Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 1

Question:

Differentiate with respect to x.
$\sinh 2x$

Solution:

$$\frac{d}{dx}(\sinh 2x) = 2\cosh 2x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 2

Question:

Differentiate with respect to x.
$\cosh 5x$

Solution:

$$\frac{d}{dx}(\cosh 5x) = \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x$$
$$= -2\frac{\sinh 2x}{\cos 2x} \times \frac{1}{\cos 2x}$$
$$= -2\tan 2x \ \text{sech} 2x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 3

Question:

Differentiate with respect to x.
$\tanh 2x$

Solution:

$$\frac{d}{dx}(\tanh 2x) = 2\text{sech}^2 2x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 4

Question:

Differentiate with respect to x.
$\sinh 3x$

Solution:

$$\frac{d}{dx}(\sinh 3x) = 3\cosh 3x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 5

Question:

Differentiate with respect to x.
$\coth 4x$

Solution:

$$\frac{d}{dx}\left(\coth 4x\right) = -4\operatorname{cosech}^2 4x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 6

Question:

Differentiate with respect to x.

$\text{sech}\, 2x$

Hint: $\text{sech}\, 2x = \dfrac{1}{\cosh 2x}$

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\text{sech}\, 2x) &= \frac{-1}{(\cosh 2x)^2} \times 2\sinh 2x \\ &= -2\frac{\sinh 2x}{\cosh 2x} \times \frac{1}{\cosh 2x} \\ &= -2\tanh 2x\, \text{sech}\, 2x\end{aligned}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 7

Question:

Differentiate with respect to x.

$e^{-x}\sinh x$

Solution:

$$\frac{d}{dx}(e^{-x}\sinh x) = -e^{-x}\sinh x + e^{-x}\cosh x$$
$$= e^{-x}(\cosh x - \sinh x)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 8

Question:

Differentiate with respect to x.
$x \cosh 3x$

Solution:

$$\frac{d}{dx}(x \cosh 3x) = \cosh 3x + 3x \sinh 3x$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise A, Question 9

Question:

Differentiate with respect to x.

$$\frac{\sinh x}{3x}$$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{3x}\right) = \frac{\cosh x}{3x} - \frac{\sinh x}{3x^2}$$

$$= \frac{x\cosh x - \sinh x}{3x^2}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 10

Question:

Differentiate with respect to x.
$x^2 \cosh 3x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \cosh 3x\right) = 2x\cosh 3x + x^2 \times 3\sinh 3x$$
$$= x\left(2\cosh 3x + 3x\sinh 3x\right)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 11

Question:

Differentiate with respect to x.
$\sinh 2x \cosh 3x$

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sinh 2x \cosh 3x) &= 2\cosh 2x \cosh 3x + \sinh 2x \times 3\sinh 3x \\ &= 2\cosh 2x \cosh 3x + 3\sinh 2x \sinh 3x\end{aligned}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 12

Question:

Differentiate with respect to x.
$\ln(\cosh x)$

Solution:

$$\frac{d}{dx}(\ln \cosh x) = \frac{1}{\cosh x} \times \sinh x$$
$$= \tanh x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 13

Question:

Differentiate with respect to x.
$\sinh x^3$

Solution:

$$\frac{d}{dx}\left(\sinh x^3\right) = 3x^2 \cosh x^3$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 14

Question:

Differentiate with respect to x.
$\cosh^2 2x$

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^2 2x\right) = 2\cosh 2x 2\sinh 2x$$
$$= 4\cosh 2x \sinh 2x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 15

Question:

Differentiate with respect to x.

$e^{\cosh x}$

Solution:

$$\frac{d}{dx}\left(e^{\cosh x}\right)=\sinh x e^{\cosh x}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 16

Question:

Differentiate with respect to x.
$\text{cosech}\, x$

Hint: $\text{cosech}\, x = \frac{1}{\sinh x}$.

Solution:

$$\frac{d}{dx}(\text{cosech} x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = \frac{0 - 1 \times \cosh x}{\sinh^2 x}$$
$$= -\coth x \,\text{cosech} x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 17

Question:

If $y = a\cosh nx + b\sinh nx$, where a and b are constants, prove that $\frac{d^2y}{dx^2} = n^2y$.

