Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

For each of the following functions, f(x), find f′(x), f″(x), f‴(x) and $f^{(n)}(x)$.

a e^{2x} **b** $(1+x)^n$ **c** xe^x **d** $\ln(1+x)$

Solution:

	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(n)}(x)$
a	$2e^{2x}$	$2^2e^{2x} = 4e^{2x}$	$2^3e^{2x} = 8e^{2x}$	$2ne^{2x}$
b	$n(1+x)^{n-1}$	$n(n-1)(1+x)^{n-2}$	$n(n-1)(n-2)(1+x)^{n-3}$	$n!$
c	$e^x + xe^x$	$e^x + (e^x + xe^x)$ $= 2e^x + xe^x$	$2e^x + (e^x + xe^x) = 3e^x + xe^x$	$ne^x + xe^x$
d	$(1+x)^{-1}$	$-(1+x)^{-2}$	$(-1)(-2)(1+x)^{-3} = 2(1+x)^{-3}$	$(-1)^{n-1}(n-1)!(1+x)^{-n}$

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Exercise A, Question 2

Question:

a Given that $y = e^{2+3x}$, find an expression, in terms of y, for $\frac{d^n y}{dx^n}$.

b Hence show that $\left(\frac{d^6 y}{dx^6}\right)_{\ln\left(\frac{1}{9}\right)} = e^2$

Solution:

a $y = e^{2+3x}$, so $\frac{dy}{dx} = 3\,e^{2+3x}$, $\frac{d^2y}{dx^2} = 3^2\,e^{2+3x}$, $\frac{d^3y}{dx^3} = 3^3\,e^{2+3x}$, and so on.

It follows that $\frac{d^n y}{dx^n} = 3^n e^{2+3x} = 3^n y$ as $y = e^{2+3x}$.

b $\frac{d^6y}{dx^6} = 3^6 y$

When $x = \ln\left(\frac{1}{9}\right) = \ln 3^{-2}$, $y = e^{2+3\ln 3^{-2}} = e^2 \times e^{3\ln 3^{-2}} = e^2 \times e^{\ln 3^{-6}} = \frac{e^2}{3^6}$ — As $e^{\ln a} = a$

So $\left(\frac{d^6y}{dx^6}\right)_{\ln\left(\frac{1}{9}\right)} = 3^6 \times \frac{e^2}{3^6} = e^2$.

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Exercise A, Question 3

Question:

Given that $y = \sin^2 3x$,

a show that $\frac{dy}{dx} = 3\sin 6x$.

b Find expressions for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$.

c Hence evaluate $\left(\frac{d^4y}{dx^4}\right)_{\frac{\pi}{6}}$.

Solution:

a $y = \sin^2 3x = (\sin 3x)^2$, so $\frac{dy}{dx} = 2(\sin 3x)(3\cos 3x)$

Use $\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}$.

$= 3(2\sin 3x\cos 3x)$

$= 3\sin 6x$

Use $\sin 2A = 2\sin A\cos A$.

b $\frac{d^2y}{dx^2} = 18\cos 6x$, $\frac{d^3y}{dx^3} = -108\sin 6x$, $\frac{d^4y}{dx^4} = -648\cos 6x$

c $\left(\frac{d^4y}{dx^4}\right)_{\frac{\pi}{6}} = -648\cos\pi = 648$

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Exercise A, Question 4

Question:

$f(x) = x^2e^{-x}$.

a Show that $f'''(x) = (6x - 6 - x^2)e^{-x}$.

b Show that $f''''(2) = 0$.

Solution:

a $f'(x) = 2xe^{-x} - x^2e^{-x}$

$f''(x) = (2e^{-x} - 2xe^{-x}) - (2xe^{-x} - x^2e^{-x}) = e^{-x}(2 - 4x + x^2)$

$f'''(x) = e^{-x}(-4 + 2x) - e^{-x}(2 - 4x + x^2) = e^{-x}(-6 + 6x - x^2)$

b $f''''(x) = e^{-x}(6 - 2x) - e^{-x}(-6 + 6x - x^2) = e^{-x}(12 - 8x + x^2)$

so $f''''(2) = e^{-2}(12 - 16 + 4) = 0$

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Exercise A, Question 5

Question:

Given that $y = \sec x$, show that

a $\frac{d^2y}{dx^2} = 2\sec^3 x - \sec x$,

b $\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = 11\sqrt{2}$.

