{l} = \omega^2$

This is given in the question.

$$\frac{d^2x}{dt^2} + 4\omega^2 x = -g$$

where $\omega = \sqrt{\left(\frac{g}{l}\right)}$

b $\frac{d^2x}{dt^2} + 4\omega^2 x = -g$

Now solve the differential equation using the methods of book FP2 chapter 5.

Auxiliary equation: $m^2 + 4\omega^2 = 0$

$m = \pm 2\mathrm{i}\omega$

Complementary function:

$x = A\sin 2\omega t + B\cos 2\omega t$

For the particular integral try

$x = p$

$0 + 4\omega^2 p = -g$

$$p = \frac{-g}{4\omega^2} = \frac{-g}{4}\times\frac{l}{g}$$

$$p = -\frac{l}{4}$$

$\therefore$ Complete solution is

$$x = A\sin 2\omega t + B\cos 2\omega t - \frac{l}{4}$$

$$t = 0, x = 0 \therefore 0 = B - \frac{l}{4}$$

$$B = \frac{l}{4}$$

$$\dot{x} = 2\omega A\cos 2\omega t - 2\omega B\sin 2\omega t$$

$$t = 0, \dot{x} = \frac{1}{2}\sqrt{(gl)}$$

$$\frac{1}{2}\sqrt{(gl)} = 2\omega A$$

$$A = \frac{1}{4\omega}\sqrt{(gl)}$$

$$= \frac{1}{4} \times \sqrt{\frac{l}{g}} \times \sqrt{gl} = \frac{1}{4}l$$

$$\therefore x = \frac{l}{4}(\sin 2\omega t + \cos 2\omega t - 1)$$

c $\dot{x} = 2\omega\frac{l}{4}(\cos 2\omega t - \sin 2\omega t)$

Use the expression for $\dot{x}$ obtained in **b** with the known values for A and B.

At rest $\dot{x} = 0$

$$\cos 2\omega t = \sin 2\omega t$$

$$\tan 2\omega t = 1$$

$$2\omega t = \frac{\pi}{4}$$

when $2\omega t = \frac{\pi}{4}$

You are finding the distance P travels before it first comes to rest. The value of t is not needed explicitly.

$$x = \frac{l}{4}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 1\right)$$

$$= \frac{l}{4}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{l}{4}\left(\frac{2}{\sqrt{2}} - 1\right)$$

P travels a distance $\frac{l}{4}(\sqrt{2} - 1)$ up the plane before first coming to rest.

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Review Exercise 2
Exercise A, Question 4

Question:

A particle P of mass m is suspended from a fixed point by a light elastic spring. The spring has natural length a and modulus of elasticity $2m\omega^2 a$, where ω is a positive constant. A time $t = 0$ the particle is projected vertically downwards with speed U from its equilibrium position. The motion of the particle is resisted by a force of magnitude $2m\omega v$, where v is the speed of the particle. At time t, the displacement of P downwards from its equilibrium position is x.

a Show that $\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0$.

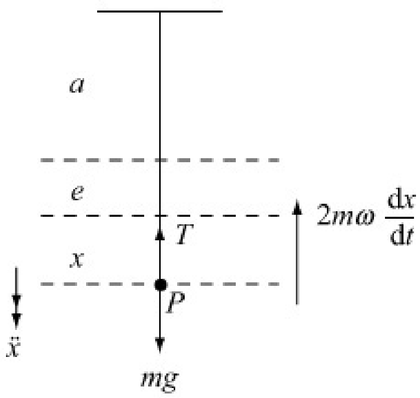
Given that the solution of this differential equation is $x = e^{-\omega t}(A\cos\omega t + B\sin\omega t)$, where A and B are constants,

b find A and B.

c Find an expression for the time at which P first comes to rest. **[E]**

Solution:

a



$l = a$

$\lambda = 2m\omega^2 a$

Hooke's law:

$$T = \frac{\lambda x}{l} = \frac{2m\omega^2 a\left(e + x\right)}{a}$$

$$F = ma$$

$$mg - T - 2m\omega\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

In equilibrium:

$$T_E = \frac{2m\omega^2 ae}{a} = mg$$

The equilibrium tension is equal to the weight of P.

$$\therefore \frac{2m\omega^2 ae}{a} - \frac{2m\omega^2 a\left(e + x\right)}{a} - 2m\omega\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -2\omega^2 x - 2\omega\frac{dx}{dt}$$

$$\therefore \frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0$$

b

$$x = e^{-\omega t}\left(A\cos\omega t + B\sin\omega t\right)$$

The general solution of the differential equation was given in the question.

$$t = 0, x = 0 \Rightarrow 0 = A$$

$$\frac{dx}{dt} = -\omega e^{-\omega t}B\sin\omega t + \omega e^{-\omega t}B\cos\omega t$$

$$t = 0, \frac{dx}{dt} = U$$

$$\rightarrow U = B\omega, B = \frac{U}{\omega}$$

$$\therefore A = 0 \text{ and } B = \frac{U}{\omega}$$

c

$$\frac{dx}{dt} = 0$$

$v = \frac{dx}{dt} = 0$ when P is at rest.

$$\Rightarrow 0 = -\omega e^{-\omega t}\frac{U}{\omega}\left(\sin\omega t - \cos\omega t\right)$$

$$\therefore \sin\omega t = \cos\omega t$$

$$\tan\omega t = 1$$

$$t = \frac{\pi}{4\omega}$$

$\therefore$ P first comes to rest when $t = \frac{\pi}{4\omega}$

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Review Exercise 2
Exercise A, Question 5

Question:

A light elastic string, of natural length $2a$ and modulus of elasticity mg, has a particle P of mass m attached to its mid-point. One end of the string is attached to a fixed point A and the other end is attached to a fixed point B which is at a distance $4a$ vertically below A.

a Show that P hangs in equilibrium at the point E where $AE = \frac{5}{2}a$.

The particle P is held at a distance $3a$ vertically below A and is released from rest at time $t = 0$. When the speed of the particle is v, there is a resistance to motion of magnitude $2mkv$, where $k = \sqrt{\left(\frac{g}{a}\right)}$.

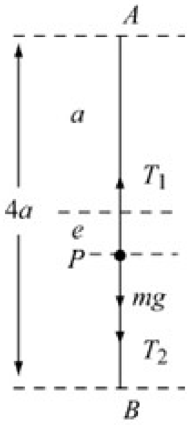
At time t the particle is at a distance $\left(\frac{5}{2}a + x\right)$ from A.

b Show that $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0$.

c Hence find x in terms of t. **[E]**

Solution:

a



$\lambda = mg$
$l = a$

Treat the two halves of the string as separate strings of natural length a.

$R(\uparrow) \quad T_1 = T_2 + mg$

Hooke's law:

$$T_1 = \frac{\lambda x}{l} = \frac{mge}{a}$$

$$T_2 = \frac{mg}{a}(2a - e)$$

The combined extensions equal $2a$.

$$\therefore \frac{mge}{a} = \frac{mg}{a}(2a - e) + mg$$

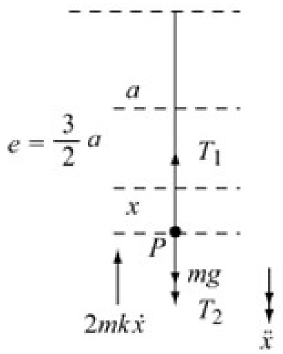
$$e = 2a - e + a$$

$$2e = 3a \quad e = \frac{3}{2}a$$

$$\therefore AE = \frac{5}{2}a$$

Remember to add the natural length of the upper 'half' string to the extension to obtain the required answer.

b



Hooke's law:

$$T_1 = \frac{mg}{a}\left(\frac{3}{2}a + x\right)$$

$$T_2 = \frac{mg}{a}\left(\frac{1}{2}a - x\right)$$

x is measured from the equilibrium level.

Equation of motion:

$$\frac{mg}{a}\left(\frac{1}{2}a - x\right) + mg - \frac{mg}{a}\left(\frac{3a}{2} + x\right) - 2mk\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Rightarrow \frac{2\not{m}g}{a}x - 2\not{m}k\frac{dx}{dt} = \not{m}\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0 \quad \text{since } k^2 = \frac{g}{a}$$

Equation of motion:

$$\frac{mg}{a}\left(\frac{1}{2}a - x\right) + mg - \frac{mg}{a}\left(\frac{3a}{2} + x\right) - 2mk\frac{dx}{dt} = m\frac{d^2x}{dt^2} \Rightarrow \frac{2\cancel{m}g}{a}x - 2\cancel{m}k\frac{dx}{dt} = \cancel{m}\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0 \quad \text{since } k^2 = \frac{g}{a}$$

c Auxiliary equation:

$$m^2 + 2km + 2k^2 = 0$$

$$m = -k \pm ki$$

Complementary function:

$$x = e^{-kt}(A\cos kt + B\sin kt)$$

$$t = 0, x = \frac{1}{2}a \Rightarrow A = \frac{1}{2}a$$

$$\frac{dx}{dt} = -ke^{-kt}(A\cos kt + B\sin kt)$$

$$+e^{-kt}(-kA\sin kt + kB\cos kt)$$

$$t = 0, \frac{dx}{dt} = 0$$

$$\Rightarrow O = -kA + kB$$

$$B = A = \frac{1}{2}a$$

$$\therefore x = \frac{1}{2}ae^{-kt}(\cos kt + \sin kt)$$

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Review Exercise 2
Exercise A, Question 6

Question:

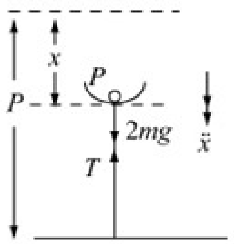
A light spring PQ is fixed at its lower end Q and is constrained to move in a vertical line. At its upper end P the spring is fixed to a small cup, of mass m, which contains a sugar lump of mass m. The spring has modulus of elasticity $8mg$ and natural length l. Given that the compression of the spring is x at time t,

a show that, while the sugar lump is in contact with the cup, $\frac{d^2x}{dt^2} + \frac{4gx}{l} = g$.

b Given that the system is released from rest when $x = \frac{3l}{4}$ and $t = 0$, show that the lump will lose contact with the cup when $t = \frac{\pi}{3}\sqrt{\frac{l}{g}}$. **[E]**

Solution:

a



$\lambda = 8mg$

Hooke's law: $T = \frac{\lambda x}{l} = \frac{8mgx}{l}$

$F = ma$

$2mg - T = 2m\ddot{x}$

$2mg - \frac{8mg}{l}x = 2m\ddot{x}$

$g - \frac{4gx}{l} = \ddot{x}$

$\frac{d^2x}{dt^2} + \frac{4gx}{l} = g$

b Auxiliary equation:

$m^2 + \frac{4g}{l} = 0$

$m = \pm 2i\sqrt{\frac{g}{l}}$

Solve the differential equation using the methods of book FP2 chapter 5.

Complementary function:

$x = A\cos\left(2\sqrt{\frac{g}{l}}t\right) + B\sin\left(2\sqrt{\frac{g}{l}}t\right)$

Particular integral: try $x = K$

$\dot{x} = 0, \ddot{x} = 0$

$\Rightarrow 4g\frac{K}{l} = g \quad K = \frac{l}{4}$

$\therefore x = A\cos\left(2\sqrt{\frac{g}{l}}t\right) + B\sin\left(2\sqrt{\frac{g}{l}}t\right) + \frac{l}{4}$

$t = 0, \quad x = \frac{3l}{4}$

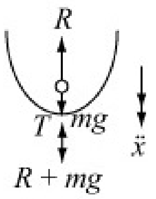
$\frac{3l}{4} = A + \frac{l}{4} \quad A = \frac{l}{2}$

$\dot{x} = 2\sqrt{\frac{g}{l}}\left[-A\sin\left(2\sqrt{\frac{g}{l}}t\right) + B\cos\left(2\sqrt{\frac{g}{l}}t\right)\right]$

$t = 0, \dot{x} = 0$

$\Rightarrow 0 = B$

$\therefore x = \frac{l}{2}\cos\left(2\sqrt{\frac{g}{l}}\,t\right) + \frac{l}{4}$



The sugar lump and the cup must be considered separately to find out when the lump leaves the cup. Be sure that you can identify the forces on each.

For sugar lump:

$F = ma$

$mg - R = m\ddot{x}$ ①

For cup:

$mg + R - T = m\ddot{x}$

$mg + R - \frac{8mgx}{l} = m\ddot{x}$ ②

When $R = 0$ — This is when the sugar lump loses contact with the cup.

$\ddot{x} = g$ — From ①

$g - \frac{8gx}{l} = g$ — From ②

$\therefore x = 0$

$x = \frac{l}{2}\cos\left(2\sqrt{\frac{g}{l}}t\right) + \frac{l}{4}$ — From the solution of the differential equation.

$x = 0 \quad \cos\left(2\sqrt{\frac{g}{l}}t\right) = -\frac{1}{2}$

$2\sqrt{\frac{g}{l}}t = \frac{2\pi}{3}$

$t = \frac{\pi}{3}\sqrt{\frac{l}{g}}$

$\therefore$ The sugar lump loses contact with the cup when $t = \frac{\pi}{3}\sqrt{\frac{l}{g}}$

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Review Exercise 2
Exercise A, Question 7

Question:

A truck is towing a trailer of mass m along a straight horizontal road by means of a tow-rope. The truck and trailer are modelled as particles and the tow-rope is modelled as a light elastic string with modulus of elasticity $4mg$ and natural length $\frac{g}{n^2}$, where n is a positive constant. The effects of friction and air resistance on the trailer are ignored. Initially the trailer is at rest and the tow-rope is slack. The truck then accelerates until the tow-rope is taut and thereafter the truck travels in a straight line with constant speed u. At time t after the tow-rope becomes taut, its extension is x, and the trailer has moved a distance y.