Solution:

$y = a\cosh nx + b\sinh nx$

Differentiate with respect to x

$\frac{dy}{dx} = an\sinh nx + nb\cosh nx$

$\frac{d^2y}{dx^2} = an^2\cosh nx + bn^2\sinh nx$

$= n^2(a\cosh nx + b\sinh nx)$

$\frac{d^2y}{dx^2} = n^2y$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 18

Question:

Find the stationary values of the curve with equation $y = 12\cosh x - \sinh x$.

Solution:

$y = 12\cosh x - \sinh x$

$\frac{dy}{dx} = 12\sinh x - \cosh x$

At stationary values $\frac{dy}{dx} = 0$

$0 = 12\sinh x - \cosh x$

$\cosh x = 12\sinh x$

$\frac{1}{12} = \tanh x$

$x = \tanh^{-1}\frac{1}{12}$

$x = 0.0835$

The stationary value is therefore $y = 12\cosh 0.0835 - \sinh 0.0835$

$= 12.13$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 19

Question:

Given that $y = \cosh 3x \sinh x$, find $\frac{d^2y}{dx^2}$.

Solution:

$$y = \cosh 3x \sinh x$$
$$\frac{dy}{dx} = 3\sinh 3x \sinh x + \cosh 3x \cosh x$$
$$\frac{d^2y}{dx^2} = 9\cosh 3x \sinh x + 3\sinh 3x \cosh x + 3\sinh 3x \cosh x + \cosh 3x \sinh x$$
$$= 10\cosh 3x \sinh x + 6\sinh 3x \cosh x$$
$$= 2(5\cosh 3x \sinh x + 3\sinh 3x \cosh x)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise A, Question 20

Question:

Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{256} - \frac{y^2}{16} = 1$ at the point $(16 \cosh q, 4 \sinh q)$.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dq}}{\frac{dx}{dq}} = \frac{4\cosh q}{16\sinh q} = \frac{\cosh q}{4\sinh q}$$

Equation of tangent

$$y - 4\sinh q = \frac{\cosh q}{4\sinh q}(x - 16\cosh q)$$

$$4y\sinh q - 16\sinh^2 q = x\cosh q - 16\cosh^2 q$$

$$4y\sinh q - x\cosh q = 16(\sinh^2 q - \cosh^2 q)$$

$$4y\sinh q - x\cosh q = -16$$

or $x\cosh q - 4y\sinh q = 16$

Equation of normal

$$y - 4\sinh q = \frac{-4\sinh q}{\cosh q}(x - 16\cosh q)$$

i.e. $y\cosh q - 4\sinh q\cosh q = -4x\sinh q + 64\sinh q\cosh q$

i.e. $y\cosh q + 4x\sinh q = 68\sinh q\cosh q$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 1

Question:

Differentiate

a $\text{arcosh } 2x$

b $\text{arsinh}(x+1)$

c $\text{artanh } 3x$

d $\text{arsech } x$

e $\text{arcosh} x^2$

f $\text{arcosh } 3x$

g $x^2 \text{arcosh} x$

h $\text{arsinh } \frac{x}{2}$

i $e^{x^3} \text{arsinh} x$

j $\text{arsinh } x \text{ arcosh } x$

k $\text{arcosh } x \text{ sech } x$

l $x \text{ arcosh } 3x$

Solution:

a Let $y = \text{arcosh}\, 2x$ then $\cosh y = 2x$

Differentiate with respect to x

$$\sinh y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\sinh y}$$

$$= \frac{2}{\sqrt{\cosh^2 y - 1}} \text{ but } \cosh y = 2x$$

$$\text{so } \frac{dy}{dx} = \frac{2}{\sqrt{4x^2 - 1}}$$

b Let $y = \text{arsinh}(x+1)$ then $\sinh y = x + 1$

$$\cosh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$= \frac{1}{\sqrt{\sinh^2 y + 1}} \text{ but } \sinh y = x + 1$$

$$\text{so } \frac{dy}{dx} = \frac{1}{\sqrt{(x+1)^2 + 1}}$$

c Let $y = \text{artanh}\, 3x$

$$\tanh y = 3x$$

$$\text{sech}^2 y \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{\text{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - \tanh^2 y}$$

