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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise A, Question 1

Question:

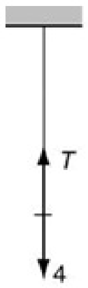
One end of a light elastic string is attached to a fixed point. A force of 4 N is applied to the other end of the string so as to stretch it. The natural length of the string is 3 m and the modulus of elasticity is λ N. Find the total length of the string when

a $\lambda = 30$,

b $\lambda = 12$,

c $\lambda = 16$.

Solution:



$(\uparrow) T = 4$

$T = \frac{\lambda x}{3}$ ← Hooke's Law

So, $\frac{\lambda x}{3} = 4$

$\Rightarrow x = \frac{12}{\lambda}$

$\therefore$ Total length of string, $L = 3 + \frac{12}{\lambda}$

a $\lambda = 30$: $L = 3 + \frac{12}{30}$
$= 3.4$ m

b $\lambda = 12$: $L = 3 + \frac{12}{12}$
$= 4$ m

c $\lambda = 16$: $L = 3 + \frac{12}{16}$
$= 3.75$ m

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Elastic strings and springs
Exercise A, Question 2

Question:

The length of an elastic spring is reduced to 0.8 m when a force of 20 N compresses it. Given that the modulus of elasticity of the spring is 25 N, find its natural length.

Solution:

by Hooke's Law,

$$20 = \frac{25(l - 0.8)}{l}$$

$$4L = 5l - 4$$

$$4 = l$$

Natural length is 4 m.

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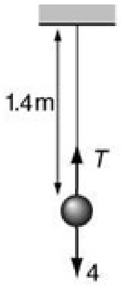
Elastic strings and springs
Exercise A, Question 3

Question:

An elastic spring of modulus of elasticity 20 N has one end fixed. When a particle of mass 1 kg is attached to the other end and hangs at rest, the total length of the spring is 1.4 m. The particle of mass 1 kg is removed and replaced by a particle of mass 0.8 kg. Find the new length of the spring.

Solution:



Let natural length be l

$$(\uparrow) T = g = 9.8$$

$$T = \frac{20(1.4 - l)}{l}$$

$$9.8 = 20\frac{(1.4 - l)}{l}$$

$$9.8l = 28 - 20l$$

$$29.8l = 28 \Rightarrow l = \frac{28}{29.8} = \frac{140}{149}$$

$$0.8g = \frac{20x}{\left(\frac{140}{149}\right)}$$

$$0.8g = \frac{\cancel{20}x \times 149}{\cancel{140}_7}$$

$$\frac{5.6g}{149} = x$$

$$x \approx 0.3683\ldots$$

Total length of string is $0.3683 + \frac{140}{149}$

$= 1.31\text{ m (3 s.f.)}$

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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs

Exercise A, Question 4

Question:

A light elastic spring, of natural length a and modulus of elasticity λ, has one end fixed. A scale pan of mass M is attached to its other end and hangs in equilibrium. A mass m is gently placed in the scale pan. Find the distance of the new equilibrium position below the old one.

Solution:

$$Mg = \frac{\lambda x_1}{a} \Rightarrow x_1 = \frac{Mga}{\lambda}$$

$$(M+m)g = \frac{\lambda x_2}{a} \Rightarrow x_2 = \frac{(M+m)ga}{\lambda}$$

$$\therefore x_2 - x_1 = \frac{ga}{\lambda}(M+m-M) = \frac{mga}{\lambda}$$

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Elastic strings and springs
Exercise A, Question 5

Question:

An elastic string has length a_1 when supporting a mass m_1 and length a_2 when supporting a mass m_2. Find the natural length and modulus of elasticity of the string.

Solution:

$$m_1 g = \frac{\lambda(a_1 - l)}{l} \quad ①$$

$$m_2 g = \frac{\lambda(a_2 - l)}{l} \quad ②$$

Dividing,

$$\frac{m_1}{m_2} = \frac{a_1 - l}{a_2 - l}$$

$$m_1(a_2 - l) = m_2(a_1 - l)$$

$$m_1 a_2 - m_2 a_1 = l(m_1 - m_2)$$

$$l = \frac{m_1 a_2 - m_2 a_1}{m_1 - m_2}$$

$$m_1 g - m_2 g = \frac{\lambda a_1}{l} - \lambda - \left(\frac{\lambda a_2}{l} - \lambda\right)$$

$$lg(m_1 - m_2) = \lambda(a_1 - a_2)$$

$$\lambda = gl\frac{(m_1 - m_2)}{(a_1 - a_2)}$$

$$= g\frac{(m_1 - m_2)}{(a_1 - a_2)}\frac{(m_1 a_2 - m_2 a_1)}{(m_1 - m_2)}$$

$$= g\frac{(m_1 a_2 - m_2 a_1)}{(a_1 - a_2)}$$

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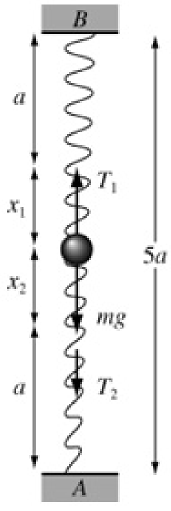
Elastic strings and springs
Exercise A, Question 6

Question:

A light elastic spring has natural length $2a$ and modulus of elasticity $2mg$. A particle of mass m is attached to the mid-point of the spring. One end of the spring, A, is attached to the floor of a room of height $5a$ and the other end is attached to the ceiling of the room at a point B vertically above A. Find the distance of the particle below the ceiling when it is in equilibrium.

Solution:



$$2a + x_1 + x_2 = 5a$$

$$x_1 + x_2 = 3a \quad ①$$

$$(\uparrow) T_1 = mg + T_2$$

$$\frac{2mgx_1}{a} = mg + \frac{2mgx_2}{a}$$

$$2x_1 = a + 2x_2$$

$$x_1 - x_2 = \frac{1}{2}a \quad ②$$

$$① + ②: \quad 2x_1 = \frac{7a}{2} \Rightarrow x_1 = \frac{7a}{4}$$

$\therefore$ Distance below ceiling is $a + \frac{7a}{4} = \frac{11a}{4}$

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Elastic strings and springs
Exercise A, Question 7

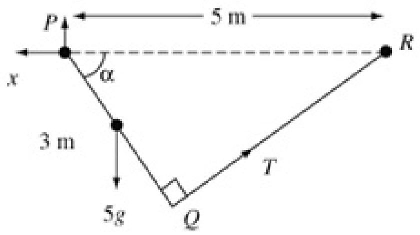
Question:

A uniform rod PQ, of mass 5 kg and length 3 m, has one end, P, smoothly hinged to a fixed point. The other end, Q is attached to one end of a light elastic string of modulus of elasticity 30 N. The other end of the string is attached to a fixed point R which is on the same horizontal level as P with $RP = 5$ m. The system is in equilibrium and $\angle PQR = 90°$. Find

a the tension in the string,

b the natural length of the string.

