Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise A, Question 1

Question:

The velocity vectors of two particles P and Q are $\mathbf{v}_P$ and $\mathbf{v}_Q$ respectively. Find the velocity of P relative to Q and the relative speed of Q to P in each of the following cases:

a $\mathbf{v}_P = (5\mathbf{i} + 6\mathbf{j})\text{m s}^{-1}$, $\mathbf{v}_Q = (4\mathbf{i} - 3\mathbf{j})\text{m s}^{-1}$

b $\mathbf{v}_P = 6\mathbf{j}\text{m s}^{-1}$, $\mathbf{v}_Q = (-2\mathbf{i} + \mathbf{j})\text{m s}^{-1}$

c $\mathbf{v}_P = (5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})\text{m s}^{-1}$, $\mathbf{v}_Q = (\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})\text{m s}^{-1}$.

Solution:

a $${}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = (5\mathbf{i} + 6\mathbf{j}) - (4\mathbf{i} - 3\mathbf{j}) = (\mathbf{i} + 9\mathbf{j})\text{ m s}^{-1}$$
$$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = |\mathbf{i}+9\mathbf{j}| = \sqrt{82}\text{ m s}^{-1}$$

b $${}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = 6\mathbf{j} - (-2\mathbf{i} + \mathbf{j}) = (2\mathbf{i} + 5\mathbf{j})\text{ m s}^{-1}$$
$$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = |(2\mathbf{i}+5\mathbf{j})| = \sqrt{2^2+5^2} = \sqrt{29}\text{ m s}^{-1}$$

c $${}_P\mathbf{v}_Q = \mathbf{v}_P - \mathbf{v}_Q = (5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}) - (\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$$
$$= (4\mathbf{i} + 12\mathbf{j} - 5\mathbf{k})\text{ m s}^{-1}$$
$$|{}_Q\mathbf{v}_P| = |{}_P\mathbf{v}_Q| = \sqrt{4^2 + 12^2 + (-5)^2} = \sqrt{16 + 144 + 25}$$
$$= \sqrt{185}\text{ m s}^{-1}$$

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Relative motion
Exercise A, Question 2

Question:

A man is driving due north at 40 km h^{-1} along a straight road when he notices that the wind appears to be coming from N60°W with a speed of 40 km h^{-1}. Find the actual velocity of the wind.

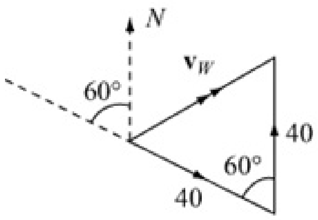
Solution:

$\mathbf{v}_M = 40 \text{ km h}^{-1}$ due N

${}_W\mathbf{v}_M = 40 \text{ km h}^{-1}$ from N60°W

${}_W\mathbf{v}_M = \mathbf{v}_W - \mathbf{v}_M \Rightarrow \mathbf{v}_W = {}_W\mathbf{v}_M + \mathbf{v}_M$

Draw the vector Δ:



Vector Δ is equilateral so $|\mathbf{v}_W| = 40 \text{ km h}^{-1}$ in direction N60°E

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Relative motion
Exercise A, Question 3

Question:

The velocity of A relative to B is $(2\mathbf{i} + 3\mathbf{j})\text{m s}^{-1}$ and the velocity of B relative to C is $(-\mathbf{i} + 4\mathbf{j})\text{m s}^{-1}$. Find the velocity of A relative to C.

Solution:

$$\left.\begin{aligned} {}_A\mathbf{v}_B &= \mathbf{v}_A - \mathbf{v}_B \\ \text{and } {}_B\mathbf{v}_C &= \mathbf{v}_B - \mathbf{v}_C \end{aligned}\right\} \text{ adding}$$

$${}_A\mathbf{v}_B + {}_B\mathbf{v}_C = \mathbf{v}_A - \mathbf{v}_C = {}_A\mathbf{v}_C$$

Hence, ${}_A\mathbf{v}_C = (2\mathbf{i} + 3\mathbf{j}) + (-\mathbf{i} + 4\mathbf{j}) = (\mathbf{i} + 7\mathbf{j})\text{ m s}^{-1}$

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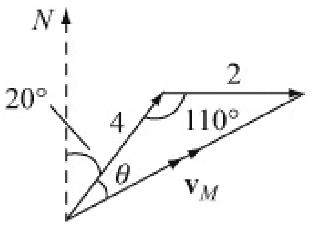
Relative motion
Exercise A, Question 4

Question:

A man who can row at 4 km h^{-1} in still water rows with his boat steering in the direction N20°E. There is a current of 2 km h^{-1} flowing due E. With what speed and in what direction does the boat actually move?

Solution:

${}_M\mathbf{v}_W$ is 4 km h^{-1} in N20°E
$\mathbf{v}_W$ is 2 km h^{-1} due E
${}_M\mathbf{v}_W = \mathbf{v}_M - \mathbf{v}_W \Rightarrow \mathbf{v}_M = {}_M\mathbf{v}_W + \mathbf{v}_W$



Draw the vector Δ:
by cosine rule,

$$|\mathbf{v}_M|^2 = 4^2 + 2^2 - 2\times 4\times 2\cos 110°$$

$$|\mathbf{v}_M| = \sqrt{20 - 16\cos 110°} = 5.05 \text{ km h}^{-1}$$

by sine rule

$$\frac{\sin\theta}{2} = \frac{\sin 110°}{5.047}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{2\sin 110°}{5.047}\right) = 21.9°$$

The boat moves at 5.05 km h^{-1} in N41.9° E

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Relative motion
Exercise A, Question 5

Question:

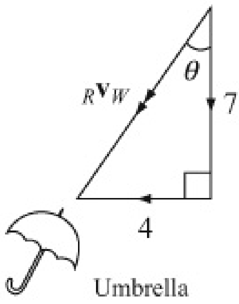
A woman is walking along a road with a speed of 4 km h^{-1}. The rain is falling vertically at 7 km h^{-1}. At what angle to the vertical should she hold her umbrella?

Solution:

$\mathbf{v}_W$ is 4 km h^{-1} horizontally $(\rightarrow)$

$\mathbf{v}_R$ is 7 km h^{-1} vertically $(\downarrow)$

${}_R\mathbf{v}_W = \mathbf{v}_R - \mathbf{v}_W$ Draw the vector Δ:



$\tan\theta = \frac{4}{7} \Rightarrow \theta = 29.7°$

Angle is $29.7°$

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Relative motion
Exercise A, Question 6

Question:

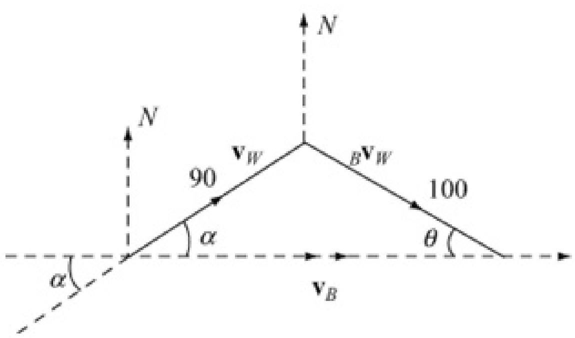
A bird can fly in still air at 100 km h^{-1}. The wind blows at 90 km h^{-1} from Wα°S, where $\tan\alpha = \frac{3}{4}$. The bird wishes to return to its nest which is due E of its present position. In which direction, relative to the air, should it fly?

Solution:

	Mag	Dir
${}_B\mathbf{v}_W$	100	?
$\mathbf{v}_W$	90	From α W of S $(\tan\alpha = \frac{3}{4})$
$\mathbf{v}_B$	?	due E

${}_B\mathbf{v}_W = \mathbf{v}_B - \mathbf{v}_W \Rightarrow \mathbf{v}_B = {}_B\mathbf{v}_W + \mathbf{v}_W$

Draw the vector Δ: (Draw $\mathbf{v}_W$ FIRST, since we have both its magnitude and direction)



$$\frac{\sin\theta}{90} = \frac{\sin\alpha}{100}$$

$$\sin\theta = \frac{9}{10} \times \frac{3}{5} = 0.54$$

$$\Rightarrow \theta = 32.68°$$

Hence, the bird should fly on a bearing of 122.68° or 32.68° S of E.

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Relative motion
Exercise A, Question 7

Question:

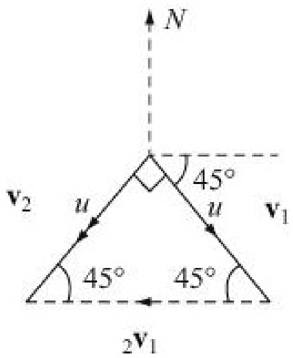
Two cars are moving at the same speed. The first is moving SE while the other appears to be approaching it from the east. Find the direction in which the second car is moving.

Solution:

	Mag	Dir
$\mathbf{v}_1$	u	SE
${}_2\mathbf{v}_1$	?	From E
$\mathbf{v}_2$	u	?

${}_2\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_1$

$\Rightarrow \mathbf{v}_2 = \mathbf{v}_1 + {}_2\mathbf{v}_1$



Draw the vector Δ:
Triangle is isosceles
Direction of $\mathbf{v}_2$ is SW

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Relative motion
Exercise A, Question 8

Question:

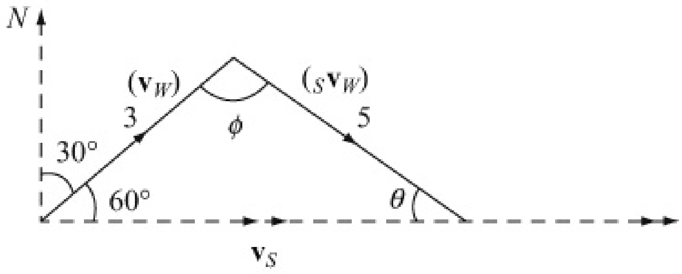
A ship has to travel 20 km due E. If the speed of the ship in still water is 5 km h^{-1} and if there is a current of 3 km h^{-1} in the direction N30°E, find how long it will take.

Solution:

	Mag	Dir
$\mathbf{v}_S$	?	E
${}_S\mathbf{v}_W$	5	?
$\mathbf{v}_W$	3	N30°E

$${}_S\mathbf{v}_W = \mathbf{v}_S - \mathbf{v}_W$$
$$\Rightarrow \mathbf{v}_S = \mathbf{v}_W + {}_S\mathbf{v}_W$$

Draw vector Δ:



$$\frac{\sin\theta}{3} = \frac{\sin 60°}{5} \Rightarrow \sin\theta = \frac{3\sqrt{3}}{10} \Rightarrow \theta = 31.3°$$
$$\Rightarrow \phi = 180° - (60° + 31.3°) = 88.7°$$
$$\therefore |\mathbf{v}_S|^2 = 3^2 + 5^2 - 2\times3\times5\cos 88.7° = 34 - 30\cos 88.7°$$
$$|\mathbf{v}_S| = 5.772 \text{ km h}^{-1}$$
$$\therefore \text{Time} = \frac{20}{5.772}\text{ h} = 3.464$$
$$= 3\text{ h } 28 \text{ minutes (nearest minute)}$$

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Relative motion
Exercise A, Question 9

Question:

An aeroplane can fly at 600 km h^{-1} in still air. It has to fly to an airport which is SW of its current position. There is a wind of 90 km h^{-1} blowing from N20° W .

a What course should the aeroplane set?

b What is the ground speed of the aeroplane?

Solution:

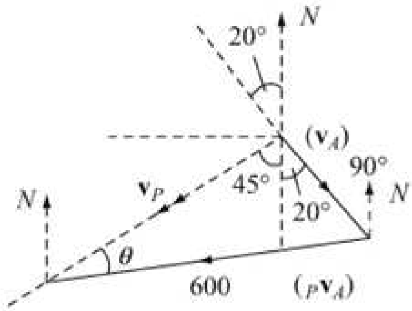
	Mag	Dir
${}_P\mathbf{v}_A$	600	?
$\mathbf{v}_A$	90	From N20° W
$\mathbf{v}_P$	?	SW

${}_P\mathbf{v}_A$ is the velocity of the plane relative to the air.