$$\frac{dy}{dx} = \frac{3}{1 - 9x^2}$$

d Let $y = \text{arsech}x$

$\text{sech}y = x$

$\frac{1}{\cosh y} = x$

$1 = x\cosh y$

Differentiate with respect to x

$0 = \cosh y + x\sinh y\frac{dy}{dx}$

$x\sinh y\frac{dy}{dx} = -\cosh y$

$\frac{dy}{dx} = \frac{-\cosh y}{x\sinh y}$

$= \frac{1}{x\tanh y}$

$= \frac{1}{x(1-\text{sech}^2 y)^{\frac{1}{2}}}$

$= \frac{-1}{x(1-x^2)^{\frac{1}{2}}}$

e Let $y = \text{arcosh}x^2$

Let $t = x^2 \quad y = \text{arcosh}t$

$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$

$\frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$

f $y = \text{arcosh}\,3x$

Let $t = 3x \quad y = \text{arcosh}t$

$\frac{dt}{dx} = 3 \quad \frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}}$

$\frac{dy}{dx} = \frac{3}{\sqrt{9x^2-1}}$

g $y = x^2\text{arcosh}x$

$\frac{dy}{dx} = 2x\text{arcosh}x + \frac{x^2}{\sqrt{x^2-1}}$

h $y = \text{arsinh}\,\frac{x}{2}$

Let $t = \frac{x}{2}$ $y = \text{arsinh}\,t$

$\frac{dt}{dx} = \frac{1}{2}$ $\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\left(\frac{x}{2}\right)^2+1}}$$

$$= \frac{1}{\sqrt{x^2+4}}$$

i $y = e^{x^3}\text{arsinh}\,x$

$$\frac{dy}{dx} = 3x^2e^{x^3}\text{arsinh}\,x + \frac{e^{x^3}}{\sqrt{x^2+1}}$$

j $y = \text{arsinh}\,x\,\text{arcosh}\,x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}\text{arcosh}\,x + \frac{1}{\sqrt{x^2-1}}\text{arsinh}\,x$$

k $y = \text{arcosh}\,x\,\text{sech}\,x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}\text{sech}\,x - \text{arcosh}\,x\tanh x\,\text{sech}\,x$$

$$= \text{sech}\,x\left(\frac{1}{\sqrt{x^2-1}} - \text{arcosh}\,x\tanh x\right)$$

l $y = x\,\text{arcosh}\,3x$

$$\frac{dy}{dx} = \text{arcosh}\,3x + x \times \frac{3}{\sqrt{9x^2-1}}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 2

Question:

Prove that

a $\frac{d}{dx}(\text{arcosh}x) = \frac{1}{\sqrt{x^2 - 1}}$

b $\frac{d}{dx}(\text{artanh}x) = \frac{1}{1 - x^2}$

Solution:

a

$$y = \text{arcosh}x$$
$$\cosh y = x$$
$$\sinh y \frac{dy}{dx} = 1 \Rightarrow$$
$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}}$$

but $\cosh y = x$ so

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

b

$$y = \text{artanh}x$$
$$\tanh y = x$$
$$\text{sech}^2 y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1 - \tanh^2 y}$$

but $\tanh y = x$ so

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

Solutionbank FP3

Edexcel AS and A Level Modular Mathematics

Differentiation

Exercise B, Question 3

Question:

Given that $y = \text{artanh}\left(\frac{e^x}{2}\right)$, prove that $(4 - e^{2x})\frac{dy}{dx} = 2e^x$.

Solution:

$y = \text{artanh}\frac{e^x}{2}$

Let $t = \frac{e^x}{2}$ $\quad y = \text{artanh}\,t$

$\frac{dt}{dx} = \frac{e^x}{2} \quad \frac{dy}{dt} = \frac{1}{1-t^2}$

Then $\frac{dy}{dx} = \frac{1}{1-t^2} \times \frac{e^x}{2}$

$$= \frac{1}{1-\left(\frac{e^x}{2}\right)^2} \times \frac{e^x}{2}$$

$$= \frac{\frac{e^x}{2}}{\frac{4-e^{2x}}{4}}$$

$$\frac{dy}{dx} = \frac{2e^x}{4-e^{2x}}$$

$$(4 - e^{2x})\frac{dy}{dx} = 2e^x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 4

Question:

Given that $y = \text{arsinh}x$, show that

$$(1+x^2)\frac{d^3y}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Solution:

$$y = \text{ar}\sinh x$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{x^2+1}}=(x^2+1)^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{2}(x^2+1)^{-\frac{3}{2}}\times 2x$$

$$=\frac{-x}{(x^2+1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3}=\frac{-1(x^2+1)^{\frac{3}{2}}-\frac{3}{2}(x^2+1)^{\frac{1}{2}}\times 2x\times -x}{(x^2+1)^3}$$

$$=\frac{3x^2(x^2+1)^{\frac{1}{2}}-(x^2+1)^{\frac{3}{2}}}{(x^2+1)^3}$$

$$=\frac{3x^2}{(x^2+1)^{\frac{5}{2}}}-\frac{1}{(x^2+1)^{\frac{3}{2}}}$$

$$(x^2+1)\frac{d^3y}{dx^3}=\frac{3x^2}{(x^2+1)^{\frac{3}{2}}}-\frac{1}{(x^2+1)^{\frac{1}{2}}}$$

$$=-3x\frac{d^2y}{dx^2}-\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^3y}{dx^3}+3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 5

Question:

If $y = (\text{arcosh}x)^2$, find $\frac{d^2y}{dx^2}$.

Solution:

$$y = (\text{arcosh}x)^2$$
$$\frac{dy}{dx} = 2\text{arcosh}x \times \frac{1}{\sqrt{x^2-1}}$$
$$= 2(x^2-1)^{-\frac{1}{2}}\text{arcosh}x$$
$$\frac{d^2y}{dx^2} = -(x^2-1)^{-\frac{3}{2}}2x\text{arcosh}x + 2(x^2-1)^{-\frac{1}{2}} \times \frac{1}{\sqrt{x^2-1}}$$
$$= \frac{-2x\text{arcosh}x}{(x^2-1)^{\frac{3}{2}}} + \frac{2}{x^2-1}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise B, Question 6

Question:

Find the equation of the tangent at the point where $x = \frac{12}{13}$ on the curve with equation $y = \text{artanh}\, x$.

Solution:

$y = \text{artanh}x \quad x = \frac{12}{13} \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln 25 = \ln 5$

$\frac{dy}{dx} = \frac{1}{1-x^2} = \frac{1}{1-\left(\frac{12}{13}\right)^2} = \frac{169}{25}$

Tangent is

$(y - \ln 5) = \frac{169}{25}\left(x - \frac{12}{13}\right)$

$25y - 25\ln 5 = 169x - 156$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 1

Question:

Given that $y = \arccos x$ prove that

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Solution:

$$y = \arccos x$$
$$\cos y = x$$
$$-\sin y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{\sin y}$$
$$= \frac{-1}{\sqrt{1-\cos^2 y}}$$

since $\cos y = x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 2

Question:

Differentiate with respect to x

a $\arccos 2x$

b $\arctan \frac{x}{2}$

c $\arcsin 3x$

d $\text{arccot}\, x$

e $\text{arcsec}\, x$

f $\text{arccosec}\, x$

g $\arcsin\left(\frac{x}{x-1}\right)$

h $\arccos x^2$

i $e^x \arccos x$

j $\arcsin x \cos x$

k $x^2 \arccos x$

l $e^{\arctan x}$

Solution:

a Let $y = \arccos 2x$

Let $t = 2x \quad y = \arccos t$

then $\frac{dt}{dx} = 2 \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \times 2$

$= \frac{-2}{\sqrt{1-4x^2}}$

b Let $y = \arctan \frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \arctan t$

$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{1+t^2}$

$\frac{dy}{dx} = \frac{1}{1+t^2} \times \frac{1}{2} = \frac{1}{2\left(1+\frac{x^2}{4}\right)} = \frac{2}{4+x^2}$ or $\frac{2}{x^2+4}$