Solution:



$PQR = 90° \Rightarrow QR = 4$ m

$\cos\alpha = \frac{3}{5}$; $\sin\alpha = \frac{4}{5}$

a $m(P)$, $5g \times \frac{3}{2}\cos\alpha = 3T$

$5g \times \frac{3}{2} \times \frac{3}{5} = 3T$

$T = \frac{3g}{2} = 14.7$ N

Tension is 14.7 N.

b $14.7 = \frac{30(4-l)}{l}$

$14.7l = 120 - 30l$

$44.7l = 120$

$l = 2.68\ldots$

Natural length is 2.7 m (2 s.f.)

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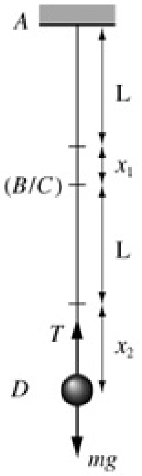
Elastic strings and springs
Exercise A, Question 8

Question:

A light elastic string AB has natural length l and modulus of elasticity $2mg$. Another light elastic string CD has natural length l and modulus of elasticity $4mg$. The strings are joined at their ends B and C and the end A is attached to a fixed point. A particle of mass m is hung from the end D and is at rest in equilibrium. Find the length AD.

Solution:



$(\uparrow) T = mg$ (throughout the length)

so, $mg = \frac{2mgx_1}{l} \Rightarrow x_1 = \frac{1}{2}l$

and $mg = \frac{4mgx_2}{l} \Rightarrow x_2 = \frac{1}{4}l$

$\therefore AD = 2l + x_1 + x_2$

$= \frac{11l}{4}$

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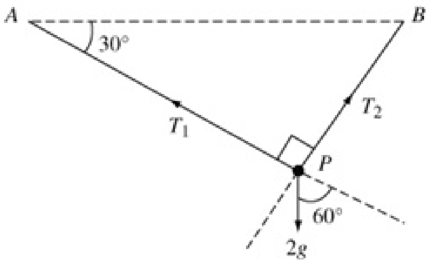
Elastic strings and springs
Exercise A, Question 9

Question:

An elastic string PA has natural length 0.5 m and modulus of elasticity 9.8 N. The string PB is inextensible. The end A of the elastic string and the end B of the inextensible string are attached to two fixed points which are on the same horizontal level. The end P of each string is attached to a 2 kg particle. The particle hangs in equilibrium below AB, with PA making an angle of $30°$ with AB and PA perpendicular to PB. Find

a the length of PA,
b the length of PB,
c the tension of PB.

Solution:



a (↖ along PA),

$T_1 = 2g\cos 60° = g = 9.8$ N

so, $\frac{9.8x_1}{0.5} = 9.8$

$x_1 = 0.5$

$\therefore AP = 0.5 + 0.5$

$= 1$ m

b $\frac{PB}{1} = \tan 30° = \frac{1}{\sqrt{3}}$ m

≈ 0.577 m

$= 0.58$ m (2 s.f.)

c (↗ along PB),

$T_2 = 2g\cos 30°$

$= 2g\frac{\sqrt{3}}{2}$

$= g\sqrt{3}$ N

≈ 17 N(2 s.f.)

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Elastic strings and springs
Exercise A, Question 10

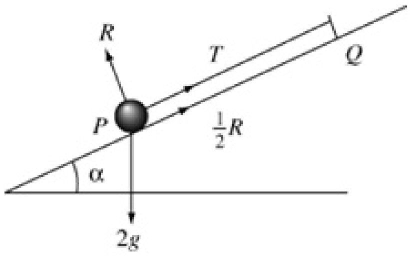
Question:

A particle of mass 2 kg is attached to one end P of a light elastic string PQ of modulus of elasticity 20 N and natural length 0.8 m. The end Q of the string is attached to a point on a rough plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$. The coefficient of friction between the particle and the plane is $\frac{1}{2}$. The particle rests in limiting equilibrium, on the point of sliding down the plane, with PQ along a line of greatest slope. Find

a the tension in the string,
b the length of the string.

Solution:

a



$(\nwarrow) R = 2g\cos\alpha = \frac{8g}{5}$

$\therefore F = \frac{1}{2} \times \frac{8g}{5} = \frac{4g}{5}$

$(\nearrow) T + \frac{4g}{5} = 2g\sin\alpha = \frac{6g}{5}$

$T = \frac{2g}{5}$

$= 3.92\text{ N}$

$= 3.9\text{ N} (2\text{ s.f.})$

b $3.92 = \frac{20x}{0.8}$

$x = 0.1568\text{ m}$

$\therefore$ Length of string is $0.9568 = 0.96\text{ m } (2\text{ s.f.})$

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Elastic strings and springs
Exercise B, Question 1

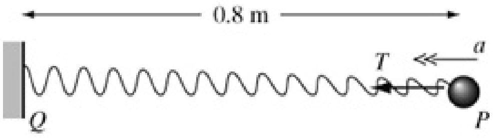
Question:

A particle of mass 4 kg is attached to one end P of a light elastic spring PQ, of natural length 0.5 m and modulus of elasticity 40 N. The spring rests on a smooth horizontal plane with the end O fixed. The particle is held at rest and then released. Find the initial acceleration of the particle

a if $PQ = 0.8\text{ m}$ initially,

b if $PQ = 0.4\text{ m}$ initially.

Solution:



a $(\leftarrow) T = 4a$

$$T = \frac{40 \times 0.3}{0.5}$$
$$= 24\text{ N}$$
$$\therefore 24 = 4a$$
$$6 = a$$

initial acceleration is 6 m s^{-2}

b

$(\rightarrow) S = 4a$

$$S = \frac{40 \times 0.1}{0.5}$$
$$= 8\text{ N}$$
$$\therefore 8 = 4a$$
$$2 = a$$

initial acceleration is 2 m s^{-2}

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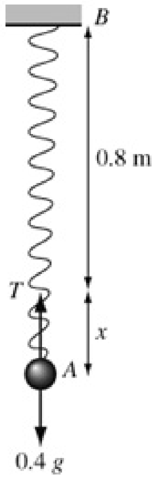
Elastic strings and springs
Exercise B, Question 2

Question:

A particle of mass 0.4 kg is fixed to one end A of a light elastic spring AB, of natural length 0.8 m and modulus of elasticity 20 N. The other end B of the spring is attached to a fixed point. The particle hangs in equilibrium. It is then pulled vertically downwards through a distance 0.2 m and released from rest. Find the initial acceleration of the particle.