${}_P\mathbf{v}_A = \mathbf{v}_P - \mathbf{v}_A$

$\Rightarrow \mathbf{v}_P = \mathbf{v}_A + {}_P\mathbf{v}_A$

a Draw the vector Δ:



$$\frac{\sin\theta}{90} = \frac{\sin 65°}{600}$$

$$\Rightarrow \sin\theta = \frac{9\sin 65°}{60}$$

$$\Rightarrow \theta = 7.813°$$

Course is $\leq$ S 52.8° W

b 3rd angle of vector Δ

$$= 180° - (65° + 7.813°)$$

$$= 107.187°$$

$$\frac{|\mathbf{v}_P|}{\sin 107.187°} = \frac{600}{\sin 65°}$$

$$\Rightarrow |\mathbf{v}_P| = \frac{600\sin 107.187°}{\sin 65°} = 632.46.$$

i.e. ground speed of aeroplane is 632 km h^{-1} (nearest km h^{-1})

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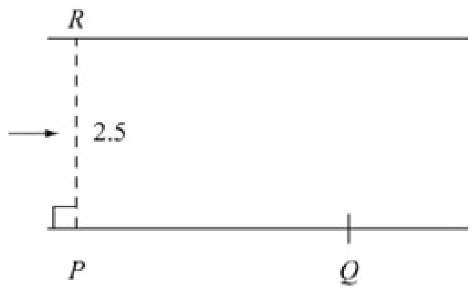
Relative motion
Exercise A, Question 10

Question:

A river flows at 2.5 m s^{-1}. A fish swims from a point P to a point Q which is directly upstream from P, and then back to P with speed 6.5 m s^{-1} relative to the water. A second fish, in the same time and with the same relative speed as the first fish, swims to the point R on the bank directly opposite to P and back to P. Find the ratio $PQ : PR$.

Solution:



$|\mathbf{v}_R| = 2.5$

$|_F\mathbf{v}_R| = 6.5$

$_F\mathbf{v}_R = \mathbf{v}_F - \mathbf{v}_R$

$\Rightarrow \mathbf{v}_F = {}_F\mathbf{v}_R + \mathbf{v}_R$

Fish 1

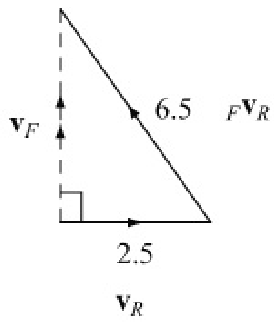
$$_Pt_Q = \left(\frac{PQ}{6.5+2.5}\right) = \frac{PQ}{9}$$

$$_Qt_P = \left(\frac{PQ}{6.5-2.5}\right) = \frac{PQ}{4}$$

$$\therefore \text{Total time} = \frac{PQ}{9} + \frac{PQ}{4} = \frac{13PQ}{36}$$

Fish 2 (P to R)

	Mag	Dir
$\mathbf{v}_R$	2.5	→
$_F\mathbf{v}_R$	6.5	?
$\mathbf{v}_F$	?	↑



$|\mathbf{v}_F| = \sqrt{6.5^2 - 2.5^2} = 6$

$|\mathbf{v}_F| = 6$ for R to P also. $\therefore \text{total time} = \frac{2PR}{6} = \frac{PR}{3}$

so, $\frac{13PQ}{36} = \frac{PR}{3} \Rightarrow PQ : PR = 12 : 13$

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Relative motion
Exercise A, Question 11

Question:

A man is cruising in a boat which is capable of a speed of 10 km h^{-1} in still water. He is heading towards a marker buoy which is NE of his position and 6 km away. The current is running at a speed of 3 km h^{-1} due E.

a What course should he set?

b How long will take to reach the buoy?

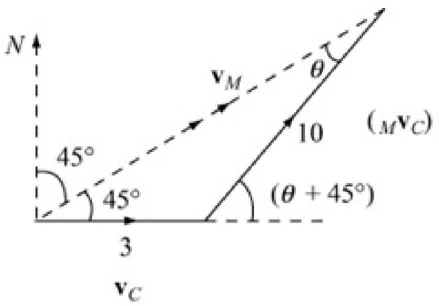
Solution:

a

	Mag	Dir
${}_M\mathbf{v}_C$	10	?
$\mathbf{v}_C$	3	due E
$\mathbf{v}_M$	?	NE

$${}_M\mathbf{v}_C = \mathbf{v}_M - \mathbf{v}_C$$
$$\Rightarrow \mathbf{v}_M = \mathbf{v}_C + {}_M\mathbf{v}_C$$

Draw the vector Δ:



$$\frac{\sin\theta}{3} = \frac{\sin 45°}{10}$$
$$\sin\theta = \frac{3\sqrt{2}}{20}$$
$$\Rightarrow \theta = 12.247°$$

Course is N $(90 - \theta - 45°)$E

i.e. N(32.753°)E

i.e. N32.8°E

b

$$\frac{|\mathbf{v}_M|}{\sin(\theta + 45°)} = \frac{10}{\sin 45°}$$

$$\Rightarrow |\mathbf{v}_M| = \frac{10\sin 57.247°}{\sin 45°} = 11.8936\ldots$$

$$\therefore \text{time} = \frac{6}{11.8936} = 30 \text{ minutes} \quad \text{(nearest minute)}$$

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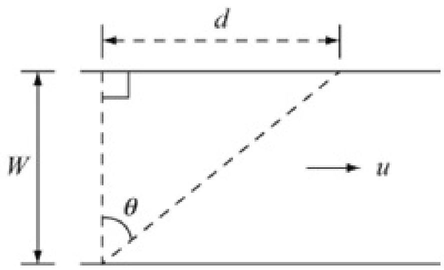
Relative motion
Exercise A, Question 12

Question:

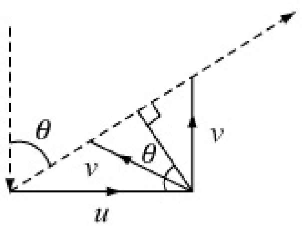
A river flows at a speed u. A boat is rowed with speed v relative to the river. The width of the river is w and the boat is to reach the opposite bank at a distance d downstream. Show that, if $\frac{uw}{\sqrt{w^2+d^2}} < v < u$, there are two directions in which the boat may be steered.

Solution:



	Mag	Dir
$\mathbf{v}_W$	u	$\rightarrow$
${}_B\mathbf{v}_W$	v	?
$\mathbf{v}_B$	?	θ

${}_B\mathbf{v}_W = \mathbf{v}_B - \mathbf{v}_W$
$\therefore \mathbf{v}_B = \mathbf{v}_W + {}_B\mathbf{v}_W$



Draw vector Δ:

Two possible positions for a vector of length v if $u > v > u\cos\theta$

From top diagram, $\cos\theta = \frac{w}{\sqrt{w^2+d^2}}$

as required.

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Relative motion
Exercise A, Question 13

Question:

A car is moving due W and the wind appears, to the driver, to be coming from a direction N60°W. When he drives due E at the same speed the wind appears to be coming from a direction N30°E. If he now travels due S at the same speed, find the apparent direction of the wind.

Solution:

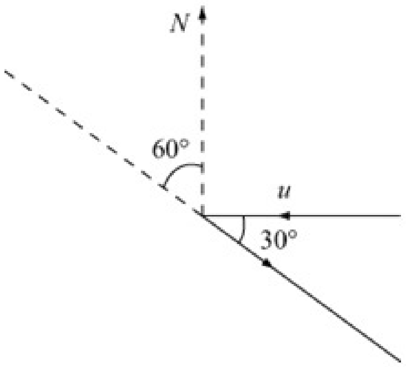
Scenario 1

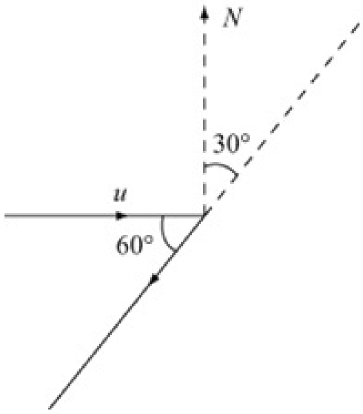
	Mag	Dir
$\mathbf{v}_C$	u	due W
${}_W\mathbf{v}_C$	?	From N60°W
$\mathbf{v}_W$	?	?

Scenario 2

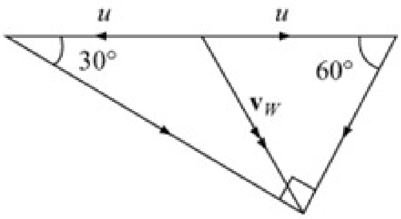
	Mag	Dir
$\mathbf{v}_C$	u	due E
${}_W\mathbf{v}_C$	?	From N30°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C \Rightarrow \mathbf{v}_W = \mathbf{v}_C + {}_W\mathbf{v}_C$$





Now, put the two triangles together, bearing in mind that the resultant, in both cases, is $\mathbf{v}_W$ i.e. will be a common side:

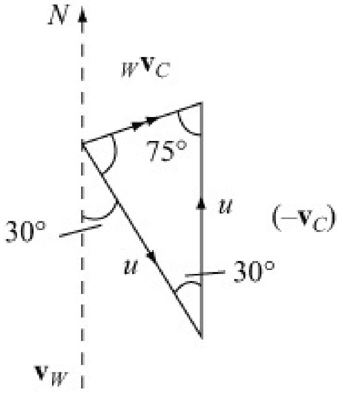


Using angle in a semi-circle is 90° property $|\mathbf{v}_W| = u$ (radius of circle). Then $RH\Delta$ is equilateral. Hence, direction of wind is on a bearing of 150°(S30°E)

Scenario 3

	Mag	Dir
$\mathbf{v}_C$	u	due S
${}_W\mathbf{v}_C$	?	?
$\mathbf{v}_W$	u	S30°E

Draw vector Δ
$_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C$



Vector Δ is isosceles.
∴ base angles are both 75°
∴ direction of $_W\mathbf{v}_C$ is N75°E

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Relative motion
Exercise A, Question 14

Question:

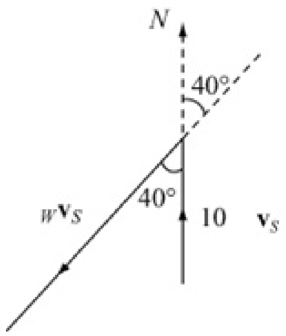
When a ship travels at 10 km h^{-1} due N the wind appears to be coming from a direction N40°E. When the speed is increased to 25 km h^{-1} the wind appears to be coming from a direction N25°E. Find the true speed and direction of the wind.

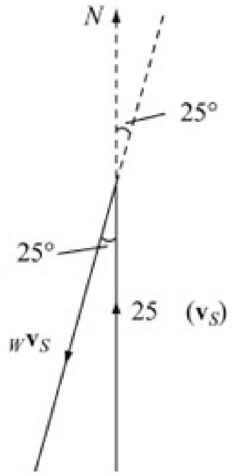
Solution:

	Mag	Dir
$\mathbf{v}_S$	10	due N
${}_W\mathbf{v}_S$	?	From N40°E
$\mathbf{v}_W$	?	?

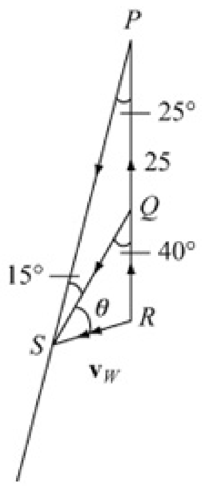
	Mag	Dir
$\mathbf{v}_S$	25	due N
${}_W\mathbf{v}_S$	?	N25°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_S = \mathbf{v}_W - \mathbf{v}_S \Rightarrow \mathbf{v}_W = \mathbf{v}_S + {}_W\mathbf{v}_S$$





We now put the two triangles together:



In ΔPQS, $PQ = 15$

$$\frac{QS}{\sin 25°} = \frac{15}{\sin 15°}$$

$$\Rightarrow QS = 24.493$$

In ΔQRS,

$$|\mathbf{v}_W|^2 = 24.493^2 + 10^2 - 2 \times 24.493 \times 10 \cos 40°$$

$$|\mathbf{v}_W| = 18.02 \text{ km h}^{-1}$$

In ΔQRS,

$$\frac{\sin\theta}{10} = \frac{\sin 40°}{18.02}$$

$$\Rightarrow \sin\theta = \frac{10\sin 40°}{18.02}$$

$$\Rightarrow \theta = 20.9°$$

$\therefore$ Speed of wind is 18.0 km h^{-1} from N60.9°E

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Relative motion
Exercise A, Question 15

Question:

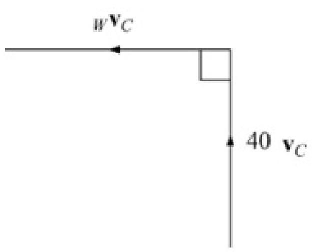
A woman cycles due N at 40 km h^{-1} and the wind seems to be blowing from the East. When she cycles due S at 50 km h^{-1}, the wind seems to be blowing from the South East. Find the true velocity of the wind.