Solution:

a Given that $y = \sec x$, so $\frac{dy}{dx} = \sec x \tan x$

$\frac{d^2y}{dx^2} = \sec x(\sec^2 x) + (\sec x \tan x)\tan x$ — Use the product rule.

$= \sec x(\sec^2 x + \tan^2 x)$

$= \sec x(\sec^2 x + \sec^2 x - 1)$ — Use $1 + \tan^2 A = \sec^2 A$.

$= 2\sec^3 x - \sec x$

b $\frac{d^3y}{dx^3} = 6\sec^2 x(\sec x \tan x) - \sec x \tan x$

$= \sec x \tan x(6\sec^2 x - 1)$

Substituting $x = \frac{\pi}{4}$ in $\frac{d^3y}{dx^3}$

$\left(\frac{d^3y}{dx^3}\right)_{\frac{\pi}{4}} = (\sqrt{2})(1)\{6(2) - 1\} = 11\sqrt{2}$

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Exercise A, Question 6

Question:

Given that y is a function of x, show that

a $\frac{d^2}{dx^2}(y^2) = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$

b Find an expression, in terms of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, for $\frac{d^3}{dx^3}(y^2)$.

Solution:

a $\frac{d}{dx}(y^2) = \frac{d}{dx}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$

Use the chain rule.

$$\frac{d^2}{dx^2}(y^2) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right) = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{dy}{dx} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

Use the product rule.

b $\frac{d^3}{dx^3}(y^2) = \frac{d}{dx}\left(2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2\right)$

$$= 2\left\{y\frac{d^3y}{dx^3} + \frac{dy}{dx} \times \frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$

$$= 2\left\{y\frac{d^3y}{dx^3} + 3\frac{dy}{dx} \times \frac{d^2y}{dx^2}\right\}$$

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Exercise A, Question 7

Question:

Given that $f(x) = \ln\{x + \sqrt{1 + x^2}\}$, show that

a $\sqrt{1 + x^2}\, f'(x) = 1$,

b $(1 + x^2)\, f''(x) + x f'(x) = 0$,

c $(1 + x^2)\, f'''(x) + 3x f''(x) + f'(x) = 0$.

d Deduce the values of $f'(0)$, $f''(0)$ and $f'''(0)$.

Solution:

$f(x) = \ln\{x + \sqrt{1 + x^2}\}$

a $f'(x) = \dfrac{1}{x + \sqrt{(1 + x^2)}} \times \left\{1 + \dfrac{x}{\sqrt{(1 + x^2)}}\right\}$,

Use $\dfrac{d}{dx}(\ln u) = \dfrac{1}{u}\dfrac{du}{dx}$.

$= \dfrac{1}{\cancel{x + \sqrt{(1 + x^2)}}} \times \left\{\dfrac{\cancel{\sqrt{(1 + x^2)} + x}}{\sqrt{(1 + x^2)}}\right\} = \dfrac{1}{\sqrt{(1 + x^2)}}$

So $\sqrt{(1 + x^2)}\, f'(x) = 1$

b Differentiating this equation w.r.t. x, using the product rule

$\sqrt{(1 + x^2)}\, f''(x) + \dfrac{x}{\sqrt{(1 + x^2)}} f'(x) = 0$

So $(1 + x^2) f''(x) + x f'(x) = 0$ ← Multiply through by $\sqrt{(1 + x^2)}$.

c Differentiating this result w.r.t. x

$\{(1 + x^2) f'''(x) + 2x f''(x)\} + \{f'(x) + x f''(x)\} = 0$

giving

$(1 + x^2) f'''(x) + 3x f''(x) + f'(x) = 0$

d $f'(0) = \dfrac{1}{\sqrt{1 + 0}} = 1$

Using $(1 + x^2) f''(x) + x f'(x) = 0$ with $x = 0$ and $f'(0) = 1$

$f''(0) + (0)(1) = 0 \Rightarrow f''(0) = 0$

Using $(1 + x^2) f'''(x) + 3x f''(x) + f'(x) = 0$ with $x = 0$, $f'(0) = 1$ and $f''(0) = 0$