Solution:



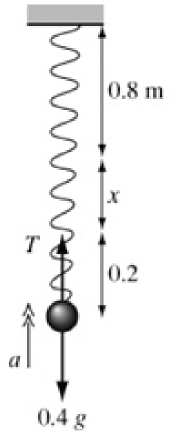
In equilibrium position

$$(\uparrow) T = 0.4g$$

$$T = \frac{20x}{0.8} = 25x$$

$$25x = 0.4g$$

$$x = \frac{2g}{125}$$



After further extension,

$$(\uparrow) T - 0.4g = 0.4a$$

$$T = \frac{20(x+0.2)}{0.8} = 25x + 5$$

So, $25x + 5 - 0.4g = 0.4a$

$$a = \frac{5}{0.4} = 12.5$$

initial acceleration is 12.5 m s^{-2}

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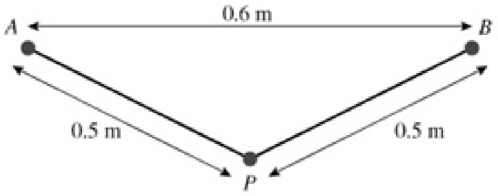
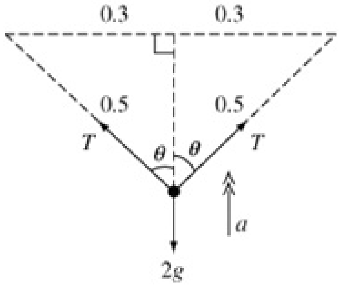
Elastic strings and springs
Exercise B, Question 3

Question:



A particle P of mass 2 kg is attached to the mid-point of a light elastic string, of natural length 0.4 m and modulus of elasticity 20 N. The ends of the elastic string are attached to two fixed points A and B which are on the same horizontal level, with $AB = 0.6$ m. The particle is held in the position shown, with $AP = BP = 0.5$ m, and released from rest. Find the initial acceleration of the particle and state its direction.

Solution:



$$(\uparrow) 2T\cos\theta - 2g = 2a$$

$$\frac{4T}{5} - g = a$$

$$T = \frac{20 \times 0.6}{0.4} = 30$$

$$\frac{4}{5} \times 30 - 9.8 = a$$

$$14.2 = a$$

initial acceleration is 14.2 m s^{-2} upwards

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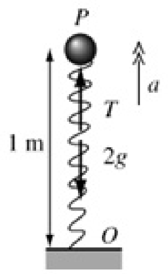
Elastic strings and springs
Exercise B, Question 4

Question:

A particle of mass 2 kg is attached to one end P of a light elastic spring. The other end Q of the spring is attached to a fixed point O. The spring has natural length 1.5 m and modulus of elasticity 40 N. The particle is held at a point which is 1 m vertically above O and released from rest. Find the initial acceleration of the particle, stating its magnitude and direction.

Solution:



$(\uparrow) T - 2g = 2a$

$$T = \frac{40 \times 0.5}{1.5}$$

$$= \frac{40}{3}$$

So, $\frac{40}{3} - 19.6 = 2a$

$$a = -3.13$$

magnitude of initial acceleration is 3.13 m s^{-2} and direction is downwards

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Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise C, Question 1

Question:

An elastic spring has natural length 0.6 m and modulus of elasticity 8 N. Find the work done when the spring is stretched from its natural length to a length of 1 m.

Solution:

$$\text{work done} = \frac{\lambda x^2}{2l} = \frac{8 \times 0.4^2}{2 \times 0.6}$$
$$= 1.06\text{ J}$$

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Elastic strings and springs
Exercise C, Question 2

Question:

An elastic spring, of natural length 0.8 m and modulus of elasticity of 4 N, is compressed to a length of 0.6 m. Find the elastic potential energy stored in the spring.

Solution:

$$\text{work done} = \frac{\lambda x^2}{2l} = \frac{4 \times 0.2^2}{2 \times 0.8}$$
$$= 0.1\text{J}$$

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Elastic strings and springs
Exercise C, Question 3

Question:

An elastic string has natural length 1.2 m and modulus of elasticity 10 N. Find the work done when the string is stretched from a length 1.5 m to a length 1.8 m.

Solution:

$$\begin{aligned}\text{work done } &= \frac{10\times 0.6^2}{2\times 1.2} - \frac{10\times 0.3^2}{2\times 1.2}\\ &= \frac{10}{2.4}(0.6^2 - 0.3^2)\\ &= \frac{10}{2.4}\times 0.9\times 0.3\\ &= 1.125\text{ J}\end{aligned}$$

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Elastic strings and springs

Exercise C, Question 4

Question:

An elastic spring has natural length 0.7 m and modulus of elasticity 20 N. Find the work done when the spring is stretched from a length

a 0.7 m to 0.9 m

b 0.8 m to 1.0 m

c 1.2 m to 1.4 m.

Note that your answer to **a**, **b** and **c** are all different.

Solution:

a $\frac{20}{2\times 0.7}(0.2^2 - 0^2) = 0.571 \text{ J (3 s.f.)}$

b $\frac{20}{2\times 0.7}(0.3^2 - 0.1^2)$

$= \frac{20}{1.4}\times 0.4\times 0.2 = 1.14 \text{ J (3 s.f.)}$

c $\frac{20}{2\times 0.7}(0.7^2 - 0.5^2)$

$= \frac{20}{1.4}\times 1.2\times 0.2 = 3.43 \text{ J (3 s.f.)}$

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Elastic strings and springs
Exercise C, Question 5

Question:

A light elastic spring has natural length 1.2 m and modulus of elasticity 10 N. One end of the spring is attached to a fixed point. A particle of mass 2 kg is attached to the other end and hangs in equilibrium. Find the energy stored in the spring.