Show that, whilst the rope remains taut,

a $y + x = ut$,

b $\frac{d^2x}{dt^2} + 4n^2x = 0.$

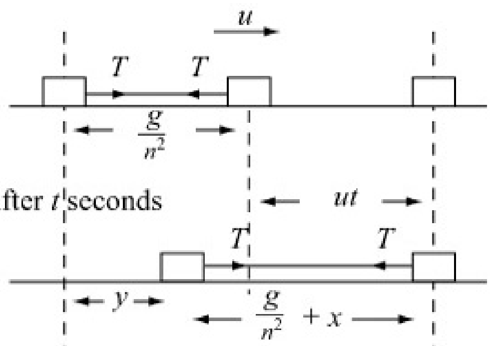
c Hence show that the tow-rope goes slack when $t = \frac{\pi}{2n}$.

d Find the speed of the trailer when $t = \frac{\pi}{3n}$.

e Find the value of t when the trailer first collides with the truck. **[E]**

Solution:

a



$\lambda = 4mg$

$l = \frac{g}{n^2}$

Draw diagrams to show the situation when the tow-rope is first taut and the situation t seconds later.

$y + \frac{g}{n^2} + x = \frac{g}{n^2} + ut$ ← Use the diagrams to equate distances.

$y + x = ut$

$y + \frac{g}{n^2} + x = \frac{g}{n^2} + ut$ ← Use the diagrams to equate distances.

$\therefore \quad y + x = ut$

b For the trailer:

$T = m\ddot{y}$ ① ← Using $F = ma$

From **a** $y + x = ut$

$\dot{y} + \dot{x} = u$

$\ddot{y} + \ddot{x} = 0$ ②

Hooke's law:

$T = \frac{\lambda x}{l}$

$T = 4mgx \times \frac{n^2}{g} = 4mn^2 x$

From ① and ②:

$T = m\ddot{y} = -m\ddot{x}$

$\therefore -m\ddot{x} = 4mn^2 x$

$\therefore \frac{d^2x}{dt^2} + 4n^2 x = 0$

c Auxiliary equation: $m^2 + 4n^2 = 0$

$m = \pm 2in$

General solution:

$x = A\cos 2nt + B\sin 2nt$

$t = 0 \;\; x = 0 \Rightarrow A = 0$

Tow-rope slack when $x = 0$

$0 = B\sin 2nt$ ← Value of B not needed here.

$\sin 2nt = 0$

$nt = \pi$

$t = \frac{\pi}{2n}$

d $\dot{x} = 2nB\cos 2nt$

$\dot{y} = u - \dot{x}$

$\therefore \dot{y} = u - 2nB\cos 2nt$

$t = 0 \quad \dot{y} = 0$

$\therefore 2nB = u$

$B = \frac{u}{2n}$

$\therefore \dot{y} = u - u\cos 2nt$

$t = \frac{\pi}{3n} \quad \dot{y} = u\left(1 - \cos\frac{2\pi}{3}\right)$

$\dot{y} = u\left(1 + \frac{1}{2}\right) = \frac{3u}{2}$

> The speed of the trailer is $\dot{y}$, not $\dot{x}$.

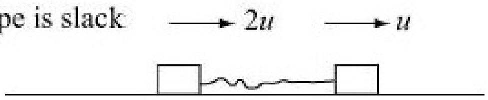
> From differentiating $y + x = ut$ (done in **b**).

e Tow-rope slack when $t = \frac{\pi}{2n}$

$\dot{y} = u - u\cos\pi = 2u$

> From **c**.

Once rope is slack



Closing speed $= 2u - u = u$

Relative distance travelled $= \frac{g}{n^2}$

> Use your knowledge of relative motion for an efficient solution!

$\therefore \text{Time} = \frac{\frac{g}{n^2}}{u} = \frac{g}{un^2}$

Total time $(t) = \frac{\pi}{2n} + \frac{g}{un^2}$

> The time from the moment the tow-rope becomes taut is required.

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Review Exercise 2
Exercise A, Question 8

Question:

Seats on a coach rest on stabilisers to enable the seats to return to their initial positions smoothly after the coach hits a bump in the road. In a mathematical model of the situation, the following assumptions are made: each stabiliser is a light elastic spring, enclosed in a viscous liquid and fixed in a vertical position; the spring exerts a force of 1.8 N for each cm by which it is extended or compressed; the seat, together with the person sitting on it, constitute a particle P attached to the upper end of the spring which is vertical, the lower end of the spring being fixed; the viscous liquid exerts a resistance to the motion of P of magnitude $240v$ N when the speed of P is v m s^{-1}.
Given that the mass of P is m kg, and the distance of P from its equilibrium position at time t seconds is x metres measured in a downwards direction,

a show that x satisfies the differential equation $m\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$.

b Show that, when P is disturbed from its equilibrium position, the resulting motion is oscillatory when $m > 80$.

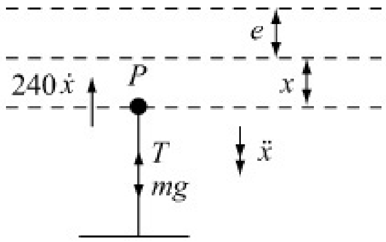
A man is sitting on the seat when the coach hits a bump in the road, giving the seat and initial upward speed of U m s^{-1}. The combined mass of the man and the seat is 80 kg.

c Find and expression for x in terms of t.

d Find the greatest displacement of the man from his equilibrium position in the subsequent motion. **[E]**

Solution:

a



When in equilibrium, $T = mg$.

Equilibrium compression $= e$ m

$\Rightarrow 1.8 \times 100e = mg$

When x m below the equilibrium level,

Be careful about the units. The tension / thrust is 1.8 N for each *cm* of extension or compression.

$$T = 1.8(e + x) \times 100$$
$$= mg + 1.8 \times 100x$$
$$F = ma$$
$$mg - T - 240\dot{x} = m\ddot{x}$$
$$mg - mg - 1.8 \times 100x - 240\dot{x} = m\ddot{x}$$
$$\therefore m\ddot{x} + 240\dot{x} + 180x = 0$$
$$m\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$$

b The motion is oscillatory if the auxiliary equation has complex roots i.e. $240^2 < 4m \times 180$

$$m > \frac{240^2}{4 \times 180}$$
$$m > 80$$

c $m = 80$

$$80\frac{d^2x}{dt^2} + 240\frac{dx}{dt} + 180x = 0$$
$$4\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 9x = 0$$

Auxiliary equation: $4m^2 + 12m + 9 = 0$

$$(2m + 3)^2 = 0$$
$$m = \frac{-3}{2} \quad \text{(twice)}$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

General solution:

$$x = (A + Bt)e^{-\frac{3}{2}t} + Be^{-\frac{3}{2}t}$$

$t = 0$ $x = 0 \Rightarrow 0 = A$

$$x = Bt\, e^{-\frac{3}{2}t}$$

$$\dot{x} = -\frac{3}{2}Bte^{-\frac{3}{2}t}$$

$t = 0$ $\dot{x} = -U \Rightarrow -U = B$

$$\therefore x = -Ute^{-\frac{3}{2}t}$$

d Maximum displacement $\Rightarrow \dot{x} = 0$

$$\therefore 0 = \frac{3}{2}Ute^{-\frac{3}{2}t} - Ue^{-\frac{3}{2}t}$$

$$Ue^{-\frac{3}{2}t}\left(\frac{3}{2}t - 1\right) = 0$$

$$t = \frac{2}{3}$$

$$\therefore x_{\max} = -U \times \frac{2}{3}e^{-1}$$

i.e. maximum displacement of the man is $\frac{2U}{3e}$

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Review Exercise 2
Exercise A, Question 9

Question:

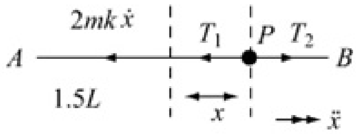
A particle P of mass m is attached to the mid-point of a light elastic string, of natural length $2L$ and modulus of elasticity $2mk^2L$, where k is a positive constant. The ends of the string are attached to points A and B on a smooth horizontal surface, where $AB = 3L$. The particle is released from rest at the point C, where $AC = 2L$ and ACB is a straight line. During the subsequent motion P experiences air resistance of magnitude $2mkv$, where v is the speed of P. At time t, $AP = 1.5L + x$.

a Show that $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4k^2x = 0$.

b Find an expression, in terms of t, k and L, for the distance AP at time t. **[E]**

Solution:

a



$\lambda = 2mk^2L$

Hooke's law:

$$T_1 = \frac{\lambda x}{l} = \frac{2mk^2L(0.5L+x)}{L}$$
$$= 2mk^2(0.5L+x)$$
$$T_2 = 2mk^2(0.5L-x)$$

$$F = ma$$
$$T_2 - T_1 - 2mk\dot{x} = m\ddot{x}$$
$$2mk^2(0.5L-x) - 2mk^2(0.5L+x) - 2mk\dot{x} = m\ddot{x}$$
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 4k^2x = 0$$

b Auxiliary equation: $m^2 + 2km + 4k^2 = 0$

$$m = \frac{-2k \pm \sqrt{(4k^2 - 16k^2)}}{2}$$
$$= -k \pm ki\sqrt{3}$$

General solution:

$$x = e^{-kt}(A\cos k\sqrt{3}t + B\sin k\sqrt{3}t)$$
$$t = 0, x = \frac{1}{2}L \Rightarrow A = \frac{1}{2}L$$
$$\dot{x} = -ke^{-kt}(A\cos k\sqrt{3}t + B\sin k\sqrt{3}t) + e^{-kt}(-k\sqrt{3}A\sin k\sqrt{3}t + k\sqrt{3}B\cos k\sqrt{3}t)$$
$$t = 0, \dot{x} = 0 \Rightarrow -kA + k\sqrt{3}B = 0$$
$$B = \frac{A}{\sqrt{3}} = \frac{1}{2\sqrt{3}}L$$
$$AP = 1.5L + x$$

Length AP is needed, not just x.

$$AP = 1.5L + e^{-kt}\left(\frac{1}{2}L\cos k\sqrt{3}t + \frac{1L}{2\sqrt{3}}\sin k\sqrt{3}t\right)$$
$$AP = 1.5L + \frac{Le^{-kt}}{2\sqrt{3}}(\sqrt{3}\cos k\sqrt{3}t + \sin k\sqrt{3}t)$$

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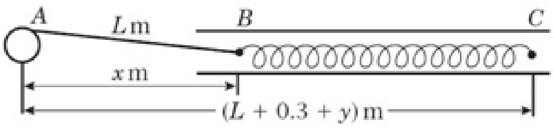
Review Exercise 2
Exercise A, Question 10

Question:



The diagram shows a sketch of a machine component consisting of a long rod AB of length L m. The end A is attached to the circumference of a flywheel centre O, radius 0.2 m, which rotates with constant angular speed 10 rad s^{-1}. The other end B is attached to a ring constrained to move in a smooth horizontal tube. The length of the rod is very much greater than the radius of the flywheel and it may be assumed that, at time t seconds, the distance x m, of B from O is given by the equation

$x = L + 0.2\cos 10t.$

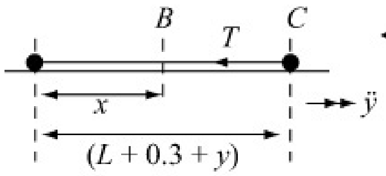
Attached to B is a light spring BC of modulus 3.75 N and natural length 0.3 m, at the other end of which is a particle C of mass 0.5 kg which is also constrained to move in the tube. When $t = 0$, the flywheel starts to rotate with B and C at rest and with the spring BC unextended.

a Show that, if the distance $OC = (L + 0.3 + y)$ m, then y satisfies the differential equation $\frac{d^2x}{dt^2} + 25y = 5\cos 10t.$

b Find an expression for y in terms of t. **[E]**

Solution:

a



Draw a diagram showing all the information. Be particularly careful about the lengths.

Length of spring $= L + 0.3 + y - x$

$= L + 0.3 + y - (L + 0.2\cos 10t)$

$= 0.3 + y - 0.2\cos 10t$

$\therefore$ Extension $= y - 0.2\cos 10t$

Use the lengths in the diagram to obtain an expression for the extension.