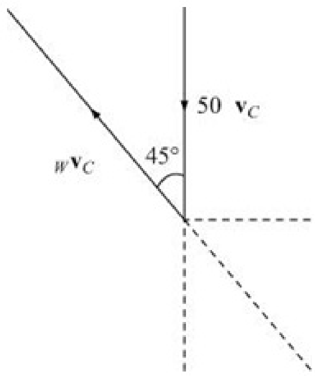
Solution:

	Mag	Dir
$\mathbf{v}_C$	40	due N
$_W\mathbf{v}_C$	?	From E
$\mathbf{v}_W$	?	?

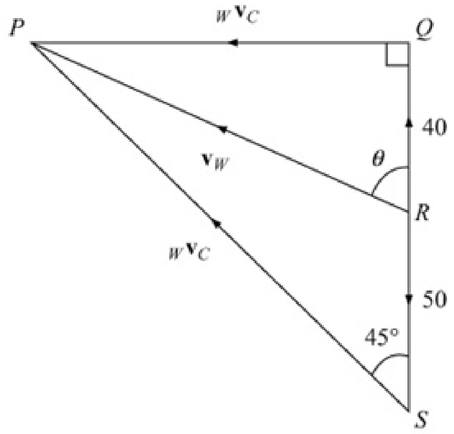
	Mag	Dir
$\mathbf{v}_C$	50	due S
$_W\mathbf{v}_C$	?	From SE
$\mathbf{v}_W$	?	?

$$_W\mathbf{v}_C = \mathbf{v}_W - \mathbf{v}_C \Rightarrow \mathbf{v}_W = \mathbf{v}_C + {_W\mathbf{v}_C}$$





We now put the two vector triangles together using the common side $(\mathbf{v}_W)$



$$Q\hat{P}S = 45° \text{ (From } \Delta PQS)$$
$$\Rightarrow PQ = 90$$
$$\Rightarrow |\mathbf{v}_W| = \sqrt{40^2 + 90^2}$$
$$= 10\sqrt{97} \simeq 98.5 \text{ km h}^{-1}$$
$$\tan\theta = \frac{90}{40}$$
$$\Rightarrow \theta = 66.0°$$

$\therefore$ Velocity of wind is 98.5 km h^{-1} from S66°E

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Relative motion
Exercise A, Question 16

Question:

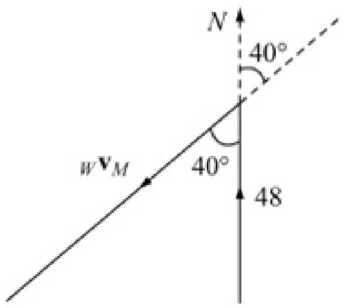
When a motorcyclist travels along a straight road at 48 km h^{-1} due N, the wind seems to be blowing from a direction N40°E. When he returns along the same road at the same speed, the wind seems to be blowing from a direction S30°E. Find the true speed and direction of the wind.

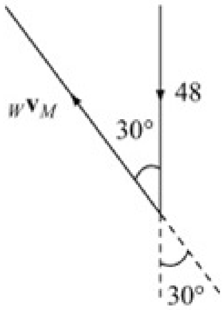
Solution:

	Mag	Dir
$\mathbf{v}_M$	48	due N
${}_W\mathbf{v}_M$	?	From N40°E
$\mathbf{v}_W$	?	?

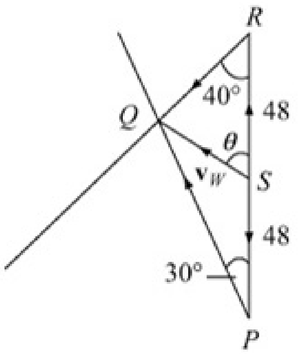
	Mag	Dir
$\mathbf{v}_M$	48	due S
${}_W\mathbf{v}_M$	?	From S30°E
$\mathbf{v}_W$	?	?

$${}_W\mathbf{v}_M = \mathbf{v}_W - \mathbf{v}_M \Rightarrow \mathbf{v}_W = \mathbf{v}_M + {}_W\mathbf{v}_M$$





Putting the two triangles together, using the common side $(\mathbf{v}_W)$



Let $Q\hat{S}R = \theta$

So $Q\hat{S}P = 180° - \theta$

In ΔPQR, $\dfrac{PQ}{\sin 40°} = \dfrac{QR}{\sin 30°} = \dfrac{96}{\sin 110°}$

$$\Rightarrow PQ = \frac{96\sin 40°}{\sin 110°} = 65.67 \text{ and } QR = \frac{96\sin 30°}{\sin 110°} = 51.08$$

In ΔPQS, $PQ^2 = 48^2 + QS^2 - 2\times 48\times QS\cos(180° - \theta)$ ①

In ΔQSR, $QR^2 = 48^2 + QS^2 - 2\times 48\times QS\cos\theta$ ②

① + ②: $PQ^2 + QR^2 = 2\times(48^2 + QS^2)$ ← since $\cos(180° - \theta) = -\cos\theta$

$$\Rightarrow QS = |\mathbf{v}_W| = \sqrt{\frac{65.67^2 + 51.08^2}{2} - 48^2}$$

$$= 34.0 \text{ km h}^{-1}$$

ΔQRS, $\dfrac{\sin\theta}{51.08} = \dfrac{\sin 40°}{34.01} \Rightarrow \sin\theta = \dfrac{51.08\sin 40°}{34.01}$

$$\Rightarrow \theta = 74.9°$$

∴ Velocity of wind is 34.0 km h^{-1} from S74.9°E

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Relative motion
Exercise B, Question 1

Question:

At 10.30 a.m. an aeroplane has position vector $(-100\mathbf{i}+220\mathbf{j})$ km and is moving with constant velocity $(300\mathbf{i}+400\mathbf{j})$km h^{-1}. At 10.45 a.m. a cargo plane has position vector $(-60\mathbf{i}+355\mathbf{j})$ km and is moving with constant velocity $(400\mathbf{i}+300\mathbf{j})$km h^{-1}.

a Show that the planes will crash if they maintain these velocities.
b Find the time at which the crash will occur.
c Find the position vector of the point at which the crash takes place.

Solution:

a position vector of aeroplane at 10.45 $=\begin{pmatrix}-100\\220\end{pmatrix}+\frac{1}{4}\begin{pmatrix}300\\400\end{pmatrix}=\begin{pmatrix}-25\\320\end{pmatrix}$

At time t h after 10.45 am:

$$\mathbf{r}_A=\begin{pmatrix}-25\\320\end{pmatrix}+t\begin{pmatrix}300\\400\end{pmatrix}$$

$$\mathbf{r}_C=\begin{pmatrix}-60\\355\end{pmatrix}+t\begin{pmatrix}400\\300\end{pmatrix}$$

$${}_A\mathbf{r}_C=\mathbf{r}_A-\mathbf{r}_C=\begin{pmatrix}35\\-35\end{pmatrix}+t\begin{pmatrix}-100\\100\end{pmatrix}=\begin{pmatrix}35-100t\\-35+100t\end{pmatrix}$$

Hence, ${}_A\mathbf{r}_C=0$ when

$t=\frac{35}{100}$ h

$=21$ minutes

b They collide at 11.06 a.m.

c $\mathbf{r}_A=\begin{pmatrix}-25\\320\end{pmatrix}+\frac{35}{100}\begin{pmatrix}300\\400\end{pmatrix}=\begin{pmatrix}80\\460\end{pmatrix}$

They collide at the point with position vector $(80\mathbf{i}+460\mathbf{j})$km

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Relative motion
Exercise B, Question 2

Question:

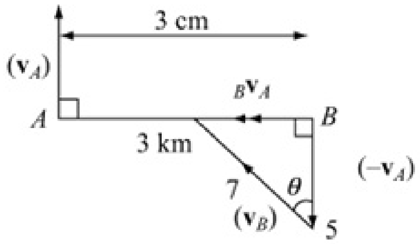
Hiker A is 3 km due W of hiker B. Hiker A walks due N at 5 km h^{-1}. Hiker B starts at the same time and walks at 7 km h^{-1}.

a In what direction should B walk in order to meet A?

b How long will it take to do so?

Solution:

Fix A (i.e. consider motion relative to A).



In velocity Δ,

$\cos\theta = \frac{5}{7}$

$\Rightarrow \theta = 44.4°$

a B should walk N44.4° W

b $|_B\mathbf{v}_A| = \sqrt{7^2 - 5^2} = \sqrt{24}$

$\therefore \text{Time} = \frac{3}{\sqrt{24}} = \frac{3}{2\sqrt{6}} = \frac{\sqrt{6}}{4} \text{ h}$

$= 36.7 \text{ minutes}$

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Relative motion
Exercise B, Question 3

Question:

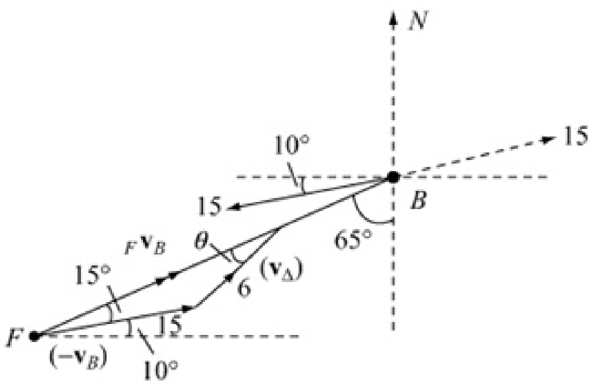
A batsman strikes a cricket ball at 15 m s^{-1} on a bearing of $260°$. A fielder is standing 45 m from the batsman on a bearing of $245°$. He runs at 6 m s^{-1} to intercept the ball.

a Find the direction in which the fielder should run in order to intercept the ball as quickly as possible.

b Find the time, to 1 decimal place, that it takes him to do so.

Solution:

a



Fix the ball
(i.e. consider motion relative to the ball)
Using sine rule on vector Δ

$$\frac{\sin\theta}{15} = \frac{\sin 15°}{6}$$

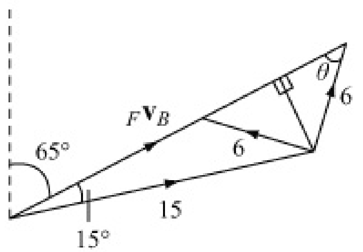
$$\sin\theta = \frac{5\sin 15°}{2}$$

$\theta = 40.32°$ (assuming θ is acute)

θ could be $180° - 40.32°$ (see below)

$\therefore$ Direction of $\mathbf{v}_F$ is $\text{N}(65° - 40.32°)\text{E}$

i.e. N24.7°E



There are 2 possible directions for $\mathbf{v}_F$, as shown is the diagram; the RH one will give the shortest interception time.

b Third angle in the vector Δ is $180° - (15° + \theta) = 124.68°$

$$\frac{|_F\mathbf{v}_B|}{\sin 124.68°} = \frac{6}{\sin 15°}$$

$$\Rightarrow |_F\mathbf{v}_B| = \frac{6\sin 124.68°}{\sin 15°} = 19.0637\ldots$$

$$\text{Time} = \frac{45}{19.0637} = 2.4 \text{ s (1 d.p.)}$$

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Relative motion
Exercise B, Question 4

Question:

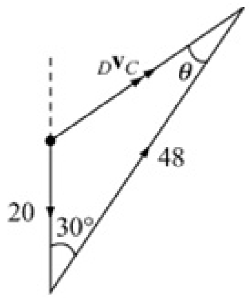
A destroyer, moving at 48 km h^{-1} in a direction N30°E, observes, at 12 noon, a cargo ship which is steaming due N at 20 km h^{-1}. The destroyer intercepts the cargo ship at 12.45 pm. Find

a the distance of the cargo ship from the destroyer at 12 noon,

b the bearing of the cargo ship from the destroyer at 12 noon.