Solution:

$$(\uparrow) T = 2g$$

$$\frac{10e}{1.2} = 2g$$

$$e = \frac{2.4g}{10} = 0.24g$$

$$\text{energy stored} = \frac{10 \times (0.24g)^2}{2 \times 1.2}$$

$$= 23.0\text{ J} = 23\text{ J (2 s.f.)}$$

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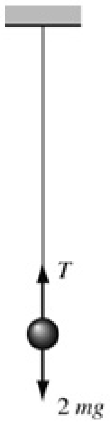
Elastic strings and springs
Exercise C, Question 6

Question:

An elastic string has natural length a. One end is fixed. A particle of mass $2m$ is attached to the free end and hangs in equilibrium, with the length of the string $3a$. Find the elastic potential energy stored in the string.

Solution:



$$(\uparrow)\ T = 2mg$$

$$\frac{\lambda \times 2a}{a} = 2mg$$

$$\Rightarrow \lambda = mg$$

$$\text{E.E.} = \frac{\lambda x^2}{2l} = \frac{mg(2a)^2}{2a} = 2mga$$

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Edexcel AS and A Level Modular Mathematics

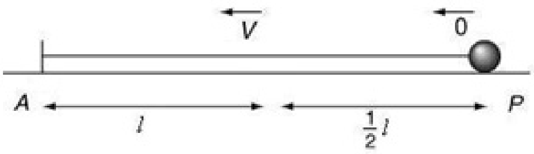
Elastic strings and springs

Exercise D, Question 1

Question:

An elastic string, of natural length l and modulus of elasticity mg, has one end fixed to a point A on a smooth horizontal table. The other end is attached to a particle P of mass m. The particle is held at a point on the table with $AP = \frac{3l}{2}$ and is released. Find the speed of the particle when the string reaches its natural length.

Solution:



Conservation of energy

K.E. gain = E.E. loss

$$\frac{1}{2}mV^2 = \frac{mg\left(\frac{1}{2}l\right)^2}{2l}$$

$$V^2 = \frac{1}{4}gl$$

$$V = \frac{1}{2}\sqrt{gl}$$

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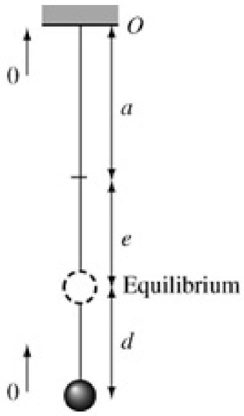
Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise D, Question 2

Question:

A particle of mass m is suspended from a fixed point O by a light elastic string, of natural length a and modulus of elasticity $4mg$. The particle is pulled vertically downwards a distance d from its equilibrium position and released from rest. If the particle just reaches O, find d.

Solution:



At equilibrium, $T = mg$

$$\frac{4mge}{a} = mg \Rightarrow e = \frac{1}{4}a$$

Conservation of energy

$$\text{P.E. gain} = \text{E.E. loss}$$

$$mg\left(a + \frac{1}{4}a + d\right) = \frac{4mg\left(\frac{1}{4}a + d\right)^2}{2a}$$

$$\frac{5a^2}{4} + ad = 2\left(\frac{a^2}{16} + \frac{ad}{2} + d^2\right)$$

$$\frac{5a^2}{4} = \frac{a^2}{8} + 2d^2$$

$$\frac{9a^2}{16} = d^2$$

$$\left(\text{ignore solution} - \frac{3a}{4}\right) \quad \frac{3a}{4} = d$$

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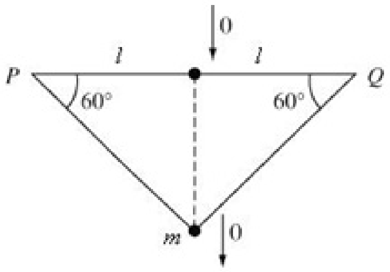
Edexcel AS and A Level Modular Mathematics

Elastic strings and springs
Exercise D, Question 3

Question:

A light elastic spring of natural length $2l$ has its ends attached to two points P and Q which are at the same horizontal level. The length PQ is $2l$. A particle of mass m is fastened to the midpoint of the spring and is held at the mid-point of PQ. The particle is released from rest and first comes to instantaneous rest when both parts of the string make an angle of $60°$ with the line PQ. Find the modulus of elasticity of the spring.

Solution:



Conservation of energy
P.E. loss = E.E. gain

$$mgl\tan 60° = \frac{2\times\lambda\left(\frac{l}{\cos 60°}-l\right)^2}{2l}$$

$$mgl\sqrt{3} = \lambda l$$

modulus is $mg\sqrt{3}$

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Elastic strings and springs
Exercise D, Question 4

Question:

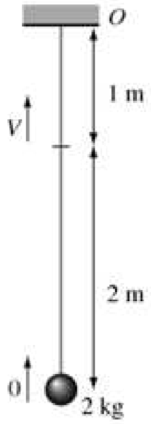
A light elastic string, of natural length 1 m and modulus of elasticity 21.6 N has one end attached to a fixed point O. A particle of mass 2 kg is attached to the other end. The particle is held at a point which is 3 m vertically below O and released from rest.

Find

a the speed of the particle when the string first becomes slack,

b the distance from O when the particle first comes to rest.

Solution:



a Conservation of energy

K.E. gain + P.E. gain = E.E. loss

$$\frac{1}{2}\times 2\times V^2 + 2g\times 2 = \frac{21.6\times 2^2}{2\times 1}$$

$$V^2 = 43.2 - 39.2$$

$$= 4$$

$$V = 2 \text{ m s}^{-1}$$

b

Conservation of energy

K.E. loss = P.E. gain

$$\frac{1}{2}mV^2 = mgd$$

$$2 = gd$$

$$0.20 \text{ m (2 s.f.)} = \frac{2}{g} = d$$

distance from O is 0.80 m (2 s.f.)

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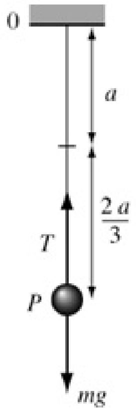
Elastic strings and springs
Exercise D, Question 5

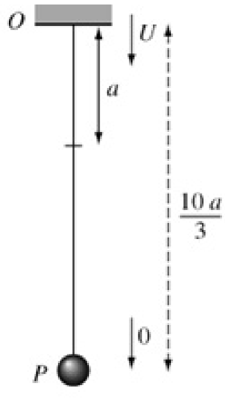
Question:

A particle P is attached to one end of a light elastic string of natural length a. The other end of the string is attached to a fixed point O. When P hangs at rest in equilibrium, the distance OP is $\frac{5a}{3}$. The particle is now projected vertically downwards from O with speed U and first comes to instantaneous rest at a distance $\frac{10a}{3}$ below O. Find U in terms of a and g.