Hooke's law:

$$T = \frac{\lambda x}{l} = \frac{3.75}{0.3}(y - 0.2\cos 10t)$$

$$T = 12.5(y - 0.2\cos 10t)$$

Consider particle C:

$$F = ma$$

$$-T = 0.5\ddot{y}$$

$$0.5\ddot{y} = -12.5y + 12.5 \times 0.2\cos 10t$$

$$\ddot{y} = -25y + 25 \times 0.2\cos 10t$$

$$\frac{d^2y}{dt^2} + 25y = 5\cos 10t$$

d Auxiliary equation: $m^2 + 25 = 0$

$m = \pm 5i$

Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$y = A\cos 5t + B\sin 5t$

Particular integral:

try $y = p\cos 10t + q\sin 10t$

$\dot{y} = -10p\sin 10t + 10q\cos 10t$

$\ddot{y} = -100p\cos 10t - 100q\sin 10t$

$\therefore -100p\cos 10t - 100q\sin 10t + 25(p\cos 10t + q\sin 10t) = 5\cos 10t$

$-75p\cos 10t - 75q\sin 10t = 5\cos 10t$

$$\Rightarrow -75p = 5 \quad p = -\frac{1}{15}$$

$$q = 0$$

Complete solution:

$$y = A\cos 5t + B\sin 5t - \frac{1}{15}\cos 10t$$

From **a**. extension $= y - 0.2\cos 10t$ and when $t = 0$, extension $= 0$.

$$t = 0\ y = 0.2 \Rightarrow 0.2 = A - \frac{1}{15}$$

$$A = \frac{1}{5} + \frac{1}{15} = \frac{4}{15}$$

$$\dot{y} = -5A\sin 5t + 5B\cos 5t + \frac{10}{15}\sin 10t$$

$$t = 0,\ \dot{y} = 0 \Rightarrow 0 = 5B, B = 0$$

$$\therefore y = \frac{4}{15}\cos 5t - \frac{1}{15}\cos 10t$$

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Review Exercise 2
Exercise A, Question 11

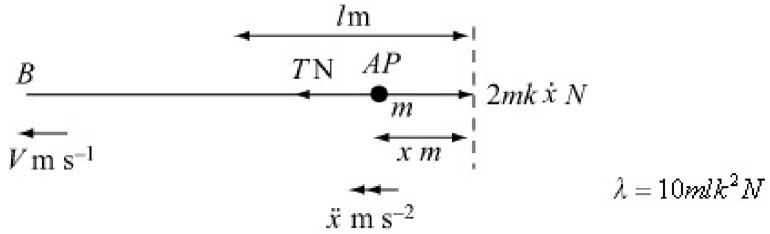
Question:

A particle P of mass m kg can move on a smooth horizontal table. It is attached to one end A of an elastic string AB, whose natural length is l metres, and whose modulus of elasticity is $10mlk^2$ newtons, where k is a positive constant. The string and particle are lying in equilibrium on the table, with $AB = l$ metres. At time $t = 0$, the end B of the string is forced to move horizontally with speed V m s^{-1} in the line of BA and in a direction away from P. The end B is forced to maintain this constant speed throughout the subsequent motion. As P moves, it experiences air resistance of magnitude $2mkv$ newtons, where v m s^{-1} is the speed of P. After t seconds, the distance of P from its initial position is x metres.
By considering the extension of the string at time t,

a show that x satisfies the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 10k^2x = 10k^2Vt$.

b Find an expression for x in terms of t, k and V. **[E]**

Solution:



a At time t:

B has moved Vt m

P has moved x m

$\therefore$ length $AB = (Vt + l - x)$ m

$\therefore$ extension $= (Vt - x)$ m

Hooke's law $\quad T = \dfrac{\lambda x}{l}$

$$T = 10mlk^2 \frac{(Vt - x)}{l}$$

$$T = 10mk^2 (Vt - x)$$

For P: $\quad F = ma$

$$T - 2mk\dot{x} = m\ddot{x}$$

$$10mk^2 (Vt - x) - 2mk\dot{x} = m\ddot{x}$$

$$\frac{d^2 x}{dt^2} + 2k\frac{dx}{dt} + 10k^2 x = 10k^2 Vt$$

b Auxiliary equation: $m^2 + 2km + 10k^2 = 0$

$$m = \frac{-2k \pm \sqrt{(4k^2 - 40k^2)}}{2}$$

$$m = -k \pm 3k\text{i}$$

Use the methods of book FP2 chapter 5 to solve the differential equation.

Complementary function:

$x = e^{-kt}(A\cos 3kt + B\sin 3kt)$

Particular integral: $\quad x = at + b$

$$\Rightarrow \dot{x} = a, \ddot{x} = 0$$

$$\therefore 2ka + 10k^2(at + b) = 10k^2 Vt$$

$$2ka + 10k^2 b = 0$$

Equating constant terms.

$$10k^2 a = 10k^2 V$$

Equating coefficients of t.

$$\Rightarrow a = V$$

$$10k^2 b = -2kV$$

$$b = \frac{-V}{5k}$$

$\therefore$ Complete solution is

$$x = e^{-kt}(A\cos 3kt + B\sin 3kt) + Vt - \frac{V}{5k}$$

$$t = 0,\ x = 0 \Rightarrow 0 = A - \frac{V}{5k}$$

$$A = \frac{V}{5k}$$

$$\dot{x} = -ek^{-kt}\left(A\cos 3kt + B\sin 3kt\right) + e^{-kt}\left(-3kA\sin 3kt + 3kB\cos 3kt\right) + V$$

$$t = 0, \dot{x} = 0 \Rightarrow 0 = -kA + 3kB + V$$

$$3kB = V - \frac{V}{5}$$

$$B = \frac{-4V}{15k}$$

$$\therefore x = e^{-kt}\left(\frac{V}{5k}\cos 3kt - \frac{4V}{15k}\sin 3kt\right) + Vt - \frac{V}{5k}$$

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Review Exercise 2
Exercise A, Question 12

Question:

A particle P of mass m is attached to one end A of a light elastic string AB, of natural length l and modulus of elasticity mln^2, where n is a constant. The string is lying at rest on a smooth horizontal table, with $AB = l$. At time $t = 0$, the end B is forced to move with constant acceleration f in the direction AB away from A. After time t, the distance of P from its initial position is y, and the extension of the string is x.

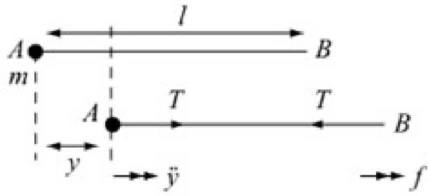
a By finding a relationship between x, y, f and t, show that, while the string remains taut, $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + n^2 x = f$.

b Hence express x and y as functions of t.

c Find the speed of P when the string is at its natural length for the first time in the ensuing motion.

d Show that the string never becomes slack. **[E]**

Solution:



$\lambda = mln^2$

Draw diagrams showing the initial situation and the situation after time t.

a At time t, distance moved by B

$= \frac{1}{2}ft^2$

$\Rightarrow$ length $AB = l + \frac{1}{2}ft^2 - y$

$\therefore$ extension $= x = \frac{1}{2}ft^2 - y$ ①

Use the lengths in the diagram to obtain the suggested relationship.

Consider particle P: $F = ma$

$T = m\ddot{y}$ ②

From ①

$y = \frac{1}{2}ft^2 - x$

$\dot{y} = ft - \dot{x}$

$\ddot{y} = f - \ddot{x}$

T will be a function of the extension, x, so use ① to obtain $\ddot{y}$ in terms of $\ddot{x}$.

Hooke's law:

$T = \frac{\lambda x}{l}$

$T = \frac{mln^2 x}{l} = mn^2 x$

In ②

$mn^2 x = m(f - \ddot{x})$

$\ddot{x} + n^2 x = f$

$\frac{d^2 x}{dt^2} + n^2 x = f$

b Auxiliary equation: $m^2 + n^2 = 0$

$m = \pm in$ ← Solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = A\cos nt + B\sin nt$

Particular integral: try $x = k$

$\dot{x} = \ddot{x} = 0$

$\Rightarrow n^2 k = f$

$k = \frac{f}{n^2}$

Complete solution is

$x = A\cos nt + B\sin nt + \frac{f}{n^2}$

$t = 0, x = 0 \Rightarrow 0 = A + \frac{f}{n^2}$

$A = \frac{-f}{n^2}$

$\dot{x} = -An\sin nt + Bn\cos nt$

$t = 0, \dot{x} = 0 \Rightarrow 0 = -Bn, \ B = 0$

$\therefore x = -\frac{f}{n^2}\cos nt + \frac{f}{n^2}$

and $y = \frac{1}{2}ft^2 - x$

$y = \frac{1}{2}ft^2 + \frac{f}{n^2}\cos nt - \frac{f}{n^2}$

c $x = 0 \Rightarrow \cos nt = 1$ ← Extension is zero when the string is at its natural length.

$nt = 0, 2\pi$

↙

at start

$\dot{y} = \frac{1}{2}ft \times 2 - \frac{f}{n^2} \times n\sin nt$ ← $\dot{x}$ is the rate of increase of the extension. The speed of P is $\dot{y}$.

$t = \frac{2\pi}{n}$ $\quad \dot{y} = f \times \frac{2\pi}{n} - \frac{f}{n}\sin 2\pi$

$\dot{y} = \frac{2f\pi}{n} - 0$

The speed of P is $\frac{2f\pi}{n}$

d extension $= x = -\frac{f}{n^2}\cos nt + \frac{f}{n^2}$

$-1 \le \cos nt \le 1$

$\Rightarrow x = \frac{f}{n^2}(1 - \cos nt)$ is never negative

$\therefore$ string never becomes slack.

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Review Exercise 2
Exercise A, Question 13

Question:

A particle P of mass 0.2 kg is attached to one end of a light elastic string of natural length 0.6 m and modulus of elasticity 0.96 N. The other end of the string is fixed to a point which is 0.6 m above the surface of a liquid. The particle is held on the surface of the liquid, with the string vertical, and then released from rest. The liquid exerts a constant upward force on P of magnitude 1.48 N, and also a resistive force of magnitude $1.2v$ N, when the speed of P is v m s^{-1}. At time t seconds, the distance travelled down by P is x metres.

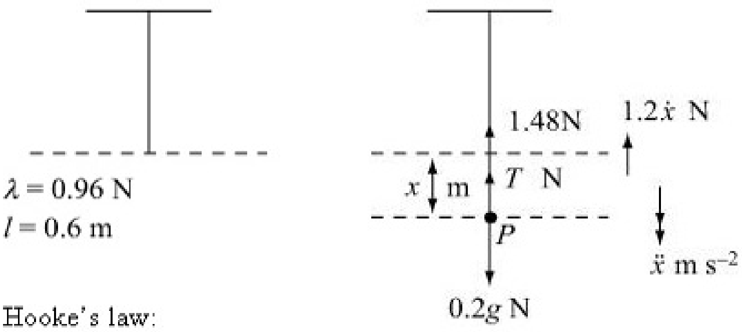
a Show that, during the time when P is moving downwards, $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 2.4$.

b Find x in terms of t.

c Show that the particle continues to move down through the liquid throughout the motion. **[E]**

Solution:

a



Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{0.96x}{0.6} = 1.6x$$

$$F = ma$$

$$0.2g - 1.6x - 1.48 - 1.2\dot{x} = 0.2\ddot{x}$$

$$\ddot{x} = 9.8 - 8x - 7.4 - 6\dot{x}$$

Use $g = 9.8$ m s^{-2}.

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 2.4$$

b Auxiliary equation:

$$m^2 + 6m + 8 = 0$$

$$(m+4)(m+2) = 0$$

$$m = -4 \quad \text{or} \quad m = -2$$

Complementary function:

$$x = Ae^{-4t} + Be^{-2t}$$

Particular integral:

$$x = a$$

$$\dot{x} = \ddot{x} = 0$$

$$\Rightarrow 8a = 2.4$$

$$a = 0.3$$

$$\therefore x = Ae^{-4t} + Be^{-2t} + 0.3$$

$$t = 0\ x = 0 \Rightarrow 0 = A + B + 0.3 \quad ①$$

$$\dot{x} = -4Ae^{-4t} - 2Be^{-2t}$$

$$t = 0\ \dot{x} = 0 \Rightarrow 0 = -4A - 2B$$

$$2A = -B \quad ②$$

$$\therefore 0 = A - 2A + 0.3$$

Solve ① and ② simultaneously.

$$A = 0.3, B = -0.6$$

$$\therefore x = 0.3e^{-4t} - 0.6e^{-2t} + 0.3$$

c

$$\dot{x} = -1.2e^{-4t} + 1.2e^{-2t}$$

$$= 1.2e^{-4t}\left(e^{2t} - 1\right)$$

For P to move downwards throughout the motion $\dot{x}$ must always be positive for

$e^{2t} > 1$ for all $t > 0$

$\therefore \dot{x} > 0$ throughout the motion (except for $t = 0$)

i.e. the particle continues to move down through the liquid throughout the motion.

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Review Exercise 2
Exercise A, Question 14

Question:

A particle P of mass m is attached to one end of a light elastic string, of natural length a and modulus of elasticity $2mak^2$, where k is a positive constant. The other end of the string is attached to a fixed point A. At time $t = 0$, P is released from rest from a point which is a distance $2a$ vertically below A. When P is moving with speed v, the air resistance has magnitude $2mkv$. At time t, the extension of the string is x.

a Show that, while the string is taut, $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = g$.

You are given that the general solution of this differential equation is

$x = e^{-kt}(C\sin kt + D\cos kt) + \frac{g}{2k^2}$, where C and D are constants.

b Find the value of C and the value of D.

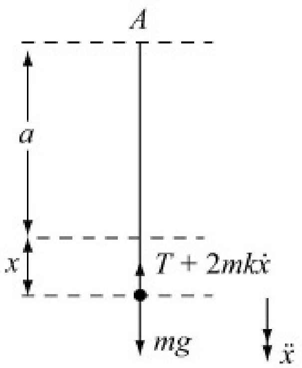
Assuming that the string remains taut,

c find the value of t when P first comes to rest,

d show that $2k^2a < g(1 + e^{\pi})$. **[E]**

Solution:

a



$l = a$

$\lambda = 2mak^2$

$F = ma$

$mg - T - 2mk\dot{x} = m\ddot{x}$

Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{2mak^2 x}{a}$$

$\therefore mg - 2mk^2 x - 2mk\dot{x} = m\ddot{x}$

$$\frac{d^2 x}{dt^2} + 2k\frac{dx}{dt} + 2k^2 x = g$$

b $x = e^{-kt}(C\sin kt + D\cos kt) + \dfrac{g}{2k^2}$ ← Given in the question.