$f'''(0) + (0)(0) + 1 = 0 \Rightarrow f'''(0) = -1$

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Exercise B, Question 1

Question:

Use the formula for the Maclaurin expansion and differentiation to show that

a $(1-x)^{-1} = 1 + x + x^2 + \ldots + x^r + \ldots$

b $\sqrt{(1+x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$

Solution:

a $f(x) = (1-x)^{-1}$ $\Rightarrow f(0) = 1$

$f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2}$ $\Rightarrow f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$ $\Rightarrow f''(0) = 2$

$f'''(x) = -3.2(1-x)^{-4}(-1) = 3.2(1-x)^{-4}$ $\Rightarrow f'''(0) = 3!$

General term: The pattern here is such that $f^{(r)}(x)$ can be written down

$f^{(r)}(x) = r(r-1)\ldots 2(1-x)^{-(r+1)} = r!(1-x)^{-(r+1)}$ $\Rightarrow f^{(r)}(0) = r!$

Using $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(r)}(0)}{r!}x^r + \ldots$

$$(1-x)^{-1} = 1 + x + \frac{2}{2!}x^2 + \ldots + \frac{r!}{r!}x^r + \ldots = 1 + x + x^2 + \ldots + x^r + \ldots$$

b $f(x) = \sqrt{(1+x)} = (1+x)^{\frac{1}{2}}$ $\Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}(-\frac{1}{2})(1+x)^{-\frac{3}{2}}$ $\Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(1+x)^{-\frac{5}{2}}$ $\Rightarrow f'''(0) = \frac{3}{8}$

Using Maclaurin's expansion

$$\sqrt{(1+x)} = 1 + \tfrac{1}{2}x + \frac{\left(-\frac{1}{4}\right)}{2!}x^2 + \frac{\left(\frac{3}{8}\right)}{3!}x^3 - \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

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Exercise B, Question 2

Question:

Use Maclaurin's expansion and differentiation to show that the first three terms in the series expansion of $e^{\sin x}$ are $1 + x + \frac{x^2}{2}$.

Solution:

a $f(x) = e^{\sin x} \qquad \Rightarrow f(0) = 1$

$f'(x) = \cos x\, e^{\sin x} \qquad \Rightarrow f'(0) = 1$

$f''(x) = \cos^2 x\, e^{\sin x} - \sin x\, e^{\sin x} \qquad \Rightarrow f''(0) = 1$

Substituting into Maclaurin's expansion gives

$$e^{\sin x} = 1 + 1x + \frac{1}{2!}x^2 + \ldots$$

$$= 1 + x + \frac{1}{2}x^2 + \ldots$$

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Exercise B, Question 3

Question:

a Show that the Maclaurin expansion for $\cos x$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$

b Using the first 3 terms of the series, show that it gives a value for $\cos 30°$ correct to 3 decimal places.

Solution:

a $f(x) = \cos x \quad \Rightarrow f(0) = 1$

$f'(x) = -\sin x \quad \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \quad \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \quad \Rightarrow f'''(0) = 0$

$f''''(x) = \cos x \quad \Rightarrow f''''(0) = 1$

The process repeats itself after every 4th derivative, like $\sin x$ does (see Example 5).
Using Maclaurin's expansion, only even powers of x are produced.

$$\cos x = 1 + \frac{(-1)}{2!}x^2 + \frac{1}{4!}x^4 + \ldots + \frac{(-1)^{r+1}}{(2r)!}x^{2r} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots$$

b Using $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ with $x = \frac{\pi}{6}$ (must be in radians)

$$\cos x \approx 1 - \frac{\pi^2}{72} + \frac{\pi^4}{31\,104} = 0.86605 \ldots \text{ which is correct to 3 d.p.}$$

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Exercise B, Question 4

Question:

Using the series expansions for e^x and $\ln(1 + x)$ respectively, find, correct to 3 decimal places, the value of

a e **b** $\ln\left(\frac{6}{5}\right)$

Solution:

a Substituting $x = 1$ into the Maclaurin expansion of e^x, gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \dots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5	6	7	8	9	10
Approx.	1	2	2.5	2.6667	2.7083	2.7167	2.7181	2.7183	2.7183	2.7183

So e = 2.718 (3 d.p.)

b Substituting $x = 0.2$ into the Maclaurin expansion of $\ln(1 + x)$, gives

$$\ln\left(\frac{6}{5}\right) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5} - \frac{(0.2)^6}{6} + \frac{(0.2)^7}{7} - \dots$$

The approximations, to 4 d.p. where necessary, using n terms of the series are

n	1	2	3	4	5
Approximation	0.2	0.18	0.1827	0.1823	0.1823

So $\ln\left(\frac{6}{5}\right) = 0.182$ (3 d.p.)

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Exercise B, Question 5

Question:

Use Maclaurin's expansion and differentiation to expand, in ascending powers of x up to and including the term in x^4,

a e^{3x} **b** $\ln(1 + 2x)$ **c** $\sin^2 x$

Solution:

a $f(x) = e^{3x},\ f^{(n)}(x) = 3^n e^{3x}$

So $f(0) = 1,\ f'(0) = 3,\ f''(0) = 3^2,\ f'''(0) = 3^3,\ f''''(0) = 3^4$

$$f(x) = e^{3x} = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4 + \ldots$$

$$= 1 + 3x + \frac{9x^2}{2} + \frac{9x^3}{2} + \frac{27}{8}x^4 + \ldots \left[\text{Note: this is } 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \ldots\right]$$

b As $f(x) = \ln(1 + 2x)$, $f(0) = \ln 1 = 0$

$f'(x) = \frac{2}{1+2x} = 2(1+2x)^{-1}$, $f'(0) = 2$

$f''(x) = -4(1+2x)^{-2}$, $f''(0) = -4$

$f'''(x) = 16(1+2x)^{-3}$, $f'''(0) = 16$

$f''''(x) = -96(1+2x)^{-4}$, $f''''(0) = -96$

$$\text{So } \ln(1+2x) = 0 + 2x + \frac{(-4)}{2!}x^2 + \frac{(16)}{3!}x^3 + \frac{(-96)}{4!}x^4 + \ldots$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots \left[\text{Note: this is } 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots\right]$$

c $f(x) = \sin^2 x$, $f(0) = 0$

$f'(x) = 2\sin x\cos x = \sin 2x$, $f'(0) = 0$

$f''(x) = 2\cos 2x$, $f''(0) = 2$

$f'''(x) = -4\sin 2x$, $f'''(0) = 0$

$f''''(x) = -8\cos 2x$, $f''''(0) = -8$

$$\text{So } f(x) = \sin^2 x = 0 + 0x + \frac{2}{2!}x^2 + 0x^3 + \frac{(-8)}{4!}x^4 + \ldots = x^2 - \frac{x^4}{3} + \ldots$$

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Exercise B, Question 6

Question:

Using the addition formula for cos $(A - B)$ and the series expansions of sin x and cos x, show that

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

Solution:

a $\cos\left(x - \frac{\pi}{4}\right) = \cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)$

Use $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

$$= \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

$$= \frac{1}{\sqrt{2}}\left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \ldots\right)$$

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Exercise B, Question 7

Question:

Given that $f(x) = (1-x)^2 \ln(1-x)$

a Show that $f''(x) = 3 + 2\ln(1-x)$.

b Find the values of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

c Express $(1-x)^2 \ln(1-x)$ in ascending powers of x up to and including the term in x^3.