Solution:



$$(\uparrow)T = mg$$
$$\frac{\lambda}{a}\times\frac{2a}{3} = mg$$
$$\lambda = \frac{3mg}{2}$$



K.E. loss + P.E. loss = E.E. gain

$$\frac{1}{2}mU^2 + mg\frac{10a}{3} = \frac{3mg}{2}\frac{\left(\frac{7a}{3}\right)^2}{2a}$$
$$\frac{U^2}{2} + \frac{10ag}{3} = \frac{3g}{4a}\frac{49a^2}{9}$$
$$\frac{U^2}{2} = \frac{49ag}{12} - \frac{10ag}{3}$$
$$U^2 = \frac{9ag\times 2}{12}$$
$$U = \sqrt{\frac{3ag}{2}}$$

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Elastic strings and springs
Exercise D, Question 6

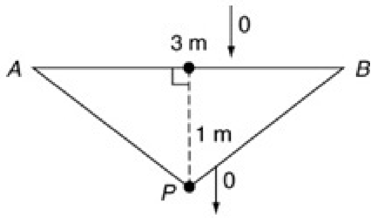
Question:

A particle P of mass 1 kg is attached to the mid-point of a light elastic string, of natural length 3 m and modulus λ N. The ends of the string are attached to two points A and B on the same horizontal level with $AB = 3\text{m}$. The particle is held at the mid-point of AB and released from rest. The particle falls vertically and comes to instantaneous rest at a point which is 1 m below the mid-point of AB.

Find **a** the value of λ,

b the speed of P when it is 0.5 m below the initial position.

Solution:



$$AP = \sqrt{1.5^2 + 1^2} = \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2}$$

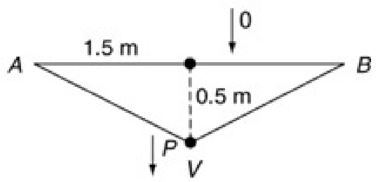
a P.E. loss = E.E. gain

$$g \times 1 = \frac{2\lambda\left(\frac{\sqrt{13}}{2} - \frac{3}{2}\right)^2}{2 \times 1.5}$$

$$\lambda = \frac{2 \times 3g}{(\sqrt{13} - 3)^2} = 80.176 \times 2$$

$$= 160 \text{ N (2 s.f.)}$$

b



$$AP = \sqrt{1.5^2 + 0.5^2} = \frac{\sqrt{10}}{2}$$

K.E. gain + E.E. gain = P.E. loss

$$\frac{1}{2}V^2 + \frac{2\lambda\left(\frac{\sqrt{10}}{2} - \frac{3}{2}\right)^2}{2 \times 1.5} = 0.5g$$

$$V^2 = g - \frac{(\sqrt{10} - 3)^2}{3} \times \lambda$$

$$V = 2.896 = 2.9 \text{ m s}^{-1} \text{ (2 s.f.)}$$

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Elastic strings and springs
Exercise D, Question 7

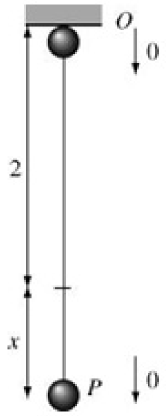
Question:

A light elastic string of natural length 2 m and modulus of elasticity 117.6 N has one end attached to a fixed point O. A particle P of mass 3 kg is attached to the other end. The particle is held at O and released from rest.

a Find the distance fallen by P before it first comes to rest.

b Find the greatest speed of P during the fall.

Solution:



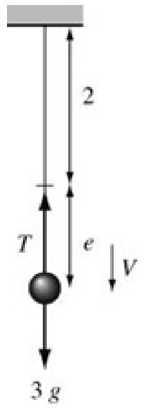
a

$$\text{P.E. loss} = \text{E.E. gain}$$
$$3g(2+x) = \frac{117.6}{4}x^2$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$x = 2(x-1)$$

Ignore negative root.

falls 4 m

b Greatest speed at equilibrium position



$$(\uparrow) T = 3g$$
$$\frac{117.6 \times e}{2} = 3g$$
$$e = 0.5\text{ m}$$
$$\text{E.E. gain} + \text{K.E. gain} = \text{P.E. loss}$$
$$\frac{117.6(0.5)^2}{2\times 2} + \frac{1}{2}\times 3V^2 = 3g \times 2.5$$
$$7.35 + 1.5V^2 = 73.5$$
$$V = 6.64$$
$$= 6.6\text{ m s}^{-1}\text{(2 s.f.)}$$

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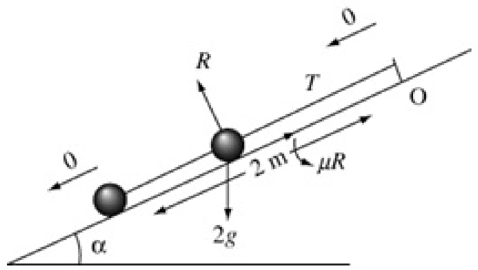
Elastic strings and springs
Exercise D, Question 8

Question:

A particle P of mass 2 kg is attached to one end of a light elastic string of natural length 1 m and modulus of elasticity 40 N. The other end of the string is fixed to a point O on a rough plane which is inclined at an angle α, where $\tan\alpha = \frac{3}{4}$. The particle is held at O and released from rest. Given that P comes to rest after moving 2 m down the plane, find the coefficient of friction between the particle and the plane.

Solution:



$$(\nwarrow) R = 2g\cos\alpha = \frac{8g}{5}$$

$$\text{Work done against friction} = \text{P.E. loss} - \text{E.E. gain}$$

$$\mu\frac{8g}{5}\times 2 = 2g\times 2\sin\alpha - \frac{40\times 1^2}{2\times 1}$$

$$\mu\frac{16g}{5} = \frac{12g}{5} - 20$$

$$\mu = \frac{12g - 100}{16g}$$

$$= 0.11\,(2\text{ s.f.})$$

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Elastic strings and springs
Exercise E, Question 1

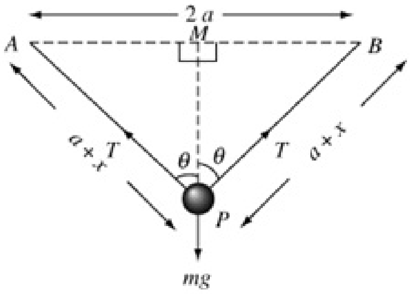
Question:

A particle of mass m is supported by two light elastic strings, each of natural length a and modulus of elasticity $\frac{15mg}{16}$. The other ends of the strings are attached to two fixed points A and B where A and B are in the same horizontal line with $AB = 2a$. When the particle hangs at rest in equilibrium below AB, each string makes an angle θ with the vertical.

a Verify that $\cos\theta = \frac{4}{5}$.

b How much work must be down to raise the particle to the mid-point of AB?