$t = 0, x = a \Rightarrow a = D + \dfrac{g}{2k^2}$

$\therefore D = a - \dfrac{g}{2k^2}$

$\dot{x} = -ke^{-kt}(C\sin kt + D\cos kt)$
$\quad + e^{-kt}(Ck\cos kt - Dk\sin kt)$

$t = 0,\ \dot{x} = 0 \Rightarrow 0 = -kD + kC$

$\therefore C = D$

$\therefore C = D = a - \dfrac{g}{2k^2}$

c $\dot{x} = 0$

$\therefore -Ck\sin kt - Dk\cos kt$
$+Ck\cos kt - Dk\sin kt = 0$ ← Avoid substituting for C and D unless it becomes unavoidable.

$C = D \Rightarrow \sin kt = 0$

$kt = \pi$

$t = \dfrac{\pi}{k}$ ← Only the first non-zero value is required.

P first comes to rest when $t = \dfrac{\pi}{k}$.

d When $t = \frac{\pi}{k}$

$$x = e^{-\pi} \times D\cos\pi + \frac{g}{2k^2}$$

$$x = -De^{-\pi} + \frac{g}{2k^2}$$

$$xe^{\pi} = \frac{g}{2k^2}e^{\pi} - \left(a - \frac{g}{2k^2}\right)$$

The expression for D must be used now.

$$xe^{\pi} = \frac{g}{2k^2}\left(e^{\pi} + 1\right) - a$$

$$x > 0 \Rightarrow g\left(e^{\pi} + 1\right) > 2k^2 a$$

String remains taut so $x > 0$.

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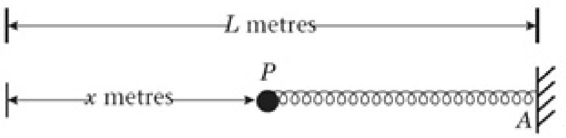
Review Exercise 2
Exercise A, Question 15

Question:



In a simple model of a shock absorber, a particle P of mass m kg is attached to one end of a light elastic horizontal spring. The other end of the spring is fixed at A and the motion of P takes place along a fixed horizontal line through A. The spring has natural length L metres and modulus of elasticity $2mL$ newtons. The whole system is immersed in a fluid which exerts a resistance on P of magnitude $3mv$ newtons, where v m s^{-1} is the speed of P at time t seconds. The compression of the spring at time t seconds is x metres, as shown in the diagram.

a Show that $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$.

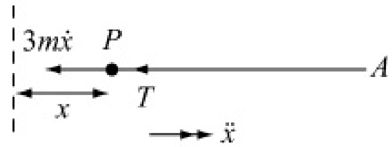
Given that when $t = 0, x = 2$ and $\frac{dx}{dt} = -4$,

b find x in terms of t.

c Sketch the graph of x against t.

d State, with a reason, whether the model is realistic. **[E]**

Solution:



$\lambda = 2mL$
$l = L$

The spring is compressed so it exerts a thrust on P.

a Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{2mLx}{L} = 2mx$$

$F = ma$:

$$-T - 3m\dot{x} = m\ddot{x}$$

$$m\ddot{x} + 2mx + 3m\dot{x} = 0$$

$$\ddot{x} + 3\dot{x} + 2x = 0$$

or $$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

b Auxiliary equation:

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, -2$$

Now solve the differential equation using the methods of book FP2. Chapter 5.

General solution:

$$x = Ae^{-t} + Be^{-2t}$$

$t = 0$, $x = 2 \Rightarrow 2 = A + B$ ①

$$\dot{x} = -Ae^{-t} - 2Be^{-2t}$$

$t = 0$, $\dot{x} = -4$ $\quad -4 = -A - 2B$

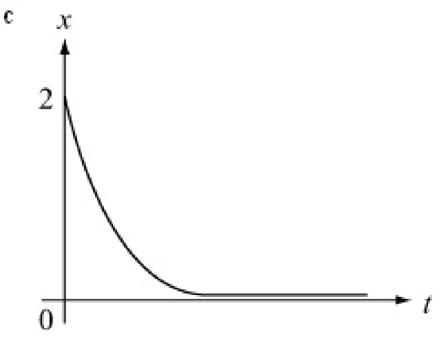
$4 = A + 2B$ ②

$\therefore 2 = B, A = 0$

$\therefore x = 2e^{-2t}$

Solving equations ① and ② simultaneously.

c



Remember t must be on the horizontal axis and you only draw the part of the curve for which $t \geq 0$.

d The model is not realistic as $\dot{x} = -4e^{-2t}$ and so P is always moving.
($\dot{x} = -4e^{-2t}$ is never zero.)

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Review Exercise 2
Exercise A, Question 16

Question:

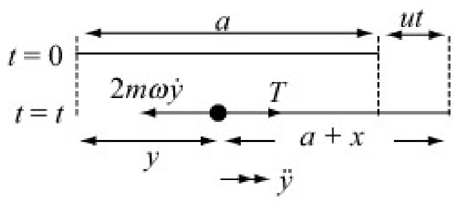
A light elastic spring, of natural length a and modulus of elasticity $5ma\omega^2$, lies unstretched along a straight line on a smooth horizontal plane. A particle of mass m is attached to one end of the spring. At time $t = 0$, the other end of the spring starts to move with constant speed U along the line of the spring and away from the particle. As the particle moves along the plane it is subject to a resistance of magnitude $2m\omega v$, where v is its speed. At time t, the extension of the spring is x and the displacement of the particle from its initial position is y. Show that

a $x + y = Ut$,

b $\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 5\omega^2 x = 2\omega U$.

c Find x in terms of ω, U and t. **[E]**

Solution:



Draw a diagram to show clearly the situation when $t = 0$ and at time t.

$\lambda = 5ma\omega^2$

$l = a$

a $a + Ut = y + a + x$

$\Rightarrow \quad x + y = Ut$

Use your diagram to establish the required equation connecting x and y.

b Hooke's law:

$$T = \frac{\lambda x}{l}$$

$$T = \frac{5ma\omega^2}{a}x$$

$$T = 5m\omega^2 x$$

$$F = ma$$

$$T - 2m\omega\dot{y} = m\ddot{y}$$

$$5m\omega^2 x - 2m\omega\dot{y} = m\ddot{y}$$

Using:

$$x + y = Ut$$

$$\dot{x} + \dot{y} = U$$

and $\ddot{x} + \ddot{y} = 0$

You need to eliminate $\dot{y}$ and $\ddot{y}$ from the equation of motion.

$$\therefore 5\omega^2 x - 2\omega(U - \dot{x}) = -\ddot{x}$$

$$\ddot{x} + 2\omega\dot{x} + 5\omega^2 x = 2\omega U$$

or $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\omega\dfrac{\mathrm{d}x}{\mathrm{d}t} + 5\omega^2 x = 2\omega U$

c Auxiliary equation:

$m^2 + 2m\omega + 5\omega^2 = 0$

$$m = \frac{-2\omega \pm \sqrt{(4\omega^2 - 20\omega^2)}}{2}$$

$$m = -\omega \pm 2\mathrm{i}\,\omega$$

Now solve the differential equation using the methods of book FP2 Chapter 5.

Complementary function:

$x = \mathrm{e}^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t)$

Particular integral:

try $x = k$

$$\dot{x} = \ddot{x} = 0$$

$$\therefore 5\omega^2 k = 2\omega U$$

$$k = \frac{2U}{5\omega}$$

Complete solution:

$$x = e^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t) + \frac{2U}{5\omega}$$

$$t = 0\ x = 0 \Rightarrow 0 = A + \frac{2U}{5\omega}$$

$$A = -\frac{2U}{5\omega}$$

$$\dot{x} = -\omega e^{-\omega t}(A\cos 2\omega t + B\sin 2\omega t) + e^{-\omega t}(-2\omega A\sin \omega t + 2\omega B\cos \omega t)$$

$$t = 0\ \dot{y} = 0 \Rightarrow \dot{x} = U$$

$$\therefore U = -\omega A + 2\omega B$$

$$U = -\omega \times \left(-\frac{2u}{5\omega}\right) + 2\omega B$$

When $t = 0$ P is at rest and $\dot{y}$ is the speed of P.

$$2\omega B = U - \frac{2}{5}U$$

$$B = \frac{3U}{10\omega}$$

$$\therefore x = e^{-\omega t}\left(\frac{3U}{10\omega}\sin 2\omega t - \frac{2U}{5\omega}\cos 2\omega t\right) + \frac{2U}{5\omega}$$

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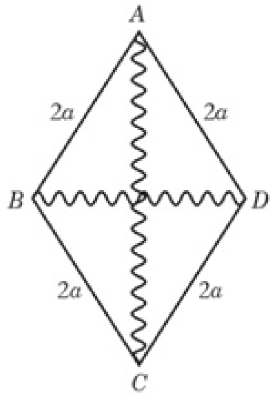
Review Exercise 2
Exercise A, Question 17

Question:



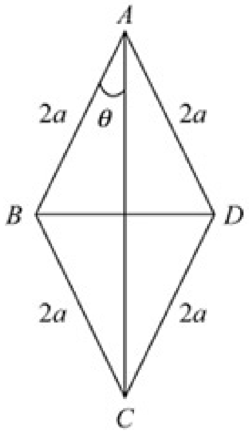
$ABCD$ is a rhombus consisting of four freely jointed uniform rods, each of mass m and length $2a$. The rhombus is freely suspended from A and is prevented from collapsing by two light springs, each of natural length a and modulus of elasticity $2mg$. One spring joins A and C and the other joins B and D, as shown in the diagram.

a Show that when AB makes an angle θ with the downward vertical, the potential energy V of the system is given by $V = -8mga(\sin\theta + 2\cos\theta) + \text{constant}$.

b Hence find the value of θ, in degrees to one decimal place, for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



a length $BD = 2 \times 2a \sin\theta$

$\therefore$ Energy in $BD = \frac{1}{2}\frac{\lambda x^2}{l}$

$= \frac{1}{2} \times \frac{2mg}{a}(4a\sin\theta - a)^2$

$= mga(4\sin\theta - 1)^2$

BD is an elastic spring and so the elastic potential energy must be found.

length $AC = 2 \times 2a\cos\theta$

$\therefore$ Energy in $AC = mga(4\cos\theta - 1)^2$

AC is an identical elastic spring. Use the work done above to write down the E.P.E.

Gravitational P.E. of rhombus
$= -4mg \times 2a\cos\theta$

Take A as the zero level as A is fixed.

$\therefore V = mga(4\sin\theta - 1)^2 + mga(4\cos\theta - 1)^2$
$-8mga\cos\theta + \text{constant}$

V is the sum of the potential energies found above. Including a 'constant' removes the need to specify a zero level.

$V = mga(16\sin^2\theta - 8\sin\theta + 1 + 16\cos^2\theta - 8\cos\theta + 1)$
$-8mga\cos\theta + \text{constant}$

$V = mga(18 - 8\sin\theta - 8\cos\theta)$
$-8mga\cos\theta + \text{constant}$

Use $\sin^2\theta + \cos^2\theta = 1$

$V = -mga(8\sin\theta + 16\cos\theta) + \text{constant}$

$V = -8mga(\sin\theta + 2\cos\theta) + \text{constant}$

$18mga$ can be absorbed into the 'constant'.

b $\frac{dV}{d\theta} = -8mga(\cos\theta - 2\sin\theta)$

$\frac{dV}{d\theta} = 0 \Rightarrow \cos\theta = 2\sin\theta$

$\tan\theta = \frac{1}{2}$

$\theta = 26.6°$

Equilibrium occurs when V has a maximum or minimum value.

c $\frac{d^2V}{d\theta^2} = -8mga(-\sin\theta - 2\cos\theta)$

$\frac{d^2V}{d\theta^2} = 8mga(\sin\theta + 2\cos\theta)$

Determine whether V is maximum or minimum when $\theta = 26.6°$

When $\theta = 26.6°$, $\frac{d^2V}{d\theta^2} > 0$

θ is acute so $\frac{d^2V}{d\theta^2}$ is positive. There is no need to evaluate $\frac{d^2V}{d\theta^2}$.

$\Rightarrow$ Equilibrium is stable

V has a minimum value when $\theta = 26.6°$ so equilibrium is stable.

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Review Exercise 2
Exercise A, Question 18

Question:

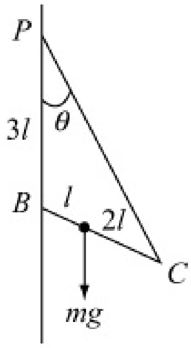
A non-uniform rod BC has mass m and length $3l$. The centre of mass of the rod is at distance l from B. The rod can turn freely about a fixed smooth horizontal axis through B. One end of a light elastic string, of natural length l and modulus of elasticity $\frac{mg}{6}$, is attached to C. The other end of the string is attached to a point P which is at a height $3l$ vertically above B.

a Show that, while the string is stretched, the potential energy of the system is $mgl(\cos^2\theta - \cos\theta) + \text{constant}$, where θ is the angle between the string and the downward vertical and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

b Find the values of θ for which the system is in equilibrium with the string stretched. **[E]**

Solution:

a



$$\lambda = \frac{mg}{6}$$

natural length $= l$

length $PC = 2 \times 3l \cos\theta$ ← ΔPBC is isosceles.

$$\text{E.P.E. in } PC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2} \times \frac{mg}{6l}(6l\cos\theta - l)^2$$

$$= \frac{mgl}{12}(6\cos\theta - 1)^2$$

$$\text{G.P.E. of rod} = -mgl\cos 2\theta$$

← Take B as the zero level as B is fixed. ΔPBC is isosceles, so the angle between BC and the downward vertical is 2θ.

$$\therefore V = \frac{mgl}{12}\left(36\cos^2\theta - 12\cos\theta + 1\right)$$

$$-mgl\cos 2\theta + \text{constant}$$

$$V = mgl\left(3\cos^2\theta - \cos\theta + 1\right)$$

$$-mgl\left(2\cos^2\theta - 1\right) + \text{constant}$$

← The required answer does not contain 2θ, so use $\cos 2\theta = 2\cos^2\theta - 1$ to change to $\cos^2\theta$.

$$V = mgl\left(3\cos^2\theta - \cos\theta - 2\cos^2\theta\right)$$

$$+mgl + mgl + \text{constant}$$

$$V = mgl\left(\cos^2\theta - \cos\theta\right) + \text{constant}$$

← $2mgl$ can be absorbed into the constant.

b

$$\frac{dV}{d\theta} = mgl\left(-2\cos\theta\sin\theta + \sin\theta\right)$$

← When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta\left(-2\cos\theta + 1\right) = 0$$

$\sin\theta = 0 \quad \theta = 0$

or $2\cos\theta = 1$

$$\cos\theta = \frac{1}{2} \quad \theta = \pm\frac{\pi}{3}$$

← If $\theta = \pm\frac{\pi}{3}$ ΔPBC is equilateral, so the string has length $3l$ and is stretched.