Solution:

a Fix the cargo ship (i.e. consider motion relative to the cargo ship)
i.e. apply a vector of magnitude 20 due S to both.



by cosine rule,

$$|_D\mathbf{v}_C|^2 = 20^2 + 48^2 - 2 \times 20 \times 48 \cos 30°$$

$$|_D\mathbf{v}_C| = 32.268 \text{ km h}^{-1}$$

$$\text{Distance} = 0.75 \times 32.268$$

$$= 24.2 \text{ km}$$

b $\dfrac{\sin\theta}{20} = \dfrac{\sin 30°}{32.268} \Rightarrow \sin\theta = \dfrac{10}{32.268}$

$\Rightarrow \theta = 18.053°$

$\therefore$ Bearing is $(30° + \theta) = 48.1°$

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Relative motion
Exercise B, Question 5

Question:

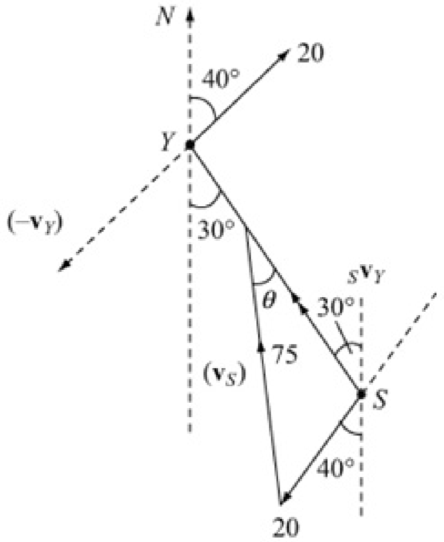
A speedboat moving at 75 km h^{-1} wishes to intercept a yacht which is moving at 20 km h^{-1} in a direction $040°$. Initially the speedboat is 10 km from the yacht on a bearing of 150°.

a Find the course that the speedboat should set in order to intercept the yacht.

b Find how long the journey will take.

Solution:

a



Fix the yacht (i.e. consider the motion relative to the yacht)

$$\frac{\sin 110°}{75} = \frac{\sin\theta}{20}$$

$$\frac{4\sin 110°}{15} = \sin\theta \Rightarrow \theta = 14.512°$$

Third angle of vector Δ is

$180° - 110° - 14.512° = 55.488°$

Course is N15.5°W

b $$\frac{|_S\mathbf{v}_Y|}{\sin 55.488°} = \frac{75}{\sin 110°} \Rightarrow |_S\mathbf{v}_Y| = 65.7667\ldots$$

$$\therefore \text{Time} = \frac{10}{65.7667}\text{ h} = 9.1\text{ minutes (1 d.p.)}$$

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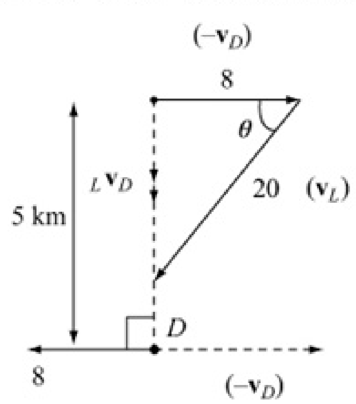
Relative motion
Exercise B, Question 6

Question:

A lifeboat sets out from a harbour at 10.10 a.m. to go to the assistance of a dinghy which is, at that time, 5 km due S of the harbour and drifting at 8 km h^{-1} due W. The lifeboat can travel at 20 km h^{-1}. Find the course that it should set in order to reach the yacht as quickly as possible and find the time when it arrives.

Solution:

Fix the dinghy (i.e. consider the motion relative to the dinghy)



$\cos\theta = \frac{8}{20} = 0.4$

$\Rightarrow \theta = 66.42°$

$90° - \theta = 23.58°$

Course is S23.6°W

$|{}_L\mathbf{v}_D| = \sqrt{20^2 - 8^2} = \sqrt{336}$

$\therefore \text{Time} = \frac{5}{\sqrt{336}} = 16.4$ minutes.

$\therefore$ Arrives at 10.26 a.m.

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Relative motion
Exercise B, Question 7

Question:

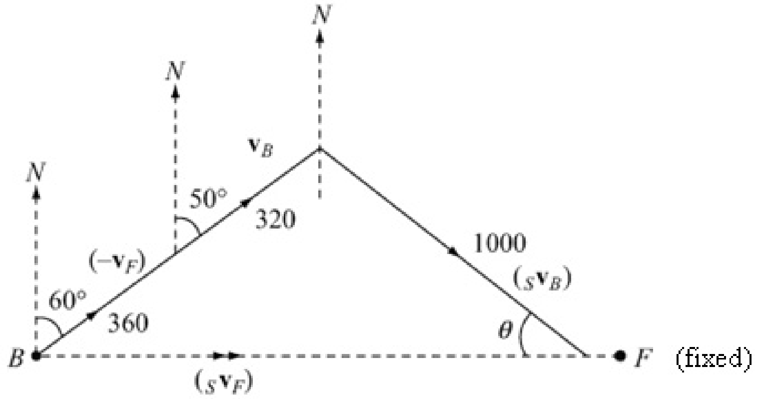
A gunner in a bomber, which is flying N50°E at 320 m s^{-1} wishes to fire at a fighter plane which is flying S60°W at 360 m s^{-1}. If the gun fires its shell at 1000 m s^{-1}, in what direction should the gun be aimed when the fighter is due E of the bomber?

Solution:

Fix the fighter by applying a vector 360 m s^{-1} N60°E

Then ${}_B\mathbf{v}_F + {}_S\mathbf{v}_B = {}_S\mathbf{v}_F$

i.e. $\mathbf{v}_B - \mathbf{v}_F + {}_S\mathbf{v}_B = {}_S\mathbf{v}_F$



$360\cos 60° + 320\cos 50° - 1000\sin\theta = 0$

$$\Rightarrow \sin\theta = \frac{180 + 320\cos 50°}{1000}$$

$$\Rightarrow \theta = 22.7° \Rightarrow 90° - \theta = 67.3°$$

Direction of gun is S67.3° E

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Relative motion
Exercise C, Question 1

Question:

The position vectors and velocity vectors of two ships P and Q at 9 a.m. are as follows

$\mathbf{r}_P = (2\mathbf{i}+\mathbf{j})\text{km} \qquad \mathbf{v}_P = (3\mathbf{i}+\mathbf{j})\text{km h}^{-1}$

$\mathbf{r}_Q = (-\mathbf{i}-4\mathbf{j})\text{km} \qquad \mathbf{v}_Q = (11\mathbf{i}+3\mathbf{j})\text{km h}^{-1}$

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

a

$$\mathbf{r}_P = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}3\\1\end{pmatrix}, t \text{ hrs after 9a.m.}$$

$$\mathbf{r}_Q = \begin{pmatrix}-1\\-4\end{pmatrix} + t\begin{pmatrix}11\\3\end{pmatrix}, t \text{ hrs after 9a.m.}$$

$$\Rightarrow {}_P\mathbf{r}_Q = \begin{pmatrix}3\\5\end{pmatrix} + t\begin{pmatrix}-8\\-2\end{pmatrix} = \begin{pmatrix}3-8t\\5-2t\end{pmatrix}$$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 = (3-8t)^2 + (5-2t)^2 = X \text{ say}$$

$$\frac{dX}{dt} = -16(3-8t) - 4(5-2t) = 0 \quad \text{for a minimum}$$

$$\Rightarrow 12 - 32t + 5 - 2t = 0$$

$$\Rightarrow 17 = 34t$$

$$\Rightarrow \frac{1}{2} = t$$

a and **b** $\therefore X_{\min} = (-1)^2 + 4^2 = 17$

$\therefore$ closest distance is $\sqrt{17}$ km at 9.30 a.m.

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Relative motion
Exercise C, Question 2

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (\mathbf{i}+4\mathbf{j})\text{km} \qquad \mathbf{v}_P = (4\mathbf{i}+8\mathbf{j})\text{km h}^{-1}$ at 9 a.m.

$\mathbf{r}_Q = (20\mathbf{j})\text{km} \qquad \mathbf{v}_Q = (9\mathbf{i}-2\mathbf{j})\text{km h}^{-1}$ at 8 a.m.

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

At t hours after 9 a.m.,

$$\mathbf{r}_P = \begin{pmatrix}1\\4\end{pmatrix} + t\begin{pmatrix}4\\8\end{pmatrix} = \begin{pmatrix}1+4t\\4+8t\end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix}0\\20\end{pmatrix} + (t+1)\begin{pmatrix}9\\-2\end{pmatrix} = \begin{pmatrix}9+9t\\18-2t\end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix}-8-5t\\-14+10t\end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix}-5\\10\end{pmatrix} \quad \text{(Differentiating with respect to } t\text{)}$$

Closest when ${}_P\mathbf{r}_Q \cdot {}_P\mathbf{v}_Q = 0$

i.e. $\begin{pmatrix}-8-5t\\-14+10t\end{pmatrix} \cdot \begin{pmatrix}-5\\10\end{pmatrix} = 0$

$$40+25t-140+100t = 0$$

$$125t = 100$$

$$t = 0.8\text{ h} \quad \text{(48 minutes)}$$

so, ${}_P\mathbf{r}_Q = \begin{pmatrix}-8-4\\-14+8\end{pmatrix} = \begin{pmatrix}-12\\-6\end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q| = 6\sqrt{1^2+2^2} = 6\sqrt{5}$

a $\therefore$ Least distance between P and Q is $6\sqrt{5}$ km

b This occurs at 9.48 a.m.

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Relative motion
Exercise C, Question 3

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (8\mathbf{i} - \mathbf{j})\text{km} \quad \mathbf{v}_P = (3\mathbf{i} + 7\mathbf{j})\text{km h}^{-1}$ at 3 p.m.

$\mathbf{r}_Q = (3\mathbf{i} + \mathbf{j})\text{km} \quad \mathbf{v}_Q = (2\mathbf{i} + 3\mathbf{j})\text{km h}^{-1}$ at 2 p.m.

Assuming that these velocities remain constant, find

a the least distance between P and Q in the subsequent motion,

b the time at which this least separation occurs.

Solution:

At t hours after 3 p.m.:

$$\mathbf{r}_P = \begin{pmatrix} 8 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 8+3t \\ -1+7t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (t+1)\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5+2t \\ 4+3t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} +3+t \\ -5+4t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} +3+t \\ -5+4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} = 0 \text{ for closest approach}$$

$$+3+t-20+16t = 0$$

$$17t = 17$$

$$t = 1$$

Then $${}_P\mathbf{r}_Q = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q| = \sqrt{17}\text{ km}$$

Least distance is $\sqrt{17}$ km at 4 p.m.

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Relative motion
Exercise C, Question 4

Question:

The position vectors and velocity vectors of two ships P and Q at 3 p.m. are as follows

$\mathbf{r}_P = (3\mathbf{i} - 5\mathbf{j})\text{km} \qquad \mathbf{v}_P = (15\mathbf{i} + 14\mathbf{j})\text{km h}^{-1}$

$\mathbf{r}_Q = (13\mathbf{i} + 5\mathbf{j})\text{km} \qquad \mathbf{v}_Q = (3\mathbf{i} - 10\mathbf{j})\text{ km h}^{-1}$

Assuming that these velocities remain constant,

a find the least distance between P and Q in the subsequent motion.

Ship Q has guns with a range of up to 5 km.

b Find the length of time for which ship P is within the range of ship Q's guns.

Solution:

At t hours after 3 p.m.:

$$\mathbf{r}_P = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + t\begin{pmatrix} 15 \\ 14 \end{pmatrix} = \begin{pmatrix} 3+15t \\ -5+14t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 13 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -10 \end{pmatrix} = \begin{pmatrix} 13+3t \\ 5-10t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} -10+12t \\ -10+24t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} -10+12t \\ -10+24t \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 24 \end{pmatrix} = 0, \quad \text{for closest approach}$$

$$-120 + 144t - 240 + 576t = 0$$

$$720t = 360$$

$$t = \frac{1}{2}$$

a Then, ${}_P\mathbf{r}_Q = \begin{pmatrix} -4 \\ 2 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q|_{\min} = \sqrt{20} = 2\sqrt{5}$ km

b

$$\text{Need } |{}_P\mathbf{r}_Q| \leq 5$$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 \leq 25$$

$$\Rightarrow (12t-10)^2 + (24t-10)^2 \leq 25$$

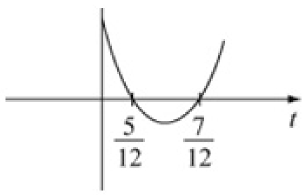
$$\Rightarrow 144t^2 - 240t + 100 + 576t^2 - 480t + 100 - 25 \leq 0$$

$$\Rightarrow 720t^2 - 720t + 175 \leq 0$$

$$\Rightarrow 144t^2 - 144t + 35 \leq 0$$

$$\Rightarrow (12t-7)(12t-5) \leq 0$$

$$\Rightarrow \frac{5}{12} \leq t \leq \frac{7}{12}.$$



$$\therefore \text{Length of time} = \frac{7}{12} - \frac{5}{12} = \frac{1}{6} \text{ hour}$$

i.e. 10 minutes

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Relative motion
Exercise C, Question 5

Question:

The position vectors and velocity vectors of two ships P and Q at certain times are as follows

$\mathbf{r}_P = (-2\mathbf{i} + 3\mathbf{j})\text{km}$ $\quad \mathbf{v}_P = (12\mathbf{i} - 4\mathbf{j})\text{km h}^{-1}$ $\quad$ at 2.45 p.m.