Solution:



$(\uparrow) 2T\cos\theta = mg$ ①

by Hooke's Law

$T = \dfrac{15mgx}{16a}$ ②

$\sin\theta = \dfrac{a}{a+x}$ ③

a If $\cos\theta = \dfrac{4}{5}, T = \dfrac{5mg}{8}$ from ①

$$\text{so, } \frac{5mg}{8} = \frac{15mgx}{16a} \quad \text{from ②}$$

$$\frac{2a}{3} = x$$

$$\text{than } \frac{3}{5} = \frac{a}{a+\frac{2a}{3}} \quad \text{from ③}$$

which is true.

b work done on particle = overall gain in energy

= P.E. gain − E.E. loss

$$PM = (a+x)\cos\theta$$

$$= \left(a + \frac{2a}{3}\right)\frac{4}{5}$$

$$= \frac{4a}{3}$$

$$\therefore \text{P.E. gain } = mg\frac{4a}{3}$$

E.E. loss = initial E.E. − final E.E.

$$= \frac{15mg}{16\times 2a}\left(2x\left(\frac{2a}{3}\right)^2 - 0^2\right)$$

$$= \frac{15mg4a^2\times 2}{16\times 2a\times 9}$$

$$= \frac{5mga}{12}$$

$$\text{So, work done} = \frac{4mga}{3} - \frac{5mga}{12}$$

$$= \frac{mga}{12}(16-5)$$

$$= \frac{11mga}{12}$$

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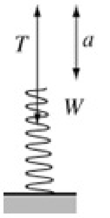
Elastic strings and springs
Exercise E, Question 2

Question:

A light elastic spring is such that a weight of magnitude W resting on the spring produces a compression a. The weight W is allowed to fall onto the spring from a height of $\frac{3a}{2}$ above it. Find the maximum compression of the spring in the subsequent motion.

Solution:

Let l be the natural length of the spring.
Let λ be the modulus of the spring.

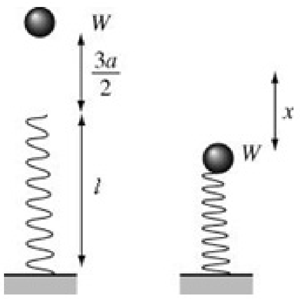


$(\uparrow) T = W$

by Hooke's Law,

$$T = \frac{\lambda a}{l}$$

$$\therefore W = \frac{\lambda a}{l} \text{ i.e. } \frac{W}{a} = \frac{\lambda}{l}$$



Using conservation of energy,
P.E. loss of W = E.E. gain of spring

$$W\left(\frac{3a}{2} + x\right) = \frac{\lambda x^2}{2l}$$

$$\text{so, } W\left(\frac{3a}{2} + x\right) = \frac{Wx^2}{2a}$$

Substitute for $\frac{\lambda}{l}$ from above.

$$3a^2 + 2ax = x^2$$

$$0 = x^2 - 2ax - 3a^2$$

$$0 = (x - 3a)(x + a)$$

$$\therefore x = 3a \text{ or } -a$$

$\therefore$ maximum compression is $3a$

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Elastic strings and springs
Exercise E, Question 3

Question:

A light elastic string of natural length 0.5 m is stretched between two points P and Q on a smooth horizontal table. The distance PQ is 0.75 m and the tension in the string is 15 N.

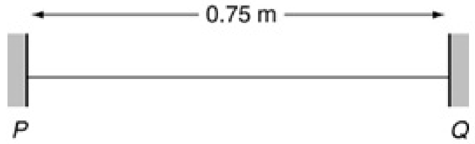
a Find the modulus of elasticity of the string.

A particle of mass 0.5 kg is attached to the mid-point of the string. The particle is pulled 0.1 m towards Q and released from rest.

b Find the speed of the particle as it passes through the mid-point of PQ.

Solution:

a

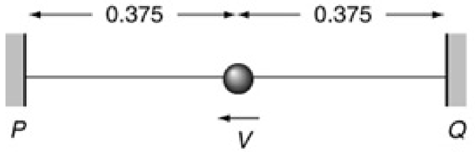


$x = 0.75 - 0.5 = 0.25$

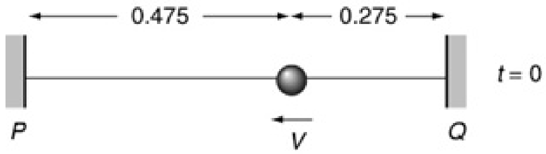
by Hooke's Law, $15 = \dfrac{\lambda \times 0.25}{0.5}$

$\Rightarrow \lambda = 30\text{ N}$

b



(final)



(initial)

Using conservation of energy

K.E. gain = E.E. loss

$$\begin{aligned}
\text{E.E. loss} &= \text{initial E.E.} - \text{final E.E.} \\
&= \frac{30}{2\times 0.25}(0.225^2 + 0.025^2 - 2\times 0.125^2) \\
&= 60(0.05125 - 0.03125) \\
&= 1.2\text{ J}
\end{aligned}$$

$$\frac{1}{2}\times\frac{1}{2}\times v^2 = 1.2$$

$$\text{So, } v^2 = 4.8$$

$$v = 2.19\text{ m s}^{-1}\ (3\text{ s.f.})$$

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Elastic strings and springs
Exercise E, Question 4

Question:

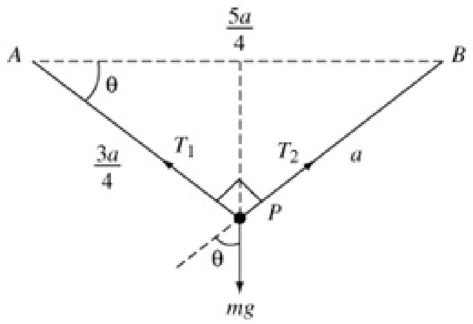
A particle P of mass m is attached to two strings AP and BP. The points A and B are on the same horizontal level and $AB = \frac{5a}{4}$.