$\therefore$ The values of θ are O and $\pm\frac{\pi}{3}$

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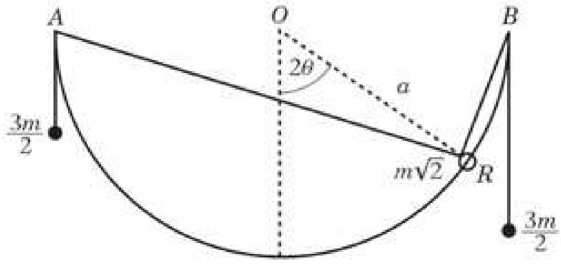
Review Exercise 2
Exercise A, Question 19

Question:



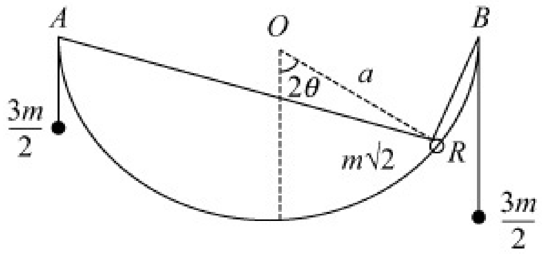
A smooth wire with ends A and B is in the shape of a semi-circle of radius a. The mid-point of AB is O. The wire is fixed in a vertical plane and hangs below AB which is horizontal. A small ring R, of mass $m\sqrt{2}$, is threaded on the wire and is attached to two light inextensible strings. The other end of each string is attached to a particle of mass $\frac{3m}{2}$. The particles hang vertically under gravity, as shown in the diagram.

a Show that, when the radius OR makes an angle 2θ with the vertical, the potential energy, V, of the system is given by $V = \sqrt{2}mga(3\cos\theta - \cos 2\theta) + \text{constant}$.

b Find the values of θ for which the system is in equilibrium.

c Determine the stability of the position of equilibrium for which $\theta > 0$. **[E]**

Solution:



a P.E. of $R = -\sqrt{2}mga\cos 2\theta$

P.E. of left hand mass

$= -\frac{3}{2}mg\left(2a - 2a\sin(45+\theta)\right)$

P.E. of right hand mass

$= -\frac{3}{2}mg\left(2a - 2a\sin(45-\theta)\right)$

Take the level of AB as the zero level for P.E. as AB is fixed.

$$\therefore V = -\sqrt{2}mga\cos 2\theta - \frac{3}{2}mga\left(2 - 2\sin(45+\theta)\right) - \frac{3}{2}mga\left(2 - 2\sin(45-\theta)\right) + \text{constant}$$

Including ' + constant' removes the need to specify a zero level.

$$V = -\sqrt{2}mga\cos 2\theta + 3mga\sin(45+\theta) + 3mga\sin(45-\theta) - 3mga - 3mga + \text{constant}$$

$$V = -\sqrt{2}mga\cos 2\theta + 3mga\left[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right] + \text{constant}$$

Expand $\sin(45+\theta)$ and $\sin(45-\theta)$ and absorb $-6mga$ into the constant.

$$V = -\sqrt{2}mga\cos 2\theta + 3mga \times 2 \times \frac{1}{\sqrt{2}}\cos\theta + \text{constant}$$

$$V = \sqrt{2}mga\left[3\cos\theta - \cos 2\theta\right] + \text{constant}$$

b

$$\frac{dV}{d\theta} = \sqrt{2}mga\left(-3\sin\theta + 2\sin 2\theta\right)$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow 2\sin 2\theta = 3\sin\theta$$

$$\sin\theta\left(4\cos\theta - 3\right) = 0$$

Use $\sin 2\theta = 2\sin\theta\cos\theta$ and factorise.

$$\sin\theta = 0, \theta = 0$$

$$\text{or } \cos\theta = \frac{3}{4}, \theta = \pm\cos^{-1}\left(\frac{3}{4}\right)$$

You can give the exact answers or the decimal equivalents ($\pm 0.723^c$).

c $\frac{d^2V}{d\theta^2} = \sqrt{2}mga\left(-3\cos\theta + 4\cos 2\theta\right)$

$\cos\theta = \frac{3}{4}$ $\cos 2\theta = 2\cos^2\theta - 1$

$= 2 \times \frac{9}{16} - 1$

$= \frac{1}{8}$

$\frac{d^2V}{d\theta^2} = \sqrt{2}mga\left(-3 \times \frac{3}{4} + 4 \times \frac{1}{8}\right) < 0$

$\therefore$ unstable equilibrium.

V is a maximum, so equilibrium is unstable.

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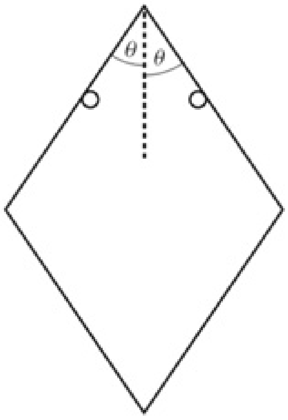
Review Exercise 2
Exercise A, Question 20

Question:



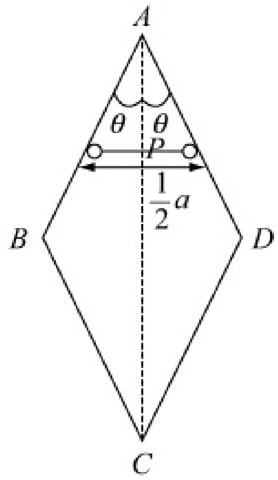
Four equal uniform rods each of length $2a$ and mass m are smoothly jointed to form a rhombus. This is used by a gardener to measure areas of lawn for treatment. When not in use it is stored resting on two smooth pegs, which are at the same level and a distance $\frac{1}{2}a$ apart, with the rhombus in a vertical plane, as shown in the diagram.

Given that each of the rods make an angle θ with the vertical,

a show that the potential energy of the system is $mga\cot\theta - 8mga\cos\theta + c$, where c is a constant.

b Hence find the value of θ when the system is in equilibrium. **[E]**

Solution:



a $AP = \frac{1}{4}a\cot\theta$

The level of the pegs must be used to calculate the P.E. as this level is fixed.

P.E. of rod $AB = -mg\left(a\cos\theta - \frac{1}{4}a\cot\theta\right)$

P.E. of rod $BC = -mg\left(3a\cos\theta - \frac{1}{4}a\cot\theta\right)$

$$\therefore \text{P.E. of system} = 2mga\left(-\cos\theta + \frac{1}{4}\cot\theta - 3\cos\theta + \frac{1}{4}\cot\theta\right) + \text{constant}$$

$$= 2mga\left(-4\cos\theta + \frac{1}{2}\cot\theta\right) + \text{constant}$$

$$= mga\cot\theta - 8mga\cos\theta + \text{constant}$$

b $\frac{dV}{d\theta} = -mga\,\text{cosec}^2\theta + 8mga\sin\theta$

When the system is in equilibrium, V has a maximum or minimum value.

$\frac{dV}{d\theta} = 0 \Rightarrow 8\sin\theta = \text{cosec}^2\theta$

$8\sin\theta = \frac{1}{\sin^2\theta}$

$\sin^3\theta = \frac{1}{8}$

$\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

The system is in equilibrium when $\theta = \frac{\pi}{6}$.

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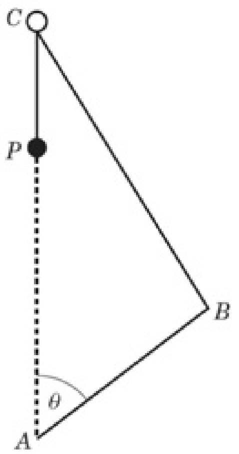
Review Exercise 2
Exercise A, Question 21

Question:

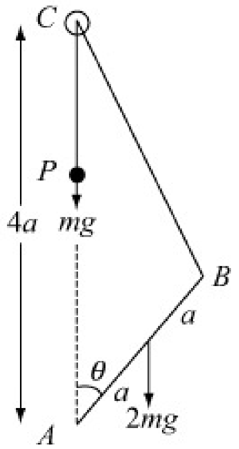


A uniform rod AB of mass $2m$ and length is freely hinged about a horizontal axis through A. The end B is attached to a light inextensible string of length b, $2a < b < 6a$, which passes through a small, smooth ring at C. A particle P, of mass m, is attached to the other end of the string and hangs freely. The point C is vertically above the point A and $AC = 4a$. The angle CAB is denoted by θ, as shown in the diagram.

a Show that the total potential energy of the system is given by $2mga(\cos\theta + \sqrt{[5 - 4\cos\theta]} + k)$, where k is a constant.

b Find, in degrees to 1 decimal place, a value of $\theta, 0 < \theta < 180°$, for which the system is in equilibrium. **[E]**

Solution:



a

$$CB^2 = (4a^2) + (2a)^2 - 2 \times 4a \times 2a\cos\theta$$

The length of CB is needed to obtain the length of CP.

$$CB^2 = 20a^2 - 16a^2\cos\theta$$

$$CB = 2a\sqrt{(5 - 4\cos\theta)}$$

$$\therefore AP = 4a - \left[b - 2a\sqrt{(5-4\cos\theta)}\right]$$

CP = total length of the string $- CB$ $= b - 2a\sqrt{(5-4\cos\theta)}$

$$\therefore \text{P.E. of } P = mg\left(4a - \left[b - 2a\sqrt{(5-4\cos\theta)}\right]\right)$$

$$\text{P.E. of rod} = 2mga\cos\theta$$

Using the level of A as the zero level for P.E. as A is fixed.

$$\therefore \text{P.E. of system} = 4mga - mgb + 2mga\sqrt{(5-4\cos\theta)} + 2mga\cos\theta + \text{constant}$$

$$= 2mga\left(\cos\theta + \sqrt{[5-4\cos\theta]}\right) + \text{constant}$$

Absorb $4mga - mgb$ into the constant

b $V = 2mga\left(\cos\theta + (5-4\cos\theta)^{\frac{1}{2}}\right) + \text{constant}$

$$\frac{dV}{d\theta} = 2mga\left(-\sin\theta + \frac{1}{2}(5-4\cos\theta)^{-\frac{1}{2}} \times 4\sin\theta\right)$$

When the system is in equilibrium V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow -\sin\theta + \frac{2\sin\theta}{\sqrt{(5-4\cos\theta)}} = 0$$

$$\sin\theta\left(1 - \frac{2}{\sqrt{(5-4\cos\theta)}}\right) = 0$$

$\sin\theta = 0$ – no solution in range $0 < \theta < 180°$

or $\dfrac{2}{\sqrt{(5-4\cos\theta)}} = 1$

$$2 = \sqrt{(5-4\cos\theta)}$$

$$4 = 5 - 4\cos\theta$$

$$4\cos\theta = 1$$

$$\cos\theta = \frac{1}{4}$$

$$\theta = 75.52$$

Only *one* value of θ is required.

$$\therefore \theta = 75.5°$$

The system is in equilibrium when $\theta = 75.5°$.

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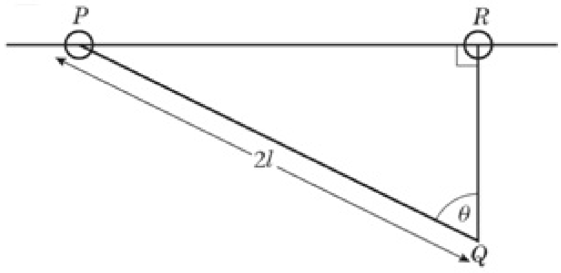
Review Exercise 2
Exercise A, Question 22

Question:



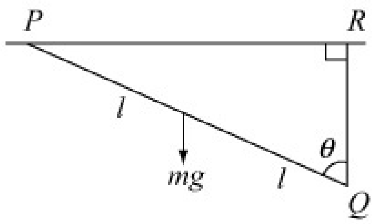
A uniform rod PQ has mass m and length $2l$. A smooth light ring is fixed to the end P of the rod. This ring is threaded on to a fixed horizontal smooth straight wire. A second small smooth light ring R is threaded on to the wire and is attached by a light elastic string, of natural length l and modulus of elasticity kmg, to the end Q of the rod, where k is a constant.

a Show that, when the rod PQ makes an angle θ with the vertical, where $0 < \theta \leq \frac{\pi}{3}$, and Q is vertically below R, as shown in the diagram, the potential energy of the system is $mgl[2k\cos^2\theta - (2k+1)\cos\theta] + \text{constant}$.