$\mathbf{r}_Q = (8\mathbf{i} + 7\mathbf{j})\text{km}$ $\quad \mathbf{v}_Q = (2\mathbf{i} - 14\mathbf{j})\text{km h}^{-1}$ $\quad$ at 3 p.m.

Assuming that these velocities remain constant,

a find the least distance between P and Q in the subsequent motion.

Ship Q has guns with a range of up to 2 km.

b Find the length of time for which ship P is within the range of ship Q's guns.

Solution:

At t hours after 2.45 p.m.,

$$\mathbf{r}_P = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ -4 \end{pmatrix} = \begin{pmatrix} -2+12t \\ 3-4t \end{pmatrix}$$

$$\mathbf{r}_Q = \begin{pmatrix} 8 \\ 7 \end{pmatrix} + \left(t-\frac{1}{4}\right)\begin{pmatrix} 2 \\ -14 \end{pmatrix} = \begin{pmatrix} 7\frac{1}{2}+2t \\ 10\frac{1}{2}-14t \end{pmatrix}$$

$${}_P\mathbf{r}_Q = \begin{pmatrix} -9\frac{1}{2}+10t \\ -7\frac{1}{2}+10t \end{pmatrix} \Rightarrow {}_P\mathbf{v}_Q = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} -9\frac{1}{2}+10t \\ -7\frac{1}{2}+10t \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 10 \end{pmatrix} = 0 \quad \text{for closest approach}$$

$$\Rightarrow -95+100t-75+100t = 0$$

$$200t = 170$$

$$t = \frac{17}{20}$$

Then $${}_P\mathbf{r}_Q = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow |{}_P\mathbf{r}_Q|_{\min} = \sqrt{2} \text{ km}$$

b Need $|{}_P\mathbf{r}_Q| \le 2$

$$\Rightarrow |{}_P\mathbf{r}_Q|^2 \le 4$$

$$\Rightarrow \left(10t-9\frac{1}{2}\right)^2 + \left(10t-7\frac{1}{2}\right)^2 \le 4$$

$$\Rightarrow 100t^2-190t+90.25+100t^2-150t+56.25-4 \le 0$$

$$\Rightarrow 200t^2-340t+142.5 \le 0$$

Roots given by $$t = \frac{340 \pm \sqrt{(340)^2 - 4\times 200\times 142.5}}{400}$$

$$= \frac{340\pm 40}{400} = \frac{15}{20} \quad \text{or} \quad \frac{19}{20}$$

$$\therefore \frac{15}{20} \le t \le \frac{19}{20}$$

$$\therefore \text{Length of time} = \frac{4}{20} = \frac{1}{5}\text{ h} = 12 \text{ minutes}$$

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Relative motion
Exercise D, Question 1

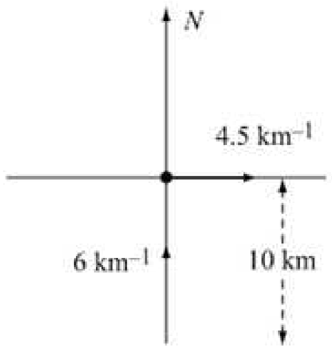
Question:

Two straight roads cross at right angles. A woman leaves the cross-roads and walks due E at 4.5 km h^{-1}. At the same time another woman leaves a point 10 km due S of the cross-roads and walks due N at 6 km h^{-1}.

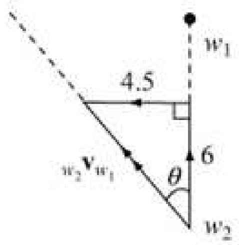
a After how long will they be closest together?

b How far apart will they then be?

Solution:

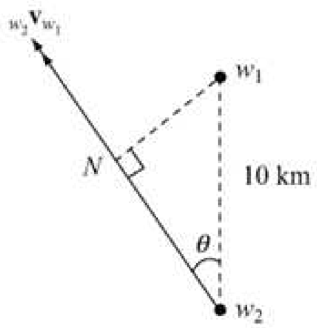


Fix first woman
(i.e. apply a velocity 4.5 km h^{-1} due W to both)



$$\tan\theta = \frac{4.5}{6} = \frac{3}{4}$$

a



N is closest approach position.

$w_2N = 10\cos\theta = 8$ km

$$\therefore \text{Time} = \frac{8}{\sqrt{4.5^2 + 6^2}} = \frac{8}{7.5} = \frac{16}{15}\text{ h}$$

$\therefore$ Closest after 1 hr 4 minutes

b $w_1N = 10\sin\theta = 10 \times \frac{3}{5} = 6$ km

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Relative motion
Exercise D, Question 2

Question:

Two trains are travelling on railway lines which cross at right angles. The first train is travelling at 45 km h^{-1} and the second is travelling at 108 km h^{-1}.

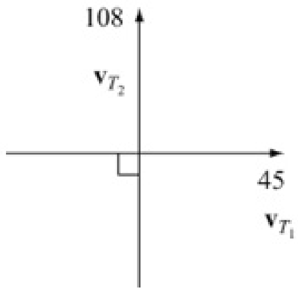
a Find their relative speed.

The slower train passes the point where the lines cross one minute before the faster train.

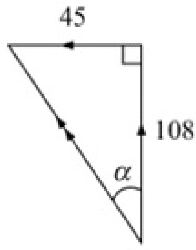
b Find the shortest distance between the trains.

Solution:

a



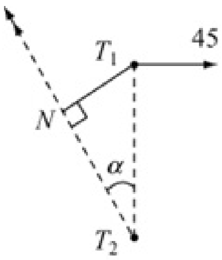
$_{T_2}\mathbf{v}_{T_1} = \mathbf{v}_{T_2} - \mathbf{v}_{T_1}$



$$\therefore \text{Relative speed} = |_{T_2}\mathbf{v}_{T_1}| = \sqrt{45^2 + 108^2} = 117 \text{ km h}^{-1}$$

b At $t = 0$:

$$T_1T_2 = \frac{108}{60} = 1.8 \text{ km}$$



T_1N is shortest distance

$$= 1.8 \sin\alpha = 1.8 \times \frac{45}{117} = \frac{9}{13} \text{ km}$$

$$\simeq 0.692 \text{ km (3 s.f.)}$$

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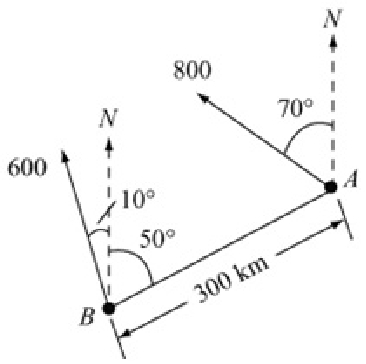
Relative motion
Exercise D, Question 3

Question:

At 10 a.m. an aircraft A is 300 km N50°E of another aircraft B. Aircraft A is flying at 800 km h^{-1} in the direction N70°W and aircraft B is flying at 600 km h^{-1} in the direction N10°W.

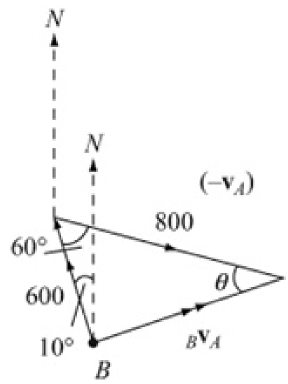
a Find the least distance between the aircraft in the subsequent motion.
b Find the time when they are closest to each other.

Solution:



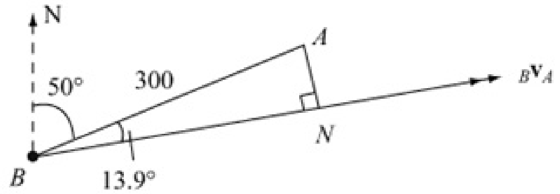
Fix A (i.e. consider motion relative to A)
Apply a vector 800 S70°E to both:
by cos rule,

$$|{}_B\mathbf{v}_A|^2 = 600^2 + 800^2 - 2\times 600\times 800\cos 60°$$
$$= 520\,000$$
$$|{}_B\mathbf{v}_A| = 100\sqrt{52}$$



$$\frac{\sin\theta}{600} = \frac{\sin 60°}{100\sqrt{52}}$$
$$\sin\theta = \frac{3\sqrt{3}}{\sqrt{52}}$$
$$\theta = 46.1°$$

Third angle is
$180° - 60° - 46.1° = 73.9°$
Direction of ${}_B\mathbf{v}_A$ is N63.9°E



N is the point of closest approach.

$AN = 300\sin 13.9° = 72.1$ km (3 s.f.)

$BN = 300\cos 13.9° = 291.21\ldots$

$$\text{Time} = \frac{291.21}{100\sqrt{52}}\text{ h} = 0.4038..$$
$$= 24.2 \text{ minutes}$$

Least distance between then is 72.1 km at 10.24 (nearest minute)

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Relative motion
Exercise D, Question 4

Question:

A ship P steams at 20 km h^{-1} on a bearing of $015°$. Another ship Q steams at 12 km h^{-1} on a bearing of $330°$.

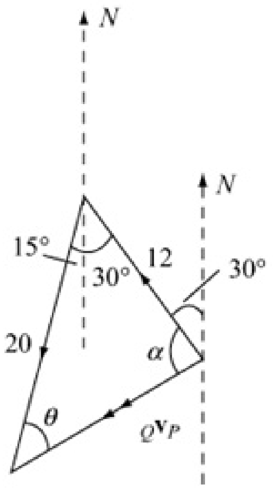
a Find the velocity of Q relative to P.

At 12 noon Q is 5 km due E of P. If they maintain their velocities,

b find the shortest distance between the ships.

Solution:

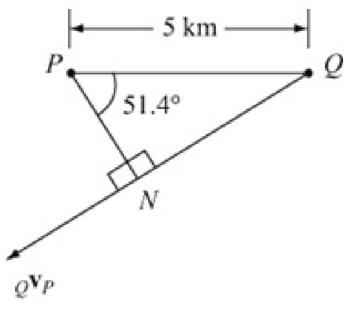
a ${}_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P$



$$|{}_Q\mathbf{v}_P|^2 = 20^2 + 12^2 - 2\times 20\times 12\cos 45°$$
$$= 544 - 240\sqrt{2}$$
$$|{}_Q\mathbf{v}_P| = 14.3 \text{ km h}^{-1}$$
$$\frac{\sin\theta}{12} = \frac{\sin 45°}{14.303\ldots} \Rightarrow \sin\theta = \frac{12\sin 45°}{14.303\ldots}$$
$$\Rightarrow \theta = 36.4°$$
$$\alpha = 180° - 45° - 36.4° = 98.6°$$

$\therefore$ Direction of ${}_Q\mathbf{v}_P$ is on a bearing $(180° + 51.4°)$ i.e. $231.4°$.

b At noon:



N is the point of closest approach.

Shortest distance between P and $Q = PN = 5\cos 51.4°$
$= 3.12$ km

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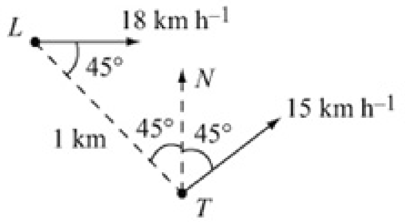
Relative motion
Exercise D, Question 5

Question:

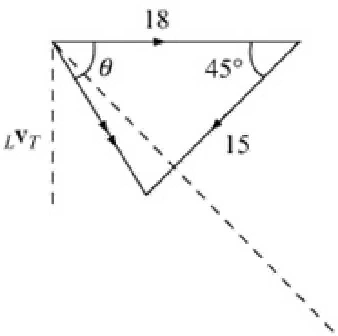
At a particular instant a liner is 1 km NW of a tanker. The liner is moving at 18 km h^{-1} due E and the tanker is moving at 15 km h^{-1} NE.

a Find the shortest distance between the ships.
b Find the interval of time that passes until they are at the point of closest approach.