Solution:

a $f(x) = (1-x)^2 \ln(1-x)$

$$f'(x) = (1-x)^2 \times \frac{(-1)}{1-x} + 2(1-x)(-1)\ln(1-x)$$

Use the product rule.

$$= x - 1 - 2(1-x)\ln(1-x)$$

$$f''(x) = 1 - 2\left[(1-x) \times \frac{(-1)}{1-x} - \ln(1-x)\right] = 1 + 2 + 2\ln(1-x) = 3 + 2\ln(1-x)$$

b $f'''(x) = \frac{-2}{1-x}$

Substituting $x = 0$ in all the results gives

$f(0) = 0,\ f'(0) = -1,\ f''(0) = 3,\ f'''(0) = -2$

c $f(x) = (1-x)^2 \ln(1-x) = 0 + (-1)x + \frac{3}{2!}x^2 + \frac{(-2)}{3!}x^3 + \ldots$

$$= -x + \frac{3x^2}{2} - \frac{1}{3}x^3$$

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Exercise B, Question 8

Question:

a Using the series expansions of $\sin x$ and $\cos x$, show that
$3\sin x - 4x\cos x + x = \frac{3}{2}x^3 - \frac{17}{120}x^5 + \ldots$

b Hence, find the limit, as $x \to 0$, of $\frac{3\sin x - 4x\cos x + x}{x^3}$.

Solution:

a Using the series expansions for $\sin x$ and $\cos x$ as far as the term in x^5,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots$$

so $3\sin x - 4x\cos x + x = 3\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \ldots\right) - 4x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \ldots\right) + x$

$$= 3x - \frac{1}{2}x^3 + \frac{1}{40}x^5 - 4x + 2x^3 - \frac{1}{6}x^5 + x + \ldots$$

$$3\sin x - 4x\cos x + x = \frac{3}{2}x^3 - \frac{17}{120}x^5 + \ldots$$

b $\frac{3\sin x - 4x\cos x + x}{x^3} = \frac{3}{2} - \frac{17}{120}x^2 +$ higher powers in x using **a**

Hence, the limit, as $x \to 0$, is $\frac{3}{2}$.

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Exercise B, Question 9

Question:

Given that $f(x) = \ln \cos x$,

a Show that $f'(x) = -\tan x$

b Find the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$.

c Express $\ln \cos x$ as a series in ascending powers of x up to and including the term in x^4.

d Show that, using the first two terms of the series for $\ln \cos x$, with $x = \frac{\pi}{4}$, gives a value for $\ln 2$ of $\frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$.

Solution:

a $f(x) = \ln \cos x \qquad \Rightarrow f(0) = 0$

$f'(x) = \frac{1}{\cos x} \times (-\sin x) \left[\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}\right] \qquad \Rightarrow f'(0) = 0$

$= -\tan x$

b $f''(x) = -\sec^2 x \qquad \Rightarrow f''(0) = -1$

$f'''(x) = -2\sec x(\sec x \tan x) = -2\sec^2 x \tan x \qquad \Rightarrow f'''(0) = 0$

$f''''(x) = -2\{\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)\} \qquad \Rightarrow f''''(0) = -2$

c Substituting into Maclaurin's expansion

$$\ln \cos x = 0 + 0x + \frac{(-1)}{2!}x^2 + 0x^3 + \frac{(-2)}{4!}x^4 + \ldots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$$

d Substituting $x = \frac{\pi}{4}$ gives $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\left(\frac{\pi^2}{16}\right) - \frac{1}{12}\left(\frac{\pi^4}{256}\right)$

but $\ln\left(\frac{1}{\sqrt{2}}\right) = \ln 2^{-\frac{1}{2}} = -\frac{1}{2}\ln 2$,

so $-\frac{1}{2}\ln 2 = -\frac{\pi^2}{2.16} - \frac{\pi^4}{12.256} + \ldots$

$\Rightarrow \ln 2 = \frac{\pi^2}{16} + \frac{\pi^4}{6.256}$, using only first two terms.

$= \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

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Exercise B, Question 10

Question:

Show that the Maclaurin series for $\tan x$, as far as the term in x^5, is $x + \frac{1}{3}x^3 + \frac{2}{15}x^5$.