Given that there is a position of equilibrium with $\theta > 0$,

b show that $k > \frac{1}{2}$. **[E]**

Solution:



$\lambda = kmg$

a length $RQ = 2l\cos\theta$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2}\times\frac{kmg}{l}(2l\cos\theta - l)^2$$

$$= \frac{kmgl}{2}(2\cos\theta - 1)^2$$

$$\text{G.P.E. of rod} = -mgl\cos\theta$$

Take level of the wire as the zero level since this is fixed.

$$\text{Total P.E. of the system} = \frac{kmgl}{2}\left(4\cos^2\theta - 4\cos\theta + 1\right) - mgl\cos\theta + \text{constant}$$

$$= mgl\left(2k\cos^2\theta - 2k\cos\theta - \cos\theta\right) + \frac{kmgl}{2} + \text{constant}$$

$$= mgl\left(2k\cos^2\theta - (2k+1)\cos\theta\right) + \text{constant}$$

Absorb $\frac{kmgl}{2}$ into the constant.

b

$$V = mgl\left(2k\cos^2\theta - (2k+1)\cos\theta\right) + \text{constant}$$

$$\frac{dV}{d\theta} = mgl\left[-4k\cos\theta\sin\theta + (2k+1)\sin\theta\right]$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta\left[-4k\cos\theta + (2k+1)\right] = 0$$

$\sin\theta = 0 \quad \theta = 0$ not applicable

Outside the given range

$$\text{or } \cos\theta = \frac{2k+1}{4k}$$

$$0 < \theta \le \frac{\pi}{3} \Rightarrow 1 > \cos\theta \ge \frac{1}{2}$$

Use the condition on θ given in the question.

$$\therefore \frac{2k+1}{4k} < 1$$

$$2k+1 < 4k$$

$$1 < 2k$$

$$k > \frac{1}{2}$$

$$\text{and } \frac{2k+1}{4k} \ge \frac{1}{2}$$

$$2k+1 \ge 2k$$

Check that the other inequality does not give rise to any problem.

always true for any k.

$$\therefore k > \frac{1}{2}$$

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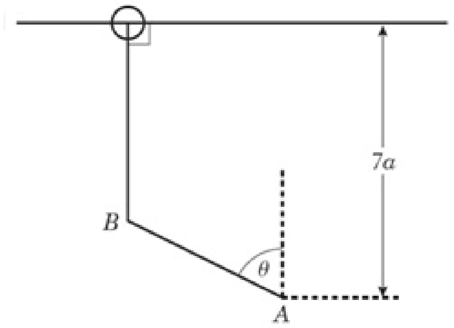
Review Exercise 2
Exercise A, Question 23

Question:



A uniform rod AB, of length $2a$ and mass $8m$, is free to rotate in a vertical plane about a fixed smooth horizontal axis through A. One end of a light elastic string, of natural length a and modulus of elasticity $\frac{4}{5}mg$, is fixed to B. The other end of the string is attached to a small ring which is free to slide on a smooth straight horizontal wire which is fixed in the same vertical plane as AB at a height $7a$ vertically above A. The rod AB makes an angle θ with the upward vertical at A, as shown in the diagram.

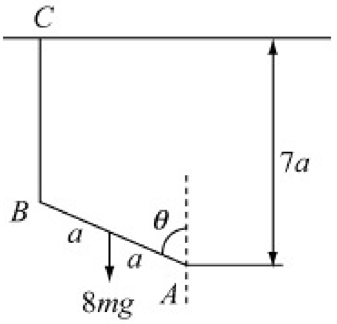
a Show that the potential energy V of the system is given by

$V = \frac{8}{5}mga(\cos^2\theta - \cos\theta) + \text{constant}$.

b Hence find the values of $\theta, 0 \le \theta \le \pi$, for which the system is in equilibrium.

c Determine the nature of these positions of equilibrium. **[E]**

Solution:



BC:

$\lambda = \frac{4}{5}mg$

$l = a$

a length of string $= 7a - 2a\cos\theta$

$\therefore \text{extension} = 6a - 2a\cos\theta$

$= 2a(3 - \cos\theta)$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2} \times \frac{4}{5}mg \times \frac{[2a(3-\cos\theta)]^2}{a}$$

$$= \frac{8}{5}mga(3-\cos\theta)^2$$

G.P.E. of rod $= 8mga\cos\theta$

Using level of A as zero level as A is fixed.

$$\therefore \text{Total P.E.} = 8mga\cos\theta + \frac{8}{5}mga\,(9 - 6\cos\theta + \cos^2\theta) + \text{constant}$$

$$V = \frac{8}{5}mga(9 - 6\cos\theta + \cos^2\theta + 5\cos\theta)$$

$+ \text{constant}$

$$V = \frac{8}{5}mga(\cos^2\theta - \cos\theta) + \text{constant}$$

Absorb $\frac{8}{5}mga \times 9$ into the constant.

b

$$\frac{dV}{d\theta} = \frac{8mga}{5}\left(-2\cos\theta\sin\theta + \sin\theta\right)$$

$$\frac{dV}{d\theta} = 0 \Rightarrow -2\cos\theta\sin\theta + \sin\theta = 0$$

When the system is in equilibrium, V has a maximum or minimum value.

$\sin\theta\,(1 - 2\cos\theta) = 0$

$\sin\theta = 0 \quad \theta = 0, \pi$

or $\cos\theta = \frac{1}{2} \quad \theta = \frac{\pi}{3}$

c

$$\frac{d^2V}{d\theta^2} = \frac{8mga}{5}\left(-2\cos^2\theta + 2\sin^2\theta + \cos\theta\right)$$

$\theta = 0 \quad \frac{d^2V}{d\theta^2} = -\frac{8mga}{5} < 0$

$\therefore V$ is maximum and equilibrium is unstable

$\theta = \frac{\pi}{2} \quad \frac{d^2V}{d\theta^2} = -3 \times \frac{8mga}{5} < 0$

$\therefore$ unstable

$$\theta = \frac{\pi}{3} \quad \frac{d^2V}{d\theta^2} = \frac{8mga}{5}\left(-2 \times \frac{1}{4} + 2 \times \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\right)$$

$$= \frac{8mga}{5} \times \frac{3}{2} > 0$$

$\therefore V$ is a minimum and the equilibrium is stable.

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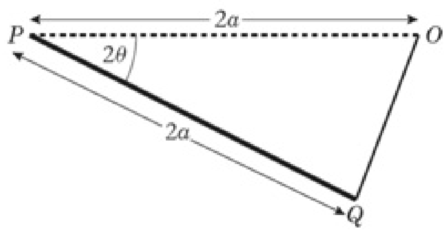
Review Exercise 2
Exercise A, Question 24

Question:



A uniform rod PQ, of length $2a$ and mass m, is free to rotate in a vertical plane about a fixed smooth horizontal axis through the end P. The end Q is attached to one end of a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{2\sqrt{3}}$. The other end of the string is attached to a fixed point O, where OP is horizontal and $OP = 2a$, as shown in the diagram. $\angle OPQ$ is denoted by 2θ.

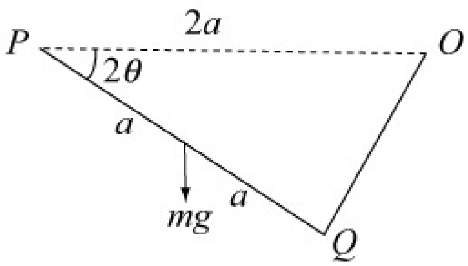
a Show that, when the string is taut, the potential energy of the system is

$$-\frac{mga}{\sqrt{3}}(2\cos 2\theta + \sqrt{3}\sin 2\theta + 2\sin\theta) + \text{constant}.$$

b Verify that there is a position of equilibrium at $\theta = \frac{\pi}{6}$.

c Determine whether this is a position of stable equilibrium. **[E]**

Solution:



OQ:

$\lambda = \frac{mg}{2\sqrt{3}}$

$l = a$

a length $OQ = 2 \times 2a \sin\theta$ since ΔOPQ is isoceles

$\therefore$ extension $= a(4\sin\theta - 1)$

$$\text{E.P.E} = \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2}\frac{mg}{2\sqrt{3}}\frac{a^2(4\sin\theta-1)^2}{a}$$

$$= \frac{mga}{4\sqrt{3}}(4\sin\theta - 1)^2$$

G.P.E. of $PQ = -mga\sin 2\theta$ ← Using the level of OP as the zero level as this is fixed.

$$\therefore \text{Total P.E.} = \frac{mga}{4\sqrt{3}}(16\sin^2\theta - 8\sin\theta + 1) - mga\sin 2\theta + \text{constant}$$

$$= \frac{mga}{\sqrt{3}}(4\sin^2\theta - 2\sin\theta) - mga\sin 2\theta + \text{constant}$$ ← $\frac{mga}{4\sqrt{3}} \times 1$ can be absorbed into the constant.

$$= \frac{mga}{\sqrt{3}}\left[2(1-\cos 2\theta) - 2\sin\theta\right] - mga\sin 2\theta + \text{constant}$$ ← Use $\cos 2\theta = 1 - 2\sin^2\theta$ to remove $\sin^2\theta$ from the expression.

$$= -\frac{mga}{\sqrt{3}}\left[2\cos 2\theta + 2\sin\theta + \sqrt{3}\sin 2\theta\right] + \text{constant}$$ ← $\frac{2mga}{\sqrt{3}}$ can be absorbed into the constant.

b $$\frac{dV}{d\theta} = \frac{-mga}{\sqrt{3}}\left[-4\sin 2\theta + 2\cos\theta + 2\sqrt{3}\cos 2\theta\right]$$ ← When the system is in equilibrium, V has a maximum or minimum value.

$\theta = \frac{\pi}{6}$

$$\frac{dV}{d\theta} = \frac{-mga}{\sqrt{3}}\left[-4\sin\frac{\pi}{3} + 2\cos\frac{\pi}{6} + 2\sqrt{3}\cos\frac{\pi}{3}\right]$$

$$= \frac{-mga}{\sqrt{3}}\left[-4 \times \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} + 2\sqrt{3} \times \frac{1}{2}\right]$$

$$= 0$$

$\therefore$ There is a position of equilibrium when $\theta = \frac{\pi}{6}$

c $\dfrac{d^2V}{d\theta^2} = \dfrac{-mga}{\sqrt{3}}\left[-8\cos 2\theta - 2\sin\theta - 4\sqrt{3}\sin 2\theta\right]$

$\theta = \dfrac{\pi}{6}$

$$\frac{d^2V}{d\theta^2} = \frac{-mga}{\sqrt{3}}\left[-8\cos\frac{\pi}{3} - 2\sin\frac{\pi}{6} - 4\sqrt{3}\sin\frac{\pi}{3}\right]$$

$$= \frac{-mga}{\sqrt{3}}\left[-8\times\frac{1}{2} - 2\times\frac{1}{2} - 4\sqrt{3}\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{-mga}{\sqrt{3}}\left[-4 - 1 - 6\right] = \frac{11mga}{\sqrt{3}}$$

$\dfrac{d^2V}{d\theta^2} > 0 \therefore V$ is a minimum and equilibrium is stable.

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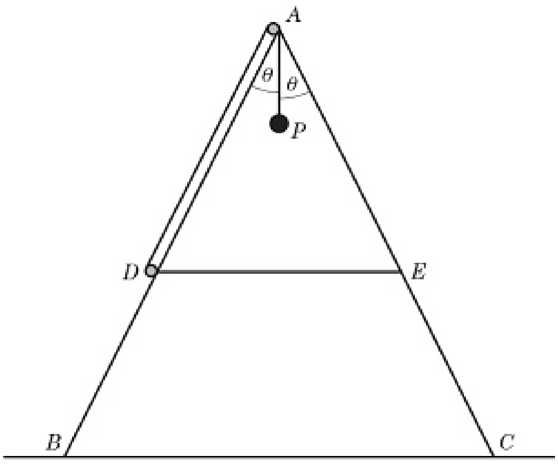
Review Exercise 2
Exercise A, Question 25

Question:



Two uniform rods AB and AC, each of mass $2m$ and length $2L$, are freely jointed at A. The mid-points of the rods are D and E respectively. A light inextensible string of length s is fixed to E and passes round small, smooth light pulleys at D and A. A particle P of mass m is attached to the other end of the string and hangs vertically. The points A, B and C lie in the same vertical plane with B and C on a smooth horizontal surface. The angles PAB and PAC are each equal to θ $(\theta > 0)$, as shown in the diagram.

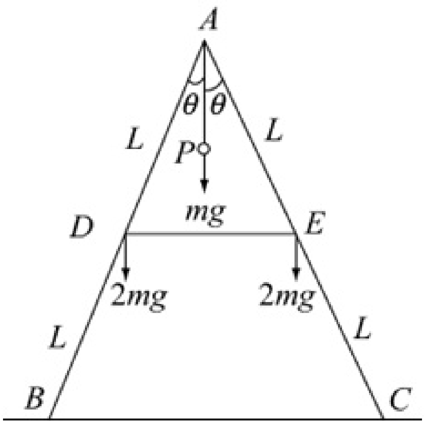
a Find the length of AP in terms of s, L and θ.

b Show that the potential energy V of the system is given by
$V = 2mgL(3\cos\theta + \sin\theta) + \text{constant}$.

c Hence find the value of θ for which the system is in equilibrium.

d Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



a $AP = s - (AD + DE)$

$= s - (L + 2L\sin\theta)$

b $V = 2 \times 2mg \times L\cos\theta + mg(2L\cos\theta - AP) + \text{constant}$

$V = 4mgL\cos\theta + 2mgL\cos\theta - mg(s - L - 2L\sin\theta) + \text{constant}$

$V = 4mgL\cos\theta + 2mgL\cos\theta + 2mgL\sin\theta - mg(s - L) + \text{constant}$

$V = 2mgL(3\cos\theta + \sin\theta) + \text{constant}$

Using the level of BC as zero level as this is fixed.