c Let $y = \arcsin 3x$

$\sin y = 3x$

$\cos y \frac{dy}{dx} = 3$

$\frac{dy}{dx} = \frac{3}{\cos y} = \frac{3}{\sqrt{1-\sin^2 y}}$

$= \frac{3}{\sqrt{1-9x^2}}$

$= \frac{3}{\sqrt{1-9x^2}}$

d Let $y = \operatorname{arccot} x$

$\cot y = x$

$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y}$

$= \frac{-1}{1+\cot^2 y}$

$= \frac{-1}{1+x^2}$

e Let $y = \operatorname{arcsec} x$

$\sec y = x$

$\sec y \tan y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$

$= \frac{1}{\sec y \sqrt{\sec^2 y - 1}}$

$= \frac{1}{x\sqrt{x^2-1}}$

f Let $y = \operatorname{arccosec} x$

$\operatorname{cosec} y = x$

$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y}$

$= \frac{-1}{\operatorname{cosec} y \sqrt{(\operatorname{cosec}^2 y - 1)}}$

$= \frac{-1}{x\sqrt{x^2-1}}$

g Let $y = \arcsin\left(\frac{x}{x-1}\right)$

$\sin y = \frac{x}{x-1}$

$$\cos y \frac{dy}{dx} = \frac{-1}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{1 - \frac{x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\sqrt{\frac{(x-1)^2 - x^2}{(x-1)^2}}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{1}{\frac{\sqrt{1-2x}}{x-1}} \times \frac{-1}{(x-1)^2}$$

$$= \frac{-1}{(x-1)\sqrt{1-2x}}$$

h Let $y = \arccos x^2$

Let

$t = x^2 \quad y = \arccos t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \times 2x$$

$$= \frac{-2x}{\sqrt{1-x^4}}$$

i Let $y = e^x \arccos x$

$$\frac{dy}{dx} = e^x \arccos x - e^x \frac{1}{\sqrt{1-x^2}}$$

$$= e^x \left(\arccos x - \frac{1}{\sqrt{1-x^2}} \right)$$

j Let $y = \arcsin x \cos x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cos x + \arcsin x \times - \sin x$$

$$= \frac{\cos x}{\sqrt{1-x^2}} - \sin x \arcsin x$$

k Let $y = x^2 \arccos x$

$$\frac{dy}{dx} = 2x \arccos x - x^2 \times \frac{1}{\sqrt{1-x^2}}$$

$$= 2x \arccos x - \frac{x^2}{\sqrt{1-x^2}}$$

$$= x \left(2\arccos x - \frac{x}{\sqrt{1-x^2}} \right)$$

l Let $y = e^{\arctan x}$

$$\frac{dy}{dx} = \frac{e^{\arctan x}}{1+x^2}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 3

Question:

If $\tan y = x \arctan x$, find $\frac{dy}{dx}$.

Solution:

$$\tan y = x \arctan x$$

$$\sec^2 y \frac{dy}{dx} = \arctan x + \frac{x}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\left(\arctan x + \frac{x}{1+x^2}\right)$$

$$= \frac{1}{1+x^2(\arctan x)^2}\left(\arctan x + \frac{x}{1+x^2}\right)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 4

Question:

Given that $y = \arcsin x$ prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$ **[E]**

Solution:

$$y = \arcsin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{0 - \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \times -2x}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{x\left(1-x^2\right)^{-\frac{1}{2}}}{\left(1-x^2\right)}$$

$$= \frac{x}{\sqrt{1-x^2}\left(1-x^2\right)}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} = x\frac{dy}{dx}$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise C, Question 5

Question:

Find an equation of the tangent to the curve with equation $y = \arcsin 2x$ at the point where $x = \frac{1}{4}$.

Solution:

$$y = \arcsin 2x \quad x = \frac{1}{4} \quad y = \arcsin\left(\frac{2}{4}\right) = \frac{\pi}{6}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{4}{\sqrt{3}}$$

Tangent is

$$\left(y - \frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\left(x - \frac{1}{4}\right)$$

$$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = 4x - 1$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 1

Question:

Given $y = \cosh 2x$, find $\frac{dy}{dx}$.