Solution:

a $f(x) = \tan x \quad \Rightarrow f(0) = 0$

$f'(x) = \sec^2 x \quad \Rightarrow f'(0) = 1$

$f''(x) = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x \quad \Rightarrow f''(0) = 0$

$f'''(x) = 2[\sec^2 x(\sec^2 x) + \tan x(2\sec^2 x \tan x)] \quad \Rightarrow f'''(0) = 2$

$= 2(\sec^4 x + 2\sec^2 x \tan^2 x)$

$f''''(x) = 2(\{4\sec^3 x(\sec x \tan x)\} + 2\{\sec^2 x(2\tan x \sec^2 x) + \tan^2 x(2\sec^2 x \tan x)\} \quad \Rightarrow f''''(0) = 0$ as $\tan(0) = 0$

$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$= 8\sec^2 x \tan x(2\sec^2 x + \tan^2 x)$

$f'''''(x) = 8\sec^2 x \tan x(4\sec^2 x \tan x + 2\tan x \sec^2 x) + 8(\sec^4 x + 2\sec^2 x \tan^2 x)(2\sec^2 x + \tan^2 x)$

$\Rightarrow f'''''(0) = 16$ as $\tan(0) = 0$, $\sec(0) = 1$

Substitute into Maclaurin's expansion gives

$$\tan x = 0 + 1x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^3 + \frac{16}{5!}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

Use the series expansions of e^x, $\ln(1 + x)$ and $\sin x$ to expand the following functions as far as the fourth non-zero term. In each case state the interval in x for which the expansion is valid.

a $\frac{1}{e^x}$

b $\frac{e^{2x} \times e^{3x}}{e^x}$

c e^{1+x}

d $\ln(1 - x)$

e $\sin\left(\frac{x}{2}\right)$

f $\ln(2 + 3x)$

Solution:

a $\frac{1}{e^x} = e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} +$

$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$ valid for all values of x

b $\frac{e^{2x} \times e^{3x}}{e^x} = e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} +$

$= 1 + 4x + 8x^2 + \frac{32x^3}{3} + \ldots$ valid for all values of x

c $e^{1+x} = e \times e^x = e\left\{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right\}$ valid for all values of x

d $\ln(1 - x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + \frac{(-x)^4}{4} + \ldots$ $[-1 < -x \leqslant 1]$

$\Rightarrow 1 > x \geqslant -1$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} -$ $-1 \leqslant x < 1$

e $\sin\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right) - \frac{\left(\frac{x}{2}\right)^3}{3!} + \frac{\left(\frac{x}{2}\right)^5}{5!} - \frac{\left(\frac{x}{2}\right)^7}{7!} + \ldots$

$= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} - \frac{x^7}{645\,120} +$ valid for all values of x

f $\ln(2 + 3x) = \ln\left\{2\left(1 + \frac{3x}{2}\right)\right\} = \ln 2 + \ln\left(1 + \frac{3x}{2}\right)$

$= \ln 2 + \frac{3x}{2} - \frac{\left(\frac{3x}{2}\right)^2}{2} + \frac{\left(\frac{3x}{2}\right)^3}{3} +$ $\left[-1 < \frac{3x}{2} \leqslant 1\right]$

$= \ln 2 + \frac{3x}{2} - \frac{9x^2}{8} + \frac{9x^3}{8} + \ldots$ $-\frac{2}{3} < x \leqslant \frac{2}{3}$

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

a Using the Maclaurin expansion of $\ln(1+x)$, show that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right),\ -1 < x < 1.$$

b Deduce the series expansion for $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$, $-1 < x < 1$.

c By choosing a suitable value of x, and using only the first three terms of the series in **a**, find an approximation for $\ln\left(\frac{2}{3}\right)$, giving your answer to 4 decimal places.

d Show that the first three terms of your series in **b**, with $x = \frac{3}{5}$, gives an approximation for $\ln 2$, which is correct to 2 decimal places.

Solution:

a $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots, \qquad -1 < x \leqslant 1$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots, \qquad -1 \leqslant x < 1$

$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots\right)$

$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \dots$

$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$

As x must be in both the intervals $-1 < x \leqslant 1$ and $-1 \leqslant x < 1$ this expansion requires x to be in the interval $-1 < x < 1$.

b $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

so $\ln\sqrt{\left(\frac{1+x}{1-x}\right)} = \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right), -1 < x < 1.$

c Solving $\left(\frac{1+x}{1-x}\right) = \frac{2}{3}$ gives $3 + 3x = 2 - 2x$

$5x = -1$

$x = -0.2$

This is a valid value of x.

So an approximation to $\ln\left(\frac{2}{3}\right)$ is $2\left(-0.2 - \frac{0.008}{3} - \frac{0.00032}{5}\right)$

$= 2(-0.2 - 0.0026666 - 0.000064)$

$= -0.4055$ (4 d.p.)

This is accurate to 4 d.p.

d $\ln\sqrt{\left(\frac{1+x}{1-x}\right)}$ with $x = \frac{3}{5}$ gives $\ln\sqrt{4} = \ln 2$

so $\ln 2 \approx 0.6 + \frac{(0.6)^3}{3} + \frac{(0.6)^5}{5}$

Use the result in **b**.

$\approx 0.687552\dots = 0.69$ (2 d.p.)

[Using the series in **a** gives $\ln 2 = 0.7424\dots$]

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Exercise C, Question 3

Question:

Show that for small values of x, $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$.

Solution:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots$$

$$e^{-x} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \ldots = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$$

So $e^{2x} - e^{-x} \approx 3x + \frac{3}{2}x^2$, if terms x^3 and above may be neglected.

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Exercise C, Question 4

Question:

a Show that $3x \sin 2x - \cos 3x = -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 - \ldots$

b Hence find the limit, as $x \to 0$, of $\left(\frac{3x \sin 2x - \cos 3x + 1}{x^2}\right)$.

Solution:

a $3x \sin 2x = 3x\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} = 6x^2 - 4x^4 + \ldots$

$\cos 3x = \left\{1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \right\} = 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots$

So $3x \sin 2x - \cos 3x = 6x^2 - 4x^4 + \ldots - \left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \ldots\right)$

$$= -1 + \frac{21}{2}x^2 - \frac{59}{8}x^4 + \ldots$$

b $\frac{3x \sin 2x - \cos 3x + 1}{x^2} = \frac{21}{2} - \frac{59}{8}x^2 +$ terms in higher powers of x

As $x \to 0$, so $\frac{3x \sin 2x - \cos 3x + 1}{x^2}$ tends to $\frac{21}{2}$.

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Exercise C, Question 5

Question:

Find the series expansions, up to and including the term in x^4, of

a $\ln(1 + x - 2x^2)$

b $\ln(9 + 6x + x^2)$.

and in each case give the range of values of x for which the expansion is valid.

Solution:

a $\ln(1 + x - 2x^2) = \ln(1 - x)(1 + 2x) = \ln(1 - x) + \ln(1 + 2x)$

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \qquad -1 \leqslant x < 1$$

$$\ln(1 + 2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \ldots, \qquad -\tfrac{1}{2} < x \leqslant \tfrac{1}{2}$$

$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4$$

So $\ln(1 + x - 2x^2) = \ln(1 - x) + \ln(1 + 2x)$

$$= x - \frac{5x^2}{2} + \frac{7x^3}{3} - \frac{17x^4}{4} + \ldots, \qquad -\tfrac{1}{2} < x \leqslant \tfrac{1}{2} \text{ (smaller interval)}$$

b $\ln(9 + 6x + x^2) = \ln(3 + x)^2 = 2\ln(3 + x) = 2\ln 3\left(1 + \frac{x}{3}\right) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

The expansion of $\ln\left(1 + \frac{x}{3}\right)$ is $= \left(\frac{x}{3}\right) - \frac{\left(\frac{x}{3}\right)^2}{2} + \frac{\left(\frac{x}{3}\right)^3}{3} - \frac{\left(\frac{x}{3}\right)^4}{4} + \ldots, \qquad \left[-1 < \frac{x}{3} \leqslant 1\right]$

$$= \frac{x}{3} - \frac{x^2}{18} + \frac{x^3}{81} - \frac{x^4}{324} + \ldots, \qquad -3 < x \leqslant 3$$

So $\ln(9 + 6x + x^2) = 2\left\{\ln 3 + \ln\left(1 + \frac{x}{3}\right)\right\}$

$$= 2\ln 3 + \frac{2x}{3} - \frac{x^2}{9} + \frac{2x^3}{81} - \frac{x^4}{162} + \ldots, \qquad -3 < x \leqslant 3$$

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Exercise C, Question 6

Question:

a Write down the series expansion of $\cos 2x$ in ascending powers of x, up to and including the term in x^8.

b Hence, or otherwise, find the first 4 non-zero terms in the power series for $\sin^2 x$.

Solution:

a $\cos 2x = \left\{1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \ldots\right\}$

$= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \ldots$

b Using $\cos 2x = 1 - 2\sin^2 x$,

$2\sin^2 x = 1 - \cos 2x = 2x^2 - \frac{2x^4}{3} + \frac{4x^6}{45} - \frac{2x^8}{315} + \ldots$

So $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} - \frac{x^8}{315} + \ldots$

[Alternative: write out expansion of $\sin x$ as far as term in x^7, square it, and collect together appropriate terms!]

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Exercise C, Question 7

Question:

Show that the first two non-zero terms of the series expansion, in ascending powers of x, of $\ln(1 + x) + (x - 1)(e^x - 1)$ are px^3 and qx^4, where p and q are constants to be found.

Solution:

a $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$(x - 1)(e^x - 1) = (x - 1)\left(x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= x^2 + \frac{x^3}{2} + \frac{x^4}{6} \ldots - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

$$= -x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots$$

So $\ln(1 + x) + (x - 1)(e^x - 1) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + \left(-x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \ldots\right)$

$$= \frac{2x^3}{3} - \frac{x^4}{8} + \ldots \qquad \Rightarrow p = \frac{2}{3}, q = -\frac{1}{8}$$

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Exercise C, Question 8

Question:

a Expand $\frac{\sin x}{(1-x)^2}$ in ascending powers of x as far as the term in x^4, by considering the product of the expansions of $\sin x$ and $(1-x)^{-2}$.

b Deduce the gradient of the tangent, at the origin, to the curve with equation $y = \frac{\sin x}{(1-x)^2}$.

Solution:

a Only terms up to and including x^4 in the product are required, so using

$$\sin x = x - \frac{x^3}{3!} + \ldots \quad \text{(next term is } kx^5\text{)}$$

and the binomial expansion of $(1-x)^{-2}$, with terms up to and including x^3.
(It is not necessary to use the term in x^4, because it will be multiplied by expansion of $\sin x$.)

$$(1-x)^{-2} = 1 + (-2)(-x) + (-2)(-3)\frac{(-x)^2}{2!} + (-2)(-3)(-4)\frac{(-x)^3}{3!} + \ldots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

So $$\frac{\sin x}{(1-x)^2} = \left(x - \frac{x^3}{6} + \ldots\right)(1 + 2x + 3x^2 + 4x^3 + \ldots)$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \ldots - \left(\frac{x^3}{6} + \frac{x^4}{3} + \ldots\right)$$

$$= x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$$

b $$y = \frac{\sin x}{(1-x)^2} = x + 2x^2 + \frac{17x^3}{6} + \frac{11x^4}{3} + \ldots$$

So $\frac{dy}{dx} = 1 + 4x +$ higher powers of $x \Rightarrow$ at the origin the gradient of tangent $= 1$.

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Exercise C, Question 9

Question:

Using the series given on page 112, show that

a $(1 - 3x)\ln(1 + 2x) = 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \ldots$

b $e^{2x}\sin x = x + 2x^2 + \frac{11}{6}x^3 + x^4 + \ldots$

c $\sqrt{(1 + x^2)}e^{-x} = 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \ldots$

Solution:

a $(1 - 3x)\ln(1 + 2x) = (1 - 3x)\left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right)$ (see Q5a)

$$= \left(2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \ldots\right) - (6x^2 - 6x^3 + 8x^4 - \ldots)$$

$$= 2x - 8x^2 + \frac{26}{3}x^3 - 12x^4 + \ldots$$

b $e^{2x}\sin x = \left\{1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots\right\}\left\{x - \frac{x^3}{3!} + \ldots\right\}$ [only terms up to x^4]

$$= \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)$$

$$= \left(x + 2x^2 + 2x^3 + \frac{4x^4}{3}\right) + \left(-\frac{x^3}{6} - \frac{