Solution:

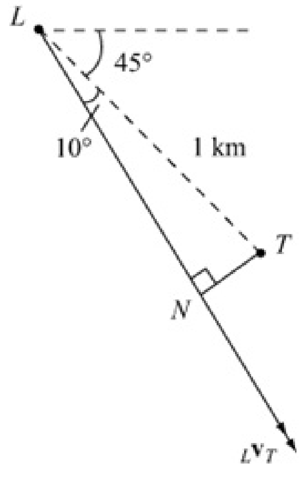


Fix the tanker
(i.e. apply a vector of 15 km h^{-1} SW to both)



by cos rule,

$$|_L\mathbf{v}_T|^2 = 18^2 + 15^2 - 2 \times 18 \times 15 \cos 45°$$
$$= 549 - 270\sqrt{2}$$
$$|_L\mathbf{v}_T| = 12.9\ (291)$$
$$\frac{\sin\theta}{15} = \frac{\sin 45°}{12.9291} \Rightarrow \sin\theta = \frac{15\sin 45°}{12.9291} \Rightarrow \theta = 55°$$



N is the point of closest approach.

$$TN = 1\sin 10°\ \text{km}$$
$$= 0.174\ \text{km}$$
$$\text{Time} = \frac{LN}{|_L\mathbf{v}_T|}$$
$$= \frac{1\cos 10°}{12.929}\ \text{h}$$
$$= 4.6\ \text{minutes}$$

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Relative motion
Exercise E, Question 1

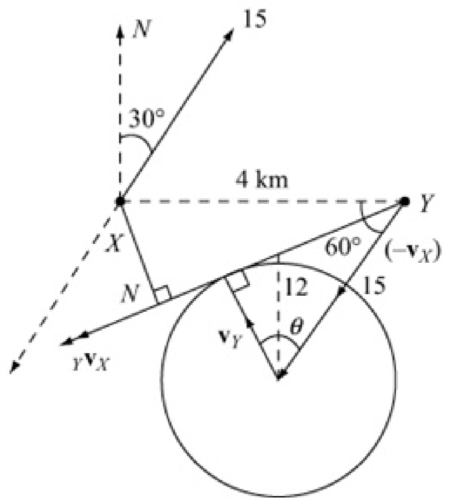
Question:

X and Y are two yachts and X is sailing at a constant speed of 15 km h^{-1} in a direction N30°E . At 2 p.m. Y is 4 km due E of X. Given that Y travels at a constant speed of 12 km h^{-1},

a show that it is not possible for Y to intercept X,
b find the course that Y should set in order to get as close as possible to X,
c find the shortest distance between the yachts,
d find the time when they are closest.

Solution:

a



Fix X (i.e. consider motion relative to X)
Since $15\sin 60° > 12$, impossible for Y to catch X.

b $\cos\theta = \frac{12}{15} = \frac{4}{5}$

$\Rightarrow \theta = 36.87°$

$\therefore$ course is $\theta - 30° = 6.87°$ W of N

Course for Y is N6.87°W

c N is the point of closest approach.

$X\hat{Y}N = 60° - (90° - \theta) = \theta - 30° = 6.87°$

$\therefore XN = 4\sin 6.87° = 0.48$ km

d $\text{Time} = \frac{NY}{|_Y\mathbf{v}_X|} = \frac{4\cos 6.87°}{\sqrt{15^2 - 12^2}} = \frac{4\cos 6.87°}{9}$ h

$= 26.5$ minutes

Time is $2.26\frac{1}{2}$ p.m.

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Relative motion
Exercise E, Question 2

Question:

Two aircraft P and Q are flying at the same altitude. At 12 noon aircraft Q is 5 km due S of aircraft P, and is flying at a constant 300 m s^{-1} in the direction N60°E. If aircraft P flies at a constant speed of 200 m s^{-1}, find

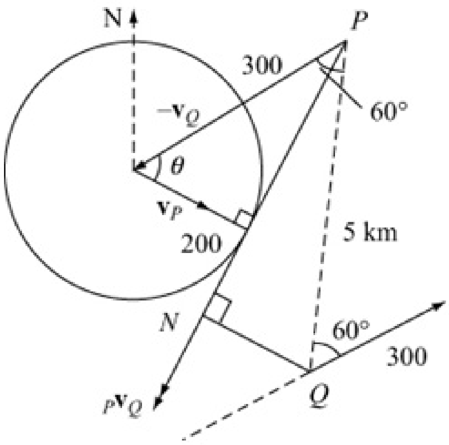
5 km due
f aircraft

a the direction in which it must fly in order to pass as close to aircraft Q as possible,
ossible,
b the distance between the planes when they are closest,
c the time when they are closest.

Solution:

At noon,
Fix Q (i.e. consider motion relative to Q)
by applying a vector of magnitude 300 m s^{-1}
in S60°W direction.
N is the point of closest approach,

$$\cos\theta = \frac{200}{300} \Rightarrow \theta = 48.19°$$



a Bearing of

$$\mathbf{v}_P = 60° + \theta$$
$$= 108° \text{(nearest degree)}$$

b Angle between ${}_P\mathbf{v}_Q$ and $PQ = 60° - (90° - \theta) = \theta - 30°$

$$= 18.19°$$

$\therefore$ Closest approach, $QN = 5\sin 18.19°$

$$= 1.56 \text{ km}$$

c $\text{Time} = \dfrac{PN}{|{}_P\mathbf{v}_Q|}$

$$= \frac{5\cos 18.19° \times 1000}{\sqrt{300^2 - 200^2}} = \frac{5\cos 18.19° \times 1000}{100\sqrt{5}} \text{ s}$$

$$= 21.2S \text{ (after 12 noon)}$$

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Relative motion
Exercise E, Question 3

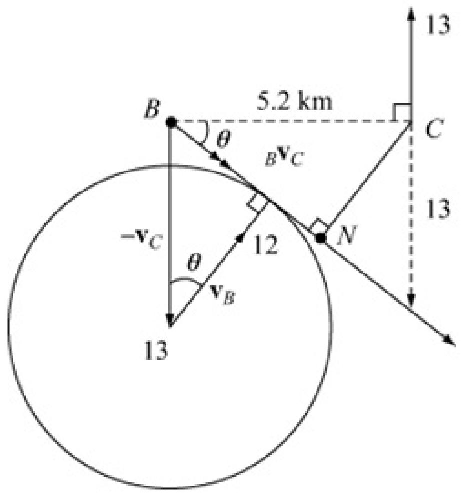
Question:

At 3 p.m. boat C is due E of boat B and $BC = 5.2$ km . Boat C is travelling due N at a constant speed of 13 km h^{-1}. Given that boat B travels at 12 km h^{-1}, find

a the course that B should set in order to get as close as possible to C,
b the shortest distance between the boats,
c the time when this occurs,
d the distance from the closest position of the boats to the initial position of B.

Solution:

a



Fix C (i.e. consider motion relative to C)

$_B\mathbf{v}_C = \sqrt{13^2 - 12^2} = 5 \text{ km h}^{-1}$

$\cos\theta = \frac{12}{13} \Rightarrow \theta = 22.62°$

Direction of B is N 22.6° E .

b Angle between $_B\mathbf{v}_C$ and $BC = 90° - (90° - \theta) = \theta$

$\therefore$ Least distance, $CN = 5.2 \sin\theta = 2$ km

c $\text{Time} = \frac{BN}{|_B\mathbf{v}_C|} = \frac{5.2\cos 22.62°}{5}$

$= 0.96$ h

$= 57.6$ minutes

$\therefore$ Time is 3.58 p.m. (nearest minute)

d Distance moved by

$B = 12 \times 0.96$

$= 11.52$ km

$= 11.5$ km (3 s.f.)

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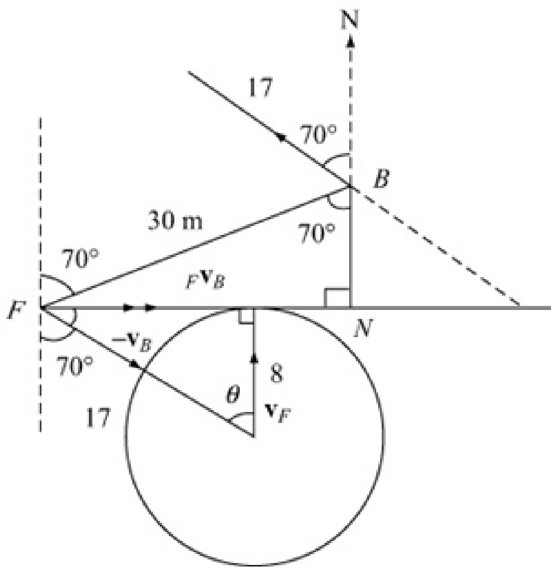
Relative motion
Exercise E, Question 4

Question:

A fielder is placed at a distance of 30 m from a batsman and on a bearing of 250°. The batsman hits the ball at 17 m s^{-1} in the direction N70°W. Given that the fielder runs at 8 m s^{-1} from the moment the ball is struck, and ignoring any change in the speed of the ball, find

a how close the fielder gets to the ball,
b the time, from the instant when the ball was struck, that it takes the fielder to get to the closest position.

Solution:



Fix the ball,
by applying a vector of magnitude 17 m s^{-1} in direction S70°E.
N is the point of closest approach.

$\cos\theta = \frac{8}{17} \Rightarrow \theta = 61.93°$

$\therefore$ Bearing of F's course is

$360° - (70 - 61.93°) = 351.93°$
$= 352° \text{ (3 s.f.)}$

a Angle $B\hat{F}N = 40° - (90° - \theta)$
$= \theta - 50° = 11.93°$

$\therefore$ Closest distance, $BN = 30 \sin 11.93° = 6.2 \text{ m}$

b $\text{Time} = \frac{FN}{\text{relative speed}} = \frac{30\cos 11.93°}{\sqrt{17^2 - 8^2}} = \frac{30\cos 11.93}{15}$
$= 2\cos 11.93°$
$= 1.96 \text{ s}$

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Relative motion
Exercise E, Question 5

Question:

At 10 a.m. a frigate F is 16 km due E of a cruiser C. The cruiser is moving at a constant speed of 40 km h^{-1} on a bearing of 030° and the frigate is moving at a constant speed of 20 km h^{-1}. Find

a the course that F should set in order to get as close as possible to C,
b the closest distance between them,
c the time when this occurs.

The guns on the frigate have a range of up to 10 km.

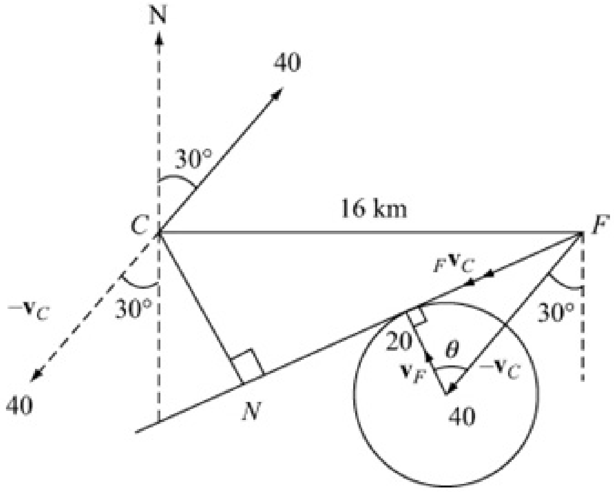
d Find the length of time for which C is within the range of ship F's guns.

The guns on the cruiser have a range of up to 9 km.

e Find the length of time for which F is within the range of ship C's guns.

Solution:

a



Fix C i.e. consider motion relative to C

$\cos\theta = \frac{20}{40} = \frac{1}{2}$

$\Rightarrow \theta = 60°$

b $\therefore$ Frigate sails an a bearing of $330°$

N is the point of closest approach.