Solution:

$$y = \cosh 2x$$
$$\frac{dy}{dx} = 2\sinh 2x$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 2

Question:

Differentiate with respect to x.

a $\text{arsinh } 3x$

b $\text{arsinh} x^2$

c $\text{arcosh } \frac{x}{2}$

d $x^2 \text{arcosh } 2x$

Solution:

a $y = \text{arsinh}\,3x$

Let $t = 3x \quad y = \text{arsinh}\,t$

$$\frac{dt}{dx} = 3 \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2+1}} \times 3$$

$$= \frac{3}{\sqrt{9x^2+1}}$$

b $y = \text{arsinh}\,x^2$

Let $t = x^2 \quad y = \text{arsinh}\,t$

$$\frac{dt}{dx} = 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{t^2+1}} \times 2x$$

$$= \frac{2x}{\sqrt{x^4+1}}$$

c $y = \text{arcosh}\,\frac{x}{2}$

Let $t = \frac{x}{2} \quad y = \text{arcosh}\,t$

$$\frac{dt}{dx} = \frac{1}{2} \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2-1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2-1}} \times \frac{1}{2}$$

$$= \frac{1}{2\sqrt{\frac{x^2}{4}-1}} = \frac{1}{\sqrt{x^2-4}}$$

d $y = x^2 \text{arcosh}\,2x$

$$\frac{dy}{dx} = 2x\,\text{arcosh}\,2x + x^2 \times \frac{2}{\sqrt{4x^2-1}}$$

$$= 2x\left(\text{arcosh}\,2x + \frac{x}{\sqrt{4x^2-1}}\right)$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 3

Question:

Given that $y = \arctan x$, prove that

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Solution:

$y = \arctan x$

then $\tan y = x$

$\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

but $\sec^2 y = 1 + \tan^2 y = 1 + x^2$

so $\frac{dy}{dx} = \frac{1}{1+x^2}$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 4

Question:

Given that $y = (\text{arsinh}x)^2$ prove that

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2=0$$

Solution:

$$y = (\text{arsinh}\,x)^2$$

$$\frac{dy}{dx} = \frac{2(\text{arsinh}x)^1}{\sqrt{x^2+1}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2}{\sqrt{x^2+1}}\times\sqrt{x^2+1}-\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\times 2x\times 2\text{arsinh}x}{\left(\sqrt{x^2+1}\right)^2}$$

$$(x^2+1)\frac{d^2y}{dx^2} = 2-2x(x^2+1)^{-\frac{1}{2}}\,\text{arsinh}x$$

$$= 2 - x\frac{dy}{dx}$$

$$(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2 = 0$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 5

Question:

Given $y = 5\cosh x - 3\sinh x$

a find $\frac{dy}{dx}$

b find the minimum turning points.

Solution:

$y = 5\cosh x - 3\sinh x$

$\frac{dy}{dx} = 5\sinh x - 3\cosh x$

At maximum and minimum $\frac{dy}{dx} = 0$

$0 = 5\sinh x - 3\cosh x$

$3\cosh x = 5\sinh x$

$\frac{3}{5} = \tanh x$

$x = \text{artanh}\frac{3}{5}$

Use $\text{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$x = \frac{1}{2}\ln\left(\frac{\frac{8}{5}}{\frac{2}{5}}\right)$

$x = \frac{1}{2}\ln 4$

$= \ln 2$

$y = 6\frac{1}{4} - 2\frac{1}{4}$

$= 4$

$\Rightarrow$ turning point is $(\ln 2, 4)$

$\frac{d^2y}{dx^2} = 5\cosh x - 3\sinh x = 4$ at $x = \ln 2$

$\therefore \frac{d^2y}{dx^2} > 0$ at $(\ln 2, 4)$ so this point is a minimum

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 6

Question:

Given that $y = (\arcsin x)^2$ show that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Solution:

$$y = (\arcsin x)^2$$

$$\frac{dy}{dx} = 2(\arcsin x)\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{2\times\frac{1}{\sqrt{1-x^2}}\times\sqrt{1-x^2} - 2\arcsin x\times\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 2 + \frac{x\times 2\arcsin x}{(1-x^2)^{\frac{1}{2}}}$$

$$= 2 + x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 7

Question:

Differentiate arcosh (sinh $2x$).

Solution:

$$y = \text{arcosh}\left(\sinh 2x\right)$$

Let $t = \sinh 2x \quad y = \text{arcosh}\, t$

$$\frac{dt}{dx} = 2\cosh 2x \quad \frac{dy}{dt} = \frac{1}{\sqrt{t^2 - 1}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{t^2 - 1}} \times 2\cosh 2x$$

$$= \frac{2\cosh 2x}{\sqrt{\sinh^2 2x - 1}}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 8

Question:

Given that $y = x - \arctan x$, prove that $\frac{d^2y}{dx^2} = 2x\left(1 - \frac{dy}{dx}\right)^2$

Solution:

$$y = x - \arctan x$$

$$\frac{dy}{dx} = 1 - \frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2} = 0 - \frac{(0-2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

$$= 2x\left(1 - \left(1 - \frac{1}{1+x^2}\right)\right)^2$$

$$\frac{d^2y}{dx^2} = 2x\left(1 - \frac{dy}{dx}\right)^2$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 9

Question:

Differentiate $\arcsin \frac{x}{\sqrt{1+x^2}}$.