The string AP is inextensible and $AP = \frac{3a}{4}$.

The string BP is elastic and $BP = a$.

The modulus of elasticity of BP is λ. Show that the natural length of BP is $\frac{5\lambda a}{3mg + 5\lambda}$.

Solution:



ΔABP is 3, 4, 5 so $A\hat{P}B = 90°$.

(↗, along PB) $T_2 = mg\cos\theta = \frac{3mg}{5}$

by Hooke's Law, $T_2 = \frac{\lambda(-l + a)}{l}$

$$\text{So,} \quad \lambda\frac{(-l + a)}{l} = \frac{3mg}{5}$$
$$5\lambda(-l + a) = 3mgl$$
$$5\lambda l + 3mgl = 5\lambda a$$
$$l(5\lambda + 3mg) = 5\lambda a$$
$$l = \frac{5\lambda a}{(5\lambda + 3mg)}$$

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Elastic strings and springs
Exercise E, Question 5

Question:

A light elastic string, of natural length a and modulus of elasticity $5mg$, has one end attached to the base of a vertical wall. The other end of the string is attached to a small ball. The ball is held at a distance $\frac{3a}{2}$ from the wall, on a rough horizontal plane, and released from rest. The coefficient of friction between the ball and the plane is $\frac{1}{5}$.

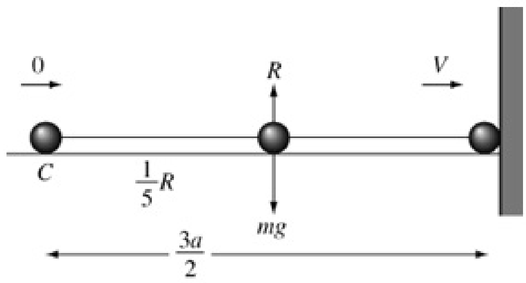
a Find, in terms of a and g, the speed V of the ball as it hits the wall.

The ball rebounds from the wall with speed $\frac{2V}{5}$.

b Find the distance from the wall at which the ball comes to rest.

Solution:

a



$(\uparrow) R = mg$

$\therefore \text{Friction} = \frac{1}{5}mg$

work done against friction = overall loss in energy
= E.E. loss − K.E. gain

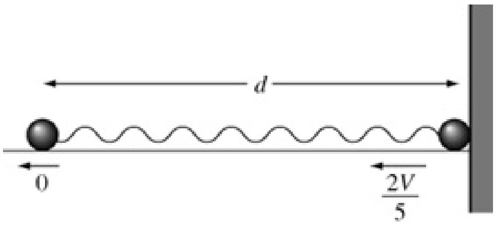
$$\frac{1}{5}mg\frac{3a}{2} = \frac{5mg\left(\frac{a}{2}\right)^2}{2a} - \frac{1}{2}mV^2$$

$$\frac{3ag}{5} = \frac{5ag}{4} - V^2$$

$$V^2 = \frac{5ag}{4} - \frac{3ag}{5} = \frac{ag(25-12)}{20}$$

$$V = \sqrt{\frac{13ag}{20}}$$

b



Friction will be same.
Assume string is still slack when ball comes to rest.

Work done against friction = K.E. loss

$$\frac{1}{5}mg\,d = \frac{1}{2}m\left(\frac{2V}{5}\right)^2 = \frac{1}{2}m\frac{4V^2}{25}$$

$$\frac{1}{5}gd = \frac{1}{2}\times\frac{4}{25}\times\frac{13ag}{20}$$

$$d = \frac{13a}{50}$$

As d is less than a, the assumption that the string is still slack is valid.

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Elastic strings and springs
Exercise E, Question 6

Question:

a Using integration, show that the work done in stretching a light elastic string of natural length l and modulus of elasticity λ, from length l to length $(l+x)$ is $\frac{\lambda x^2}{2l}$.

b The same string is stretched from a length $(l+a)$ to a length $(l+b)$ where $b > a$. Show that the work done is the product of the mean tension and the distance moved.

Solution:

a work done $= \int_0^x T \, ds = \int_0^x \frac{\lambda s}{l} \, ds$

$$= \frac{\lambda}{2l}\left[s^2\right]_0^x$$

$$= \frac{\lambda x^2}{2l}$$

b work done = E.E. gain of string

$$= \frac{\lambda}{2l}(b^2 - a^2)$$

$$= \frac{\lambda}{2l}(b+a)(b-a)$$

$$= \frac{1}{2}\left(\frac{\lambda b}{l} + \frac{\lambda a}{l}\right)(b-a)$$

$$= \frac{1}{2}(T_b + T_a)(b-a)$$

= mean of tensions × distance moved

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Elastic strings and springs
Exercise E, Question 7

Question:

A light elastic string has natural length l and modulus $2mg$. One end of the string is attached to a particle P of mass m. The other end is attached to a fixed point C on a rough horizontal plane. Initially P is at rest at a point D on the plane where $CD = \frac{4l}{3}$.

a Given that P is in limiting equilibrium, find the coefficient of friction between P and the plane.

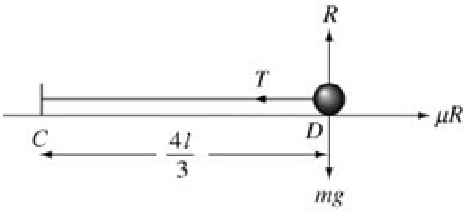
The particle P is now moved away from C to a point E on the plane where $CE = 2l$.

b Find the speed of P when the string returns to its natural length.

c Find the total distance moved by P before it comes to rest.