$-mg(s - L)$ can be absorbed into the constant.

c $\frac{dV}{d\theta} = 2mgL(-3\sin\theta + \cos\theta)$

$\frac{dV}{d\theta} = 0 \Rightarrow 3\sin\theta = \cos\theta$

$\tan\theta = \frac{1}{3}$

$\therefore \theta = 0.322^c$ (3 s.f.)

When the system is in equilibrium, V has a maximum or minimum value.

d $\frac{d^2V}{d\theta^2} = 2mgL(-3\cos\theta - \sin\theta)$

$\theta = 0.322 \Rightarrow \frac{d^2V}{d\theta^2} < 0$

θ is acute so $\cos\theta > 0$ and $\sin\theta > 0$.

$\therefore V$ is maximum and the equilibrium is unstable.

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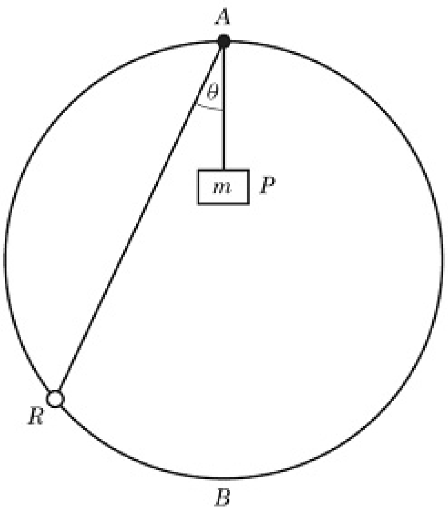
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 26

Question:



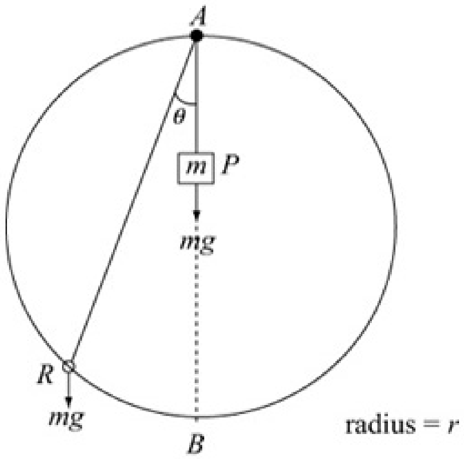
A smooth wire AB, in the shape of a circle of radius r, is fixed in a vertical plane with AB vertical. A small smooth ring R of mass m is threaded on the wire and is connected by a light inextensible string to a particle P of mass m. The length of the string is greater than the diameter of the circle. The string passes over a small smooth pulley which is fixed at the highest point A of the wire and angle $R\hat{A}P = \theta$, as shown in the diagram.

a Show that the potential energy of the system is given by $2mgr(\cos\theta - \cos^2\theta) +$ constant.

b Hence determine the values of $\theta, \theta \geq 0$, for which the system is in equilibrium.

c Determine the stability of each position of equilibrium. **[E]**

Solution:



a length $AR = 2r\cos\theta$ ← $\angle ARB = 90°$ - angle in a semicircle

P.E. of $P = -mg\,(L - 2r\cos\theta)$
where L is a constant
P.E. of $R = -mgAR\cos\theta$
$= -mg \times 2r\cos^2\theta$

The length of the string is constant - call it L.

Take the level of A as the zero level as A is fixed.

$\therefore$ P.E. of the system
$= -mgL + 2mgr\cos\theta - 2mgr\cos^2\theta + \text{constant}$
$= 2mgr\,(\cos\theta - \cos^2\theta) + \text{constant}$ ← mgL is constant, so it can be absorbed into the constant.

b $V = 2mgr\,(\cos\theta - \cos^2\theta) + \text{constant}$

$\dfrac{dV}{d\theta} = 2mgr\,(-\sin\theta + 2\cos\theta\sin\theta)$ ← When the system is in equilibrium, V has a maximum or minimum value.

$\dfrac{dV}{d\theta} = 0 \Rightarrow \sin\theta\,(2\cos\theta - 1) = 0$

$\sin\theta = 0 \quad \theta = 0$

or $\cos\theta = \dfrac{1}{2} \quad \theta = \dfrac{\pi}{3}$

c $\dfrac{d^2V}{d\theta^2} = 2mgr\,(-\cos\theta - 2\sin^2\theta + 2\cos^2\theta)$

$\theta = 0 \quad \dfrac{d^2V}{d\theta^2} = 2mgr\,(-1+2) > 0$

$\Rightarrow$ V is a minimum and equilibrium is stable.

$\theta = \dfrac{\pi}{3} \quad \dfrac{d^2V}{d\theta^2} = 2mgr\left(-\dfrac{1}{2} - 2\left(\dfrac{\sqrt{3}}{2}\right)^2 + 2\left(\dfrac{1}{2}\right)^2\right)$

$= 2mgr\left(-\dfrac{1}{2} - \dfrac{3}{2} + \dfrac{1}{2}\right) = -3mgr < 0$

$\Rightarrow$ V is a maximum and equilibrium is unstable.

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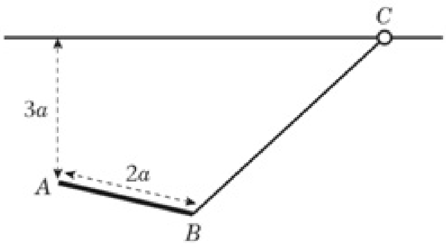
Review Exercise 2
Exercise A, Question 27

Question:



A uniform rod AB has mass m and length $2a$. One end A is freely hinged to a fixed point. One end of a light elastic string, of natural length a and modulus $\frac{1}{2}mg$, is attached to the other end B of the rod. The other end of the string is attached to a small ring C which can move freely on a smooth horizontal wire fixed at a height of $3a$ above A and in the vertical plane through A, as shown in the diagram.

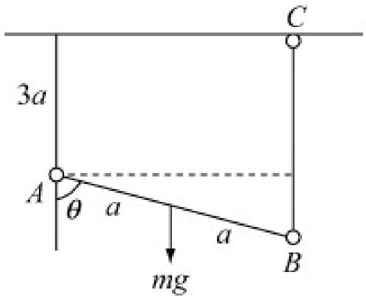
a Explain why, when the system is in equilibrium, the elastic string is vertical.

b Show that, when BC is vertical and the rod AB makes an angle θ with the downward vertical, the potential energy, V, of the system is given by
$V = mga(\cos^2\theta + \cos\theta) + \text{constant}$.

c Hence find the values of $\theta, 0 \le \theta \le \pi$, for which the system is in equilibrium.

d Determine whether each position of equilibrium is stable or unstable. **[E]**

Solution:



BC: $l = a$

$\lambda = \frac{1}{2}mg$

a Wire is smooth, so the reaction from the wire on the ring is vertical. If the ring is in equilibrium the tension in the string must be vertical as the third force on the ring is its weight.

b Length $BC = 3a + 2a\cos\theta$

$$\text{E.P.E. in } BC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2}\times\frac{1}{2}\frac{mg}{a}(2a + 2a\cos\theta)^2$$

Extension is $BC - a$

$$= mga(1 + \cos\theta)^2$$

$$\text{G.P.E. of rod} = -mga\cos\theta$$

Using level of A as the zero level as A is fixed.

$$\therefore V = mga(1 + 2\cos\theta + \cos^2\theta)$$

$$-mga\cos\theta + \text{constant}$$

$$V = mga(\cos^2\theta + \cos\theta) + \text{constant}$$

Absorb mga into the constant.

c $\frac{dV}{d\theta} = mga(-2\cos\theta\sin\theta - \sin\theta)$

When the system is in equilibrium, V has a maximum or minimum value.

$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta(2\cos\theta + 1) = 0$

$\sin\theta = 0 \quad \theta = 0, \pi$

$\cos\theta = -\frac{1}{2} \quad \theta = \frac{2\pi}{3}$

;

d $\frac{d^2V}{d\theta^2} = mga(2\sin^2\theta - 2\cos^2\theta - \cos\theta)$

$\theta = 0 \quad \frac{d^2V}{d\theta^2} = mga \times -3 < 0$

$\Rightarrow V$ is a maximum and equilibrium is unstable

$\theta = \pi \quad \frac{d^2V}{d\theta^2} = mga \times -1 < 0$

$\Rightarrow V$ is a maximum and equilibrium is unstable.

$$\theta = \frac{2\pi}{3} \quad \frac{d^2V}{d\theta^2} = mga\left(2\times\left(\frac{\sqrt{3}}{2}\right)^2 - 2\times\left(\frac{-1}{2}\right)^2 - \left(-\frac{1}{2}\right)\right)$$

$$= mga\left(\frac{3}{2} - \frac{1}{2} + \frac{1}{2}\right) = \frac{3mga}{2} > 0$$

$\Rightarrow V$ is a minimum and equilibrium is stable.

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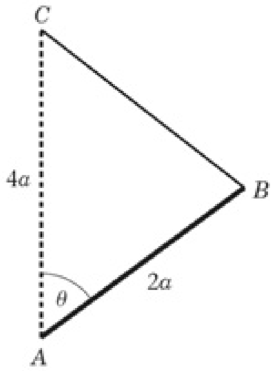
Review Exercise 2
Exercise A, Question 28

Question:

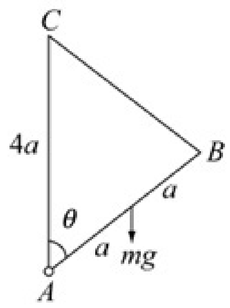


A uniform rod AB, of mass m and length $2a$, can rotate freely in a vertical plane about a fixed smooth horizontal axis through A. The fixed point C is vertically above A and $AC = 4a$. A light elastic string, of natural length $2a$ and modulus of elasticity $\frac{1}{2}mg$, joins B to C. The rod AB makes an angle θ with the upward vertical at A, as shown in the diagram.

a Show that the potential energy of the system is $-mga[\cos\theta + \sqrt{(5 - 4\cos\theta)} + \text{constant}$.

b Hence determine the values of θ for which the system is in equilibrium. **[E]**

Solution:



BC: $l = 2a$

$\lambda = \frac{1}{2}mg$

a length $BC = \sqrt{\left[(4a)^2 + (2a)^2 - 2\times 4a \times 2a\cos\theta\right]}$

$= \sqrt{\left[20a^2 - 16a^2\cos\theta\right]}$

$= 2a\sqrt{[5 - 4\cos\theta]}$

Use the cosine rule to obtain BC.

$\therefore$ E.P.E. in BC

$= \frac{1}{2}\frac{\lambda x^2}{l} = \frac{1}{2}\times\frac{1}{2}\frac{mg}{2a}\left[2a\sqrt{(5-4\cos\theta)} - 2a\right]^2$

$= \frac{mga}{2}\left[\sqrt{(5-4\cos\theta)} - 1\right]^2$

G.P.E. of rod $= mga\cos\theta$

Taking level of A as zero level since A is fixed.

$\therefore$P.E. of system $= mga\cos\theta + \frac{mga}{2}\left[(5-4\cos\theta) - 2\sqrt{(5-4\cos\theta)} + 1\right]$
+ constant

$= mga\left[\cos\theta + \frac{5}{2} - 2\cos\theta - \sqrt{(5-4\cos\theta)} + 1\right] + \text{constant}$

$= -mga\left[\cos\theta + \sqrt{(5-4\cos\theta)}\right] + \text{constant}$

Absorb $mga \times \frac{7}{2}$ into the constant.

b $V = -mga\left[\cos\theta + (5-4\cos\theta)^{\frac{1}{2}}\right] + \text{constant}$

$\frac{dV}{d\theta} = -mga\left[-\sin\theta + \frac{1}{2}(5-4\cos\theta)^{-\frac{1}{2}} \times 4\sin\theta\right]$

When the system is in equilibrium, V has a maximum or minimum value.

$\frac{dV}{d\theta} = 0$

$\Rightarrow \sin\theta\left[1 - \frac{2}{\sqrt{(5-4\cos\theta)}}\right] = 0$

$\sin\theta = 0, \pi$

or $\frac{2}{\sqrt{(5-4\cos\theta)}} = 1$

$4 = 5 - 4\cos\theta$

$\cos\theta = \frac{1}{4}$

$\theta = \cos^{-1}\left(\frac{1}{4}\right) = 1.32^\circ$ (3 s.f.)

You may give the exact answer if accuracy has not been specified.

The system is in equilibrium when $\theta = 0, 1.32^\circ, \pi$

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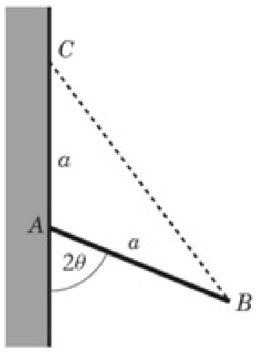
Review Exercise 2
Exercise A, Question 29

Question:



A uniform rod AB, of mass m and length a, can rotate in a vertical plane about a smooth hinge fixed at A on a vertical wall. A point C on the wall is at a height a vertically above A. One end of an elastic string, of natural length a and modulus of elasticity $2mg$, is attached to C and the other end is attached to the end B of the rod, as shown in the diagram.