Solution:

$$y = \arcsin \frac{x}{\sqrt{1+x^2}}$$

Let $t = \frac{x}{\sqrt{1+x^2}}$ $\qquad$ $y = \arcsin t$

$$\frac{dt}{dx} = \frac{1(1+x^2)^{\frac{1}{2}} - \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \times 2x \times x}{(1+x^2)} \qquad \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-t^2}}\left(\frac{(1+x^2)^{\frac{1}{2}} - x^2(1+x^2)^{-\frac{1}{2}}}{1+x^2}\right)$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}}\left((1+x^2)^{-\frac{1}{2}}\frac{[1+x^2-x^2]}{(1+x^2)}\right)$$

$$= \frac{1}{\sqrt{\frac{1}{1+x^2}}}(1+x^2)^{-\frac{1}{2}}\frac{[1]}{(1+x^2)} = \frac{1}{x^2+1}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 10

Question:

Show that the curve with equation $y = \text{sech}\, x$ has $\frac{d^2y}{dx^2} = 0$ at the point where $x = \pm \ln p$ and state a value of p.

Solution:

$$y = \operatorname{sech} x$$
$$\frac{dy}{dx} = -\tanh x \operatorname{sech} x$$
$$\frac{d^2y}{dx^2} = \operatorname{sech}^2 x \operatorname{sech} x + \tanh x(-\tanh x \sec x)$$
$$= \operatorname{sech}^3 x - \operatorname{sech} x \tanh^2 x$$
$$= \operatorname{sech} x\left(\operatorname{sech}^2 x - \tanh^2 x\right)$$
$$= \operatorname{sech} x\left(1 - \tanh^2 x - \tanh^2 x\right)$$
$$= \operatorname{sech} x(1 - 2\tanh^2 x)$$

When $\frac{d^2y}{dx^2} = 0$

$$0 = \operatorname{sech} x\left(1 - 2\tanh^2 x\right)$$

so $\tanh^2 x = \frac{1}{2} \Rightarrow \tanh x = \pm\frac{1}{\sqrt{2}}$

$$x = \operatorname{artanh} \pm\frac{1}{\sqrt{2}} = \pm\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)$$
$$x = \pm\frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$$
$$= \pm\frac{1}{2}\ln\left(\frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}}\right)$$
$$= \pm\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$
$$= \pm\frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)$$
$$= \pm\frac{1}{2}\ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right)$$
$$= \pm\frac{1}{2}\ln\left(\sqrt{2}+1\right)^2$$
$$= \pm\ln\left(\sqrt{2}+1\right) \quad p = \sqrt{2}+1 \text{(Note } p = \sqrt{2}-1 \text{ is also acceptable.)}$$

Solutionbank FP3
Edexcel AS and A Level Modular Mathematics

Differentiation
Exercise D, Question 11

Question:

Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\cosh q, b\sinh q)$.

Solution:

$x=a\cosh q \quad y=b\sinh q$

$\frac{dy}{dx}=\frac{b\cosh q}{a\sinh q}$

Equation of tangent $y-b\sinh q=\frac{b\cosh q}{a\sinh q}(x-a\cosh q)$

$ay\sinh q-ab\sinh^2 q = xb\cosh q-ab\cosh^2 q$

$ay\sinh q-xb\cosh q+ab(\cosh^2 q-\sinh^2 q)=0$

$ay\sinh q-xb\cosh q+ab = 0$

Equation of normal $y-b\sinh q=-\frac{a\sinh q}{b\cosh q}(x-a\cosh q)$

$by\cosh q-b^2\sinh q\cosh q = -ax\sinh q+a^2\sinh q\cosh q$

$ax\sinh q+by\cosh q-\sinh q\cosh q(a^2+b^2)=0$