$C\hat{F}N = 90° - (90° - \theta) - 30° = \theta - 30° = 30°$

$\therefore CN$, closest approach $= 16\sin 30° = 8$ km

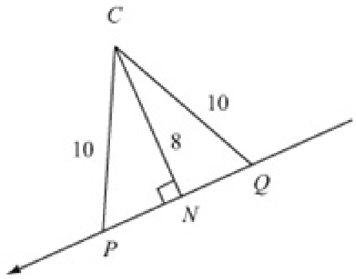
c $\text{Time} = \frac{FN}{|_F\mathbf{v}_C|} = \frac{16\cos 30°}{\sqrt{40^2 - 20^2}} = \frac{8\sqrt{3}}{10\sqrt{12}} = \frac{4}{5}\times\frac{1}{2}$

$= \frac{2}{5}$ h

$= 24$ minutes

Closest at 10.24 a.m.

d



$PQ = 2PN = 2\sqrt{10^2 - 8^2}$

$= 12$ km

$\text{Time} = \frac{12}{10\sqrt{12}}\text{ h} = 0.3464\text{ h}$

$= 20.8$ minutes

e Similarly, $\text{time} = \frac{2\sqrt{10^2 - 9^2}}{10\sqrt{12}}$

$= \frac{1}{5}\frac{\sqrt{19}}{\sqrt{12}} = 0.2516\ldots\text{ h}$

$= 15.1$ minutes

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Relative motion
Exercise F, Question 1

Question:

Particles P, Q and R move in a plane with constant velocities. At time $t = 0$ the position vectors of P, Q and R, relative to a fixed origin O, are $(\mathbf{i} + 3\mathbf{j})$ km, $(9\mathbf{i} + 9\mathbf{j})$ km and $(6\mathbf{i} + 13\mathbf{j})$ km respectively. The velocity of R relative to P is $(7\mathbf{i} - 10\mathbf{j})$ km h^{-1} and the velocity of R relative to Q is $(9\mathbf{i} - 12\mathbf{j})$ km h^{-1}.

a Find the velocity of Q relative to P.
b Show that P and Q do not collide.
c Find the shortest distance between P and Q.
d Find the time taken to reach the position of closest approach.
e Show that Q and R do collide.
f Find the distance between P and R when this collision occurs.

Solution:

a $_R\mathbf{v}_P = \mathbf{v}_R - \mathbf{v}_P = \begin{pmatrix} 7 \\ -10 \end{pmatrix}$ ①

$_R\mathbf{v}_Q = \mathbf{v}_R - \mathbf{v}_Q = \begin{pmatrix} 9 \\ -12 \end{pmatrix}$ ②

$_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P = (\mathbf{v}_R - \mathbf{v}_P) - (\mathbf{v}_R - \mathbf{v}_Q)$

$= \begin{pmatrix} 7 \\ -10 \end{pmatrix} - \begin{pmatrix} 9 \\ -12 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$_Q\mathbf{v}_P = (-2\mathbf{i} + 2\mathbf{j})\ \text{km h}^{-1}$

b $\mathbf{r}_P = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$, at $t = 0$

$\overrightarrow{QP} = -\mathbf{r}_Q + \mathbf{r}_P = -\begin{pmatrix} 9 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$

Since $_Q\mathbf{v}_P \neq k\overrightarrow{QP}$, P and Q will not collide.

c At time t,

$_Q\mathbf{r}_P = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + t\,_Q\mathbf{v}_P = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 8-2t \\ 6+2t \end{pmatrix}$

Closest when

$$_Q\mathbf{r}_P \cdot {_Q\mathbf{v}_P} = 0$$

$$\begin{pmatrix} 8-2t \\ 6+2t \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 0$$

$$-16 + 4t + 12 + 4t = 0$$

$$8t = 4$$

$$t = \frac{1}{2}\ \text{hr}$$

At $t = \frac{1}{2}$, $_Q\mathbf{r}_P = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \Rightarrow |_Q\mathbf{r}_P| = 7\sqrt{2}$ km

d $\frac{1}{2}$ h

e At $t = 0, \mathbf{r}_R - \mathbf{r}_Q = \begin{pmatrix} 6 \\ 13 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

$_Q\mathbf{v}_R = -_R\mathbf{v}_Q = \begin{pmatrix} -9 \\ 12 \end{pmatrix} = 3(\mathbf{r}_R - \mathbf{r}_Q)$ $\therefore$ collision occurs

f Collision when $t = \frac{1}{3}$

$${}_R\mathbf{r}_P = \left\{\begin{pmatrix}6\\13\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix}\right\} + t\begin{pmatrix}7\\-10\end{pmatrix}$$

When

$$t = \frac{1}{3}, {}_R\mathbf{r}_P = \begin{pmatrix}5\\10\end{pmatrix} + \frac{1}{3}\begin{pmatrix}7\\-10\end{pmatrix}$$

$$= \begin{pmatrix}\frac{22}{3}\\\frac{20}{3}\end{pmatrix} = \frac{2}{3}\begin{pmatrix}11\\10\end{pmatrix}$$

$$|{}_R\mathbf{r}_P| = \frac{2}{3}\sqrt{11^2+10^2} = \frac{2}{3}\sqrt{221} \simeq 9.91\,\text{km}$$

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Relative motion
Exercise F, Question 2

Question:

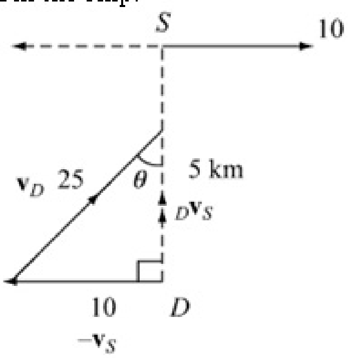
A ship is steaming due E at 10 km h^{-1}. A destroyer is 5 km due S of the ship and wishes to intercept it. If the destroyer can travel at 25 km h^{-1},

a in which direction will it travel,

b how long will it take?

Solution:

a Fix the ship.



$$\sin\theta = \frac{10}{25} = 0.4 \Rightarrow \theta = 23.6°$$

The destroyer should steer N23.6°E

b $\text{Time} = \frac{5}{\sqrt{25^2 - 10^2}} = \frac{5}{5\sqrt{5^2 - 2^2}} = \frac{1}{\sqrt{21}}\text{ h}$

$\cong 0.218\text{ h}$

$\cong 13.1\text{ minutes}$

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Relative motion
Exercise F, Question 3

Question:

Two trains S and T are moving at constant speed, S at 50 km h^{-1} NW and T at a speed $v \text{ km h}^{-1}$ due W. If the velocity of S relative T is NE in direction,

a show that it is 50 km h^{-1} in magnitude,

b find the value of v.

If the speeds of S and T are interchanged,

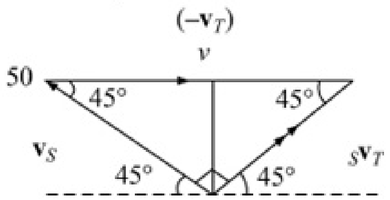
c find the velocity of S relative to T in magnitude and direction.

Solution:

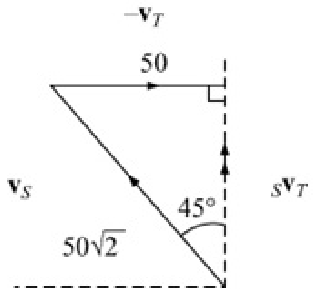
a ${}_S\mathbf{v}_T = \mathbf{v}_S - \mathbf{v}_T$: Vector Δ is isosceles,

$|{}_S\mathbf{v}_T| = 50$

b $\therefore v = 50\sqrt{2}$



c



$|{}_S\mathbf{v}_T| = 50 \text{ km h}^{-1}$
due N

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Relative motion
Exercise F, Question 4

Question:

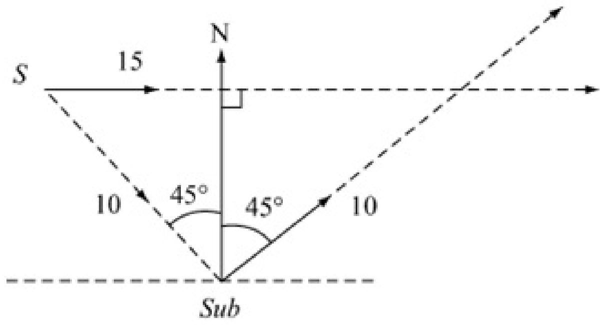
A ship is travelling due E at 15 km h^{-1} and is 10 km NW of a submarine. The submarine submerges immediately and travels at 10 km h^{-1} NE underwater.

a Show that when it crosses the ship's track, it is nearly 1 km behind.

b Find the nearest distance to which it has approached the ship.

Solution:

a



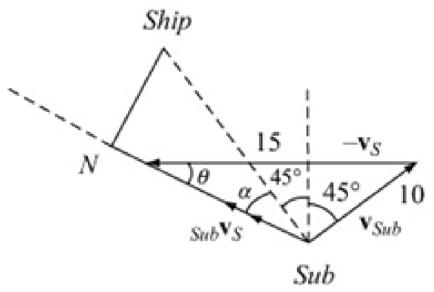
Time for sub to cross ship's track $= \frac{10}{10} = 1 \text{ h}$

Distance travelled East $= 10 \sin 45° = 5\sqrt{2} \simeq 7.07 \text{ km}$.

In 1 h, ship travels 15 km. $\therefore$distance of ship from sub $= 15 - 10\cos 45° - 5\sqrt{2}$

$15 - 10\sqrt{2} \simeq 1 \text{ km}$ i.e. sub is approximately 1 km behind.

b Fix ship; N is the point of closest approach.



cosine rule:

$$|{}_{SUB}\mathbf{v}_S|^2 = 10^2 + 15^2 - 2 \times 10 \times 15 \cos 45°$$
$$= 325 - 150\sqrt{2}$$
$$|{}_{SUB}\mathbf{v}_S| = 10.624$$
$$\frac{\sin\theta}{10} = \frac{\sin 45°}{10.624}$$
$$\Rightarrow \theta = 41.73°$$

So,

$\alpha = 180° - 135° - \theta$

$= 3.27°$

$\therefore$closest approach $= 10 \sin\alpha = 0.571 \text{ km}$

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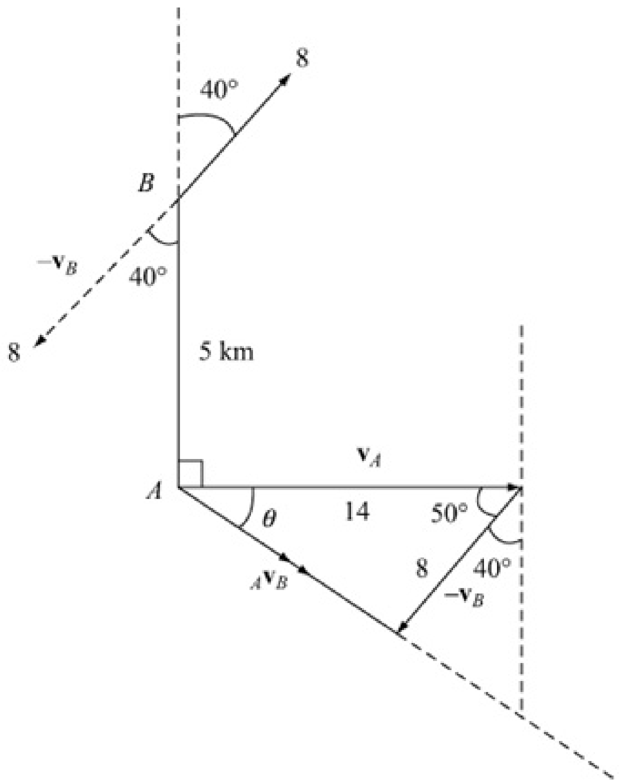
Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise F, Question 5

Question:

A ship A is moving at 14 km h^{-1} due E and a ship B is moving at 8 km h^{-1} on a bearing of $040°$. At 2 p.m., A is 5 km due S of B. If the limit of visibility is 12 km, for how long after 2 p.m. is B visible to A?

Solution:



Fix B i.e. consider motion relative to B.