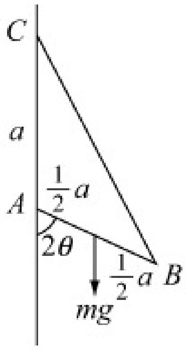
a Show that, when the rod AB makes an angle $2\theta, \theta > 0$, with the downward vertical, and the string is taut, the potential energy, V, of the system is given by

$V = -\frac{1}{2}mga\cos 2\theta + mga(2\cos\theta - 1)^2 + \text{constant}.$

b Hence determine the value of θ for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



BC: $l = a$

$\lambda = 2mg$

a length $BC = 2a\cos\theta$ ← ΔABC is isosceles with $\hat{C} = \hat{B} = \theta$

$$\text{E.P.E. in } BC = \frac{1}{2}\frac{\lambda x^2}{l}$$
$$= \frac{1}{2}\times\frac{2mg}{a}(2a\cos\theta - a)^2$$
$$= mga(2\cos\theta - 1)^2$$

$$\text{G.P.E. of } AB = -mg\times\frac{1}{2}a\cos 2\theta$$ ← Using level of A as the zero level since A is fixed.

$$\therefore V = -\frac{1}{2}mga\cos 2\theta + mga(2\cos\theta - 1)^2 + \text{constant}$$

b $$\frac{dV}{d\theta} = 2\times\frac{1}{2}mga\sin 2\theta + 2mga(2\cos\theta - 1)\times(-2\sin\theta)$$ ← When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0$$
$$\sin 2\theta - 4\sin\theta(2\cos\theta - 1) = 0$$
$$2\sin\theta\cos\theta - 4\sin\theta(2\cos\theta - 1) = 0$$
$$\sin\theta(\cos\theta - 2(2\cos\theta - 1)) = 0$$
$$\sin\theta(2 - 3\cos\theta) = 0$$

$\sin\theta = 0$ $\theta = 0, \pi$ not applicable ← $\theta > 0$ and if $\theta = \pi$, $2\theta = 2\pi$ and the situation is the same as when $\theta = 0$

$$\cos\theta = \frac{2}{3}$$

$\therefore$ Equilibrium occurs when

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) = 0.841^c$$ ← You can give the exact answer as accuracy is not specified.

c $\frac{dV}{d\theta} = mga\sin 2\theta - 8mga\sin\theta\cos\theta + 4mga\sin\theta$

$= mga(-3\sin 2\theta + 4\sin\theta)$

$\frac{d^2V}{d\theta^2} = mga(-6\cos 2\theta + 4\cos\theta)$

Simplify $\frac{dV}{d\theta}$ to make the necessary differentiation easier.

$\cos\theta = \frac{2}{3}$ $\cos 2\theta = 2\cos^2\theta - 1$

$= 2\times\frac{4}{9} - 1$

$= -\frac{1}{9}$

$\therefore \frac{d^2V}{d\theta^2} = mga\left(-6\times\frac{-1}{9} + 4\times\frac{2}{3}\right)$

$= \frac{10}{3}mga > 0$

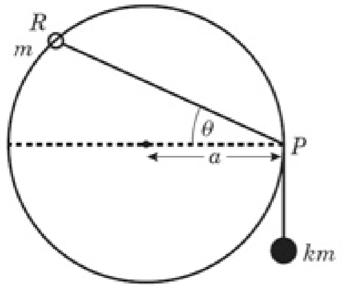
$\therefore$ V a minimum and equilibrium is stable.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 30

Question:



A small ring R, of mass m, is free to slide on a smooth wire in the shape of a circle with radius a. The wire is fixed in a vertical plane. A light inextensible string has one end attached to R and passes over a small smooth pulley at P, where P is one end of the horizontal diameter of the wire. The other end of the string is attached to a mass km $(k < 1)$ which hangs freely, as shown in the diagram. PR makes an angle θ with the horizontal.

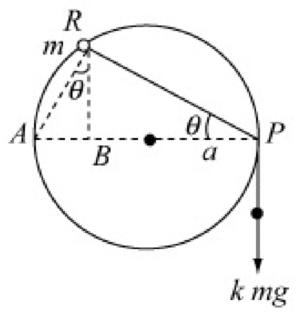
a Show that the potential energy of the system, V, is given by $V = mga(\sin 2\theta + 2k\cos\theta) + \text{constant}$.

Given that $k = \frac{1}{2}$,

b find, in radians to 3 decimal places, the values of $\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, for which the system is in equilibrium.

c Determine whether each of the positions of equilibrium is stable or unstable. **[E]**

Solution:



a $AR = 2a\sin\theta$ ← $\angle ARP = 90^\circ$ ($\angle$ in semicircle)

$\therefore RB = AR\cos\theta$

$= 2a\sin\theta\cos\theta$

$= a\sin 2\theta$

$\therefore$ P.E. of $R = mga\sin 2\theta$ ← Using AP as zero level as this level is fixed.

PE of hanging mass

$= -kmg\left(L - RP\right)$

where L is the length of the string.

$RP = 2a\cos\theta$

$\therefore V = mga\sin 2\theta - kmg\left(L - 2a\cos\theta\right) + \text{constant}$

$V = mga\left(\sin 2\theta + 2k\cos\theta\right) + \text{constant}$ ← Absorb $-kmg\,L$ into the constant.

b $k = \dfrac{1}{2}$

$\Rightarrow V = mga(\sin 2\theta + \cos\theta) + \text{constant}$

$\dfrac{dV}{d\theta} = mga(2\cos 2\theta - \sin\theta)$ ← When the system is in equilibrium, V has a maximum or minimum value.

$\dfrac{dV}{d\theta} = 0 \quad 2\cos 2\theta - \sin\theta = 0$

$2(1 - 2\sin^2\theta) - \sin\theta = 0$

$4\sin^2\theta + \sin\theta - 2 = 0$

$\sin\theta = \dfrac{-1 \pm \sqrt{(1+32)}}{8}$

$\sin\theta = \dfrac{-1 \pm \sqrt{33}}{8}$

$\theta = 0.6348\ldots$

or $\theta = -1.0029\ldots$

$\therefore$ Equilibrium occurs when $\theta = 0.635^c$ or -1.003^c (3 d.p.)

c $\dfrac{d^2V}{d\theta^2} = mga\left(-4\sin 2\theta - \cos\theta\right)$

$\theta = 0.6348 \quad \dfrac{d^2V}{d\theta^2} = mga \times (-4.63\ldots) < 0$

V is a maximum $\Rightarrow$ Unstable equilibrium when $\theta = 0.635^c$

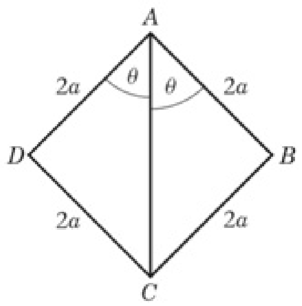
$\theta = -1.003 \quad \dfrac{d^2V}{d\theta^2} = mga \times 3.089\ldots > 0$

V is a minimum $\Rightarrow$ Stable equilibrium when $\theta = -1.003^c$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 31

Question:



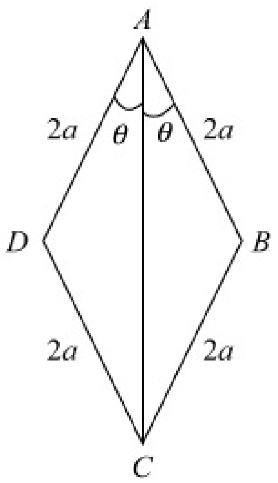
Four identical uniform rods, each of mass m and length $2a$, are freely jointed to form a rhombus $ABCD$. The rhombus is suspended from A and is prevented from collapsing by an elastic string which joins A to C, with $\angle BAD = 2\theta, 0 \le \theta \le \frac{1}{3}\pi$, as shown in the diagram. The natural length of the elastic string is $2a$ and its modulus of elasticity is $4mg$.

a Show that the potential energy, V, of the system is given by
$V = 4mga[(2\cos\theta - 1)^2 - 2\cos\theta] + \text{constant}$.

b Hence find the non-zero value of θ for which the system is in equilibrium.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:



AC:
$\lambda = 4mg$
$l = 2a$

a length $AC = 2 \times 2a\cos\theta$

$$\text{E.P.E.} = \frac{1}{2}\frac{\lambda x^2}{l}$$
$$= \frac{1}{2} \times \frac{4mg}{2a}(4a\cos\theta - 2a)^2$$
$$= mga(4\cos\theta - 2)^2$$

$$\text{G.P.E. of rods} = -2 \times mga\cos\theta - 2 \times mg \times 3a\cos\theta = -8mga\cos\theta$$

Take A as zero level for P.E. as A is fixed.

$$\therefore V = -8mga\cos\theta + 4mga(2\cos\theta - 1)^2 + \text{constant}$$
$$= 4mga\left[(2\cos\theta - 1)^2 - 2\cos\theta\right] + \text{constant}$$

b $$\frac{dV}{d\theta} = 4mga\,[2(2\cos\theta - 1) \times (-2\sin\theta) + 2\sin\theta]$$

When the system is in equilibrium, V has a maximum or minimum value.

$$\frac{dV}{d\theta} = 0$$
$$\Rightarrow -4\sin\theta(2\cos\theta - 1) + 2\sin\theta = 0$$
$$2\sin\theta[1 - 2(2\cos\theta - 1)] = 0$$
$\sin\theta = 0 \quad \theta = 0$ (not required answer)
or $1 - 4\cos\theta + 2 = 0$
$$\cos\theta = \frac{3}{4}$$

$0 \le \theta \le \frac{1}{3}\pi$ but a non-zero value is required (see question).

$$\therefore \theta = \cos^{-1} 0.75 \text{ or } 0.723^c \text{ (3 s.f.)}$$

c $\dfrac{dV}{d\theta} = 4mga\left[-8\sin\theta\cos\theta + 4\sin\theta + 2\sin\theta\right]$

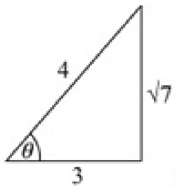
$\dfrac{d^2V}{d\theta^2} = 4mga\left[-8\cos^2\theta + 8\sin^2\theta + 6\cos\theta\right]$

$\cos\theta = \dfrac{3}{4} \Rightarrow \sin^2\theta = \dfrac{7}{16}$

$\therefore \dfrac{d^2V}{d\theta^2} = 4mga\left[-8\times\dfrac{9}{16} + 8\times\dfrac{7}{16} + 6\times\dfrac{3}{4}\right]$

$= 14mga > 0$

$\therefore V$ is a minimum and equilibrium is stable.

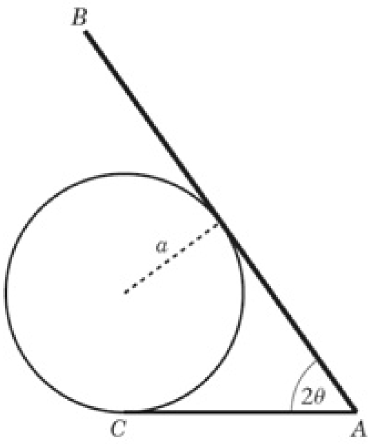
Use  to obtain the value of $\sin^2\theta$.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 2
Exercise A, Question 32

Question:



The diagram shows a uniform rod AB, of mass m and length $4a$, resting on a smooth fixed sphere of radius a. A light elastic string, of natural length a and modulus of elasticity $\frac{3}{4}mg$, has one end attached to the lowest point C of the sphere and the other end attached to A. The points A, B and C lie in a vertical plane with $\angle BAC = 2\theta$, where $\theta < \frac{\pi}{4}$.

Given that AC is always horizontal,

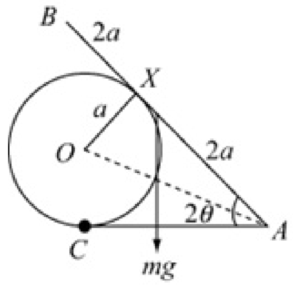
a show that the potential energy of the system is $\frac{mga}{8}(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta) + \text{constant}$,

b show that there is a value for θ for which the system is in equilibrium such that $0.535 < \theta < 0.545$.

c Determine whether this position of equilibrium is stable or unstable. **[E]**

Solution:

a



AC:

$\lambda = \frac{3}{4}mg$

$l = a$

P.E. of rod $= mg \times 2a\sin 2\theta$ ← Using AC as the zero level as this is a fixed horizontal level.

length $AC = a\cot\theta$ ← $AC = AX$ and AO bisects $\angle CAX$.

$$\therefore \text{E.P.E. in } AC = \frac{1}{2}\frac{\lambda x^2}{l}$$

$$= \frac{1}{2}\times\frac{3}{4}mg\frac{(a\cot\theta - a)^2}{a}$$

$$= \frac{3mga}{8}(\cot\theta - 1)^2$$

$\therefore$ Total P.E.

$$= 2mga\sin 2\theta + \frac{3mga}{8}\left(\cot^2\theta - 2\cot\theta + 1\right) + \text{constant}$$

$$= \frac{mga}{8}\left(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta\right) + \text{constant}$$

← Absorb $\frac{3}{8}mga$ into the constant.

b $V = \frac{mga}{8}(16\sin 2\theta + 3\cot^2\theta - 6\cot\theta) + \text{constant}$

$$\frac{dV}{d\theta} = \frac{mga}{8}(32\cos 2\theta - 6\cot\theta\operatorname{cosec}^2\theta + 6\operatorname{cosec}^2\theta)$$

← When the system is in equilibrium, V has a maximum or minimum value.

$\theta = 0.535$

$\frac{dV}{d\theta} = \frac{mga}{8}\times(-0.501..) < 0$

$\theta = 0.545$

$\frac{dV}{d\theta} = \frac{mga}{8}\times(0.299) > 0$

← Investigate the sign of $\frac{dV}{d\theta}$ at the end points of the given interval.

Change of sign $\Rightarrow \frac{dV}{d\theta} = 0$ in the given interval and so there is a position of equilibrium.

c At $\theta = 0.535$ $\frac{dV}{d\theta} < 0$

At $\theta = 0.545$ $\frac{dV}{d\theta} > 0$

← There is no need to differentiate again as you know the signs of $\frac{dV}{d\theta}$ on either side of the turning point.

$\therefore V$ is a minimum

$\Rightarrow$ This position of equilibrium is stable