Solution:

a



$(\uparrow) R = mg \quad (\rightarrow) \mu R = T$

$\mu mg = T$

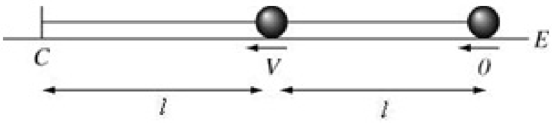
by Hooke's Law,

$$T = \frac{2mg}{l}\frac{l}{3} = \frac{2mg}{3}$$

$$\therefore \mu mg = \frac{2mg}{3}$$

$$\mu = \frac{2}{3}$$

b



$$\text{work done against friction} = \text{overall loss in energy}$$

$$= \text{E.E. loss} - \text{K.E. gain}$$

$$\frac{2}{3}mg\,l = \frac{2mgl^2}{2l} - \frac{1}{2}mV^2$$

$$\frac{1}{2}V^2 = gl - \frac{2}{3}gl - \frac{1}{3}gl$$

$$V^2 = \frac{2}{3}gl$$

$$V = \sqrt{\frac{2gl}{3}}$$

c String is now slack

$$\text{work done against friction} = \text{K.E. loss}$$

$$\frac{2}{3}mg\,d = \frac{1}{2}m \times \frac{2}{3}gl$$

$$d = \frac{1}{2}l$$

$\therefore$ total distance travelled is $\frac{3l}{2}$

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Elastic strings and springs
Exercise E, Question 8

Question:

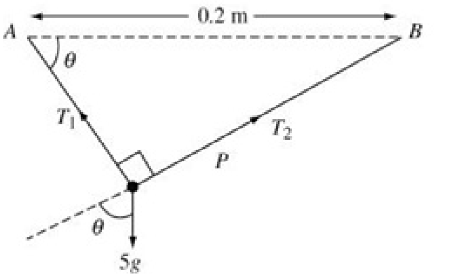
A light elastic string of natural length 0.2 m has its ends attached to two fixed points A and B which are on the same horizontal level with $AB = 0.2$ m. A particle of mass 5 kg is attached to the string at the point P where $AP = 0.15$ m. The system is released and P hangs in equilibrium below AB with $A\hat{P}B = 90°$.

a If $B\hat{A}P = \theta$, show that the ratio of the extension of AP and BP is $\frac{4\cos\theta - 3}{4\sin\theta - 1}$.

b Hence show that $\cos\theta(4\cos\theta - 3) = 3\sin\theta(4\sin\theta - 1)$.

Solution:

a



extension of $AP = 0.2\cos\theta - 0.15$

extension of $BP = 0.2\sin\theta - 0.05$

$$\therefore \text{ ratio is } \frac{0.2\cos\theta - 0.15}{0.2\sin\theta - 0.05} \times \frac{20}{20}$$

$$= \frac{4\cos\theta - 3}{4\sin\theta - 1}$$

b ($\nearrow$) along PB: $T_2 = 5g\cos\theta$

($\swarrow$) along PA: $T_1 = 5g\sin\theta$

$$\text{so, } \frac{T_2}{T_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{\lambda x_2}{0.05} \times \frac{0.15}{\lambda x_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{3x_2}{x_1} = \frac{\cos\theta}{\sin\theta}$$

$$\text{i.e. } \frac{x_1}{x_2} = \frac{3\sin\theta}{\cos\theta}$$

$$\text{i.e. } \frac{4\cos\theta - 3}{4\sin\theta - 1} = \frac{3\sin\theta}{\cos\theta}$$

$$3\sin\theta(4\sin\theta - 1) = \cos\theta(4\cos\theta - 3)$$

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Elastic strings and springs
Exercise E, Question 9

Question:

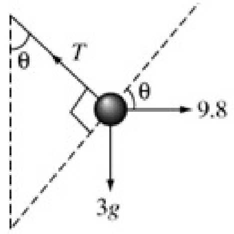
A particle of mass 3 kg is attached to one end of a light elastic string, of natural length 1 m and modulus of elasticity 14.7 N. The other end of the string is attached to a fixed point. The particle is held in equilibrium by a horizontal force of magnitude 9.8 N with the string inclined to the vertical at an angle θ.

a Find the value of θ.

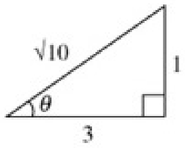
b Find the extension of the string.

c If the horizontal force is removed, find the magnitude of the least force that will keep the string inclined at the same angle.

Solution:

a 

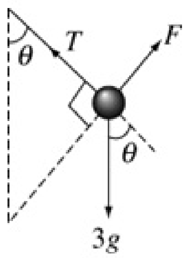
(↗ perpendicular to string)



$$9.8\cos\theta = 3g\sin\theta$$
$$\frac{1}{3} = \tan\theta$$
$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4°$$

b $(\rightarrow)\ T\sin\theta = 9.8$

$$T = 9.8\sqrt{10}$$
$$\frac{14.7 \times x}{1} = 9.8\sqrt{10}$$
$$x = \frac{2\sqrt{10}}{3}\text{ m} \approx 2.1\text{ m (2 s.f.)}$$

c 

least force will be perpendicular to string

$(\nearrow) F = 3g\sin\theta$

$$= \frac{3g}{\sqrt{10}}\text{ N}$$
$$= \frac{3g\sqrt{10}}{10}\text{ N}$$
$$= 9.3\text{ N (2 s.f.)}$$

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Elastic strings and springs
Exercise E, Question 10

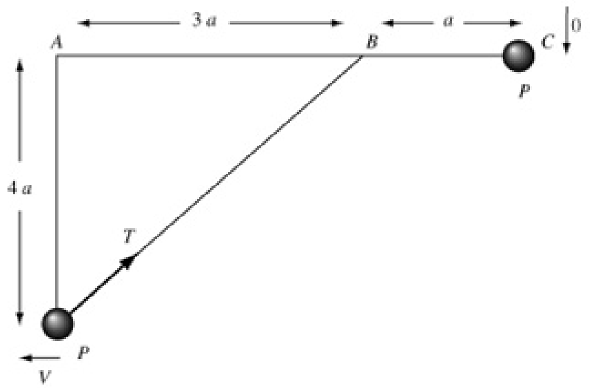
Question:

Two points A and B are on the same horizontal level with $AB = 3a$. A particle P of mass m is joined to A by a light inextensible string of length $4a$ and is joined to B by a light elastic string, of natural length a and modulus of elasticity $\frac{mg}{4}$. The particle P is held at the point C, on AB produced, such that $BC = a$ and both strings are taut. The particle P is released from rest.

a Show that when AP is vertical the speed of P is $2\sqrt{ga}$.

b Find the tension in the elastic string in this position.

Solution:



a by conservation of energy,

K.E. gain + E.E. gain = P.E. loss

$$\frac{1}{2}mv^2 + \frac{mg}{4}\frac{x^2}{2a} = mg4a$$

$BP = 5a$ (3, 4, 5 Δ)

So, $x = 4a$

$$\therefore \frac{1}{2}mv^2 + \frac{mg}{4}\cdot\frac{16a^2}{2a} = mg\,4a$$

$$v^2 + 4ga = 8ga$$

$$v^2 = 4ga$$

$$v = 2\sqrt{ga}$$

b $x = 4a : T = \frac{mg}{4} \times \frac{4a}{a}$

$= mg$