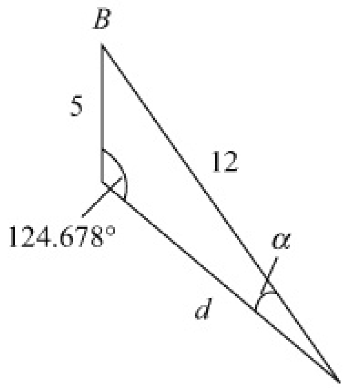
$$|_A\mathbf{v}_B|^2 = 14^2 + 8^2 - 2\times 14\times 8\cos 50^\circ$$
$$= 260 - 224\cos 50^\circ$$
$$|_A\mathbf{v}_B| = 10.771 \text{ km h}^{-1}$$

Velocity Δ

sine rule

$$\frac{\sin\theta}{8} = \frac{\sin 50^\circ}{10.771}$$
$$\Rightarrow \theta = 34.678^\circ$$

Displacement Δ



$$\frac{\sin\alpha}{5} = \frac{\sin 124.678^\circ}{12}$$
$$\Rightarrow \sin\alpha = \frac{5\sin 124.678^\circ}{12}$$
$$\Rightarrow \alpha = 20.04^\circ$$
$$\therefore \frac{d}{\sin 35.284} = \frac{12}{\sin 124.678}$$
$$\Rightarrow d = 8.4288$$
$$\therefore \text{Time} = \frac{8.4288}{|_A\mathbf{v}_B|}\text{h} = 47 \text{ minutes}$$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise F, Question 6

Question:

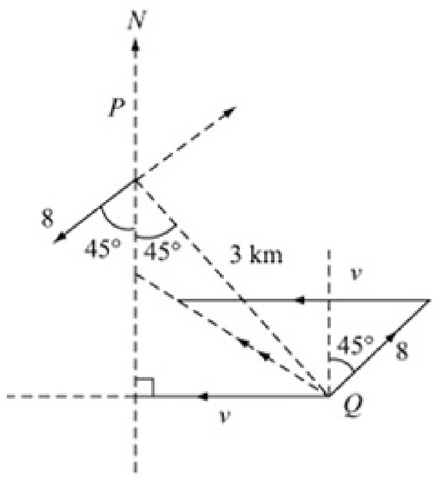
A ship P is steaming on a bearing of $225°$ at a constant speed of 8 km h^{-1}. A second ship Q is sighted, 3 km SE of P, steaming due W at a constant speed. After a certain time, Q is sighted 1 km due S of P. Find

a the time taken, from the instant when Q is first sighted, to the instant when Q is due W of P,

b the distance the ships are then apart,

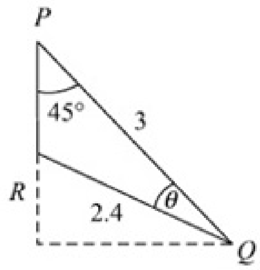
c the velocity of Q relative to P.

Solution:



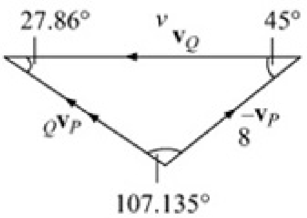
Fix P.

Displacement Δ



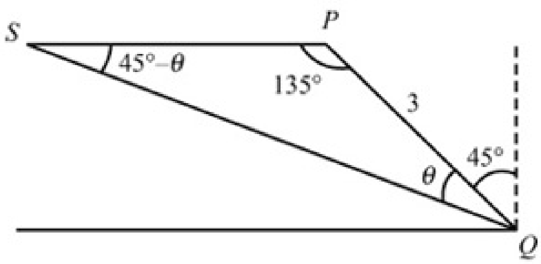
$$RQ^2 = 1^2 + 3^2 - 2\times 1\times 3\cos 45°$$
$$= 10 - 3\sqrt{2}$$
$$RQ = 2.4$$
$$\sin\theta = \frac{\sin 45°}{2.4}$$
$$\theta = 17.14°$$

Velocity Δ



$$\frac{|{}_Q\mathbf{v}_P|}{\sin 45°} = \frac{8}{\sin 27.86°} \Rightarrow |{}_Q\mathbf{v}_P| = 12.1$$

Displacement Δ



$$\frac{QS}{\sin 135°} = \frac{3}{\sin(45° - \theta)}$$
$$\Rightarrow QS = \frac{3\sin 135°}{\sin 27.865°}$$
$$= 4.539$$
$$\therefore \text{Time} = \frac{4.539}{12.1}$$
$$= 0.375 \text{ h}$$
$$\simeq 22.5 \text{ minutes}$$

$$\frac{PS}{\sin\theta} = \frac{4.539}{\sin 135°} \Rightarrow PS = 1.89 \text{ km}$$

a 22.5 minutes

b 1.89 km

c 12.1 km h^{-1} on a bearing 298°

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Edexcel AS and A Level Modular Mathematics

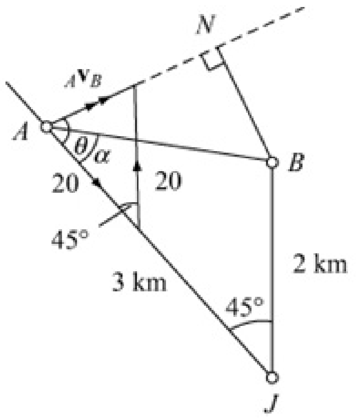
Relative motion
Exercise F, Question 7

Question:

A side road running NW joins a main road which runs due N. Two cars, A and B, each travelling at 20 km h^{-1}, are approaching the junction between the two roads. At a particular instant, A is on the side road at a distance of 3 km from the junction and B is on the main road at a distance of 2 km from the junction. Given that the speeds of the cars remain constant, find

a how close to one another they get,
b the distance of A from the junction when this occurs.

Solution:



Fix B (apply 20 km h^{-1} due N to both)
Since velocity Δ is isosceles,

$$|_A\mathbf{v}_B| = 40 \sin 22.5° = 15.307 \text{ km h}^{-1}$$

$$\theta = \frac{1}{2}(180° - 45°) = 67.5°$$

N is the point of closest approach.

$$AB^2 = 3^2 + 2^2 - 2 \times 3 \times 2 \cos 45° = 13 - 6\sqrt{2}$$

$$AB = 2.124786 \quad \text{Let } J\hat{A}B = \alpha$$

$$\frac{\sin\alpha}{2} = \frac{\sin 45°}{AB}$$

$$\sin\alpha = \frac{\sqrt{2}}{2.124786} \Rightarrow \alpha = 41.72676\ldots$$

$$\text{so, } B\hat{A}N = \theta - \alpha = 25.773°$$

$$\text{so, } BN = AB \sin 25.773° = 0.924 \text{ km}$$

$$\text{Time} = \frac{AN}{|_A\mathbf{v}_B|} = \frac{BN \cos 25.773°}{15.307} \text{ h}$$

$$= 0.05435\ldots$$

$$\therefore \text{Distance of } A \text{ from } J = 3 - (20 \times .05435..)$$

$$= 1.91 \text{ km (3 s.f.)}$$

a 0.924 km
b 1.91 km

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Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise F, Question 8

Question:

A ship is moving due W at 40 km h^{-1} and the wind appears to blow from 67.5° west of south. The ship then steams due S at the same speed and the wind then appears to blow from 22.5° east of south. Find

a the true speed of the wind,
b the true direction of the wind.

Solution:

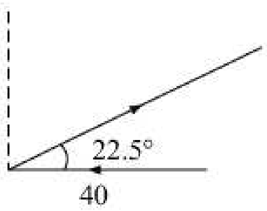
Scenario 1

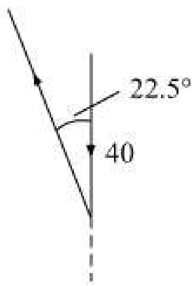
	Mag	Dir
$\mathbf{v}_S$	40	due W
$_W\mathbf{v}_S$	?	From S67.5° W
$\mathbf{v}_W$	?	?

Scenario 2

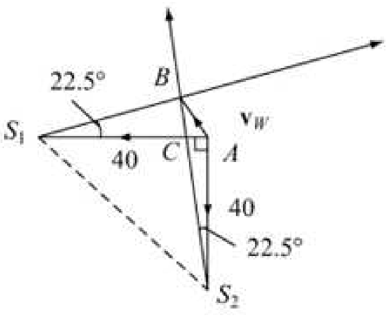
	Mag	Dir
$\mathbf{v}_S$	40	due *S*
$_W\mathbf{v}_S$	?	From S22.5°E
$\mathbf{v}_W$	?	?

$_W\mathbf{v}_S = \mathbf{v}_W - \mathbf{v}_S \Rightarrow \mathbf{v}_W = \mathbf{v}_S + {}_W\mathbf{v}_S$





Putting the two triangles together:



$S_2\hat{S}_1B = 45° + 22.5° = 67.5°$

$S_1\hat{S}_2B = 22.5° \Rightarrow S_1\hat{B}S_2 = 90°$

ΔABC is isosceles and

$A\hat{C}B = \frac{1}{2}(360° - 135°) = 112.5°$

$\therefore C\hat{A}B = \frac{1}{2} \times 67.5° = 33.75°$

$\therefore$ Direction of wind is N56.25° W. (**b**)

$S_1S_2 = 40\sqrt{2} \Rightarrow S_1B = 40\sqrt{2}\cos 67.5° = 21.6478$

$$\frac{|\mathbf{v}_W|}{\sin 22.5°} = \frac{21.6478}{\sin 33.75°} \Rightarrow |\mathbf{v}_W| = 14.9 \text{ km h}^{-1} \qquad \text{(a)}$$

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Edexcel AS and A Level Modular Mathematics

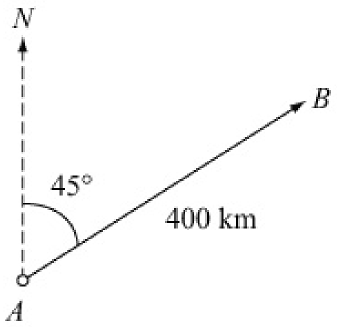
Relative motion
Exercise F, Question 9

Question:

An aeroplane, which can fly at 160 km h^{-1} in still air, starts from the point A to fly to the point B which is 400 km NE of A. If there is a wind of 40 km h^{-1} blowing from the north, find

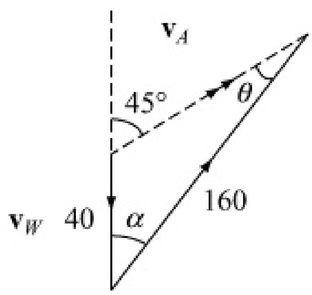
a the direction in which the aeroplane must fly,
b the time taken to reach B.

Solution:



	Mag	Dir
${}_A\mathbf{v}_W$	160	?
$\mathbf{v}_A$	?	NE
$\mathbf{v}_W$	40	From N

$$ {}_A\mathbf{v}_W = \mathbf{v}_A - \mathbf{v}_W $$
$$ \Rightarrow \mathbf{v}_A = \mathbf{v}_W + {}_A\mathbf{v}_W $$



$$ \frac{\sin\theta}{40} = \frac{\sin 135°}{160} $$
$$ \sin\theta = \frac{1}{4}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8} $$
$$ \Rightarrow \theta = 10.182° $$
$$ \alpha = 45° - \theta = 34.818° $$

a Aeroplane must fly N34.8°E

$$ \frac{|\mathbf{v}_A|}{\sin\alpha} = \frac{160}{\sin 135°} \Rightarrow |\mathbf{v}_A| = \frac{160\sin\alpha}{\sin 135°} = 129.2 \text{ km h}^{-1} $$

b $$ \text{Time} = \frac{400}{129.2}\text{ h} = 3.096 \text{ h} $$
$$ = 3 \text{ h } 6 \text{ minutes (nearest minute)} $$

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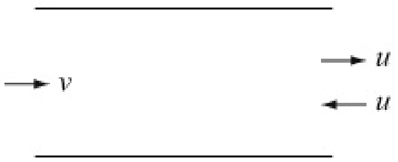
Edexcel AS and A Level Modular Mathematics

Relative motion
Exercise F, Question 10

Question:

A man can swim at a speed u relative to the water in a river which is flowing with speed v. Assuming that $u > v$, prove that it will take him $\frac{u}{\sqrt{u^2 - v^2}}$ times as long to swim a certain distance d upstream and back as it will to swim the same distance d and back in a direction perpendicular to the current, assuming that d is less than the width of the river.

Solution:

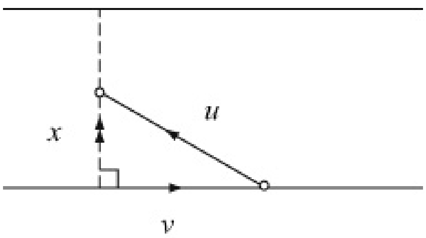


i Downstream : $t_1 = \frac{d}{u+v}$

back : $t_2 = \frac{d}{u-v}$

$$\text{Total time} = \frac{d}{u+v} + \frac{d}{u-v} = d\left(\frac{u-v+u+v}{u^2-v^2}\right)$$
$$= \frac{2du}{u^2-v^2}$$

ii



$$x = \sqrt{u^2 - v^2}$$
$$t = \frac{d}{\sqrt{u^2-v^2}}$$
$$\therefore \text{Total Time} = \frac{2d}{\sqrt{u^2-v^2}}$$

$$\therefore \text{Ratio of times} = \frac{2du}{u^2-v^2} \div \frac{2d}{\sqrt{u^2-v^2}}$$
$$= \frac{u}{u^2-v^2} \times \sqrt{u^2-v^2}$$
$$= \frac{u}{\sqrt{u^2-v^2}} \text{ as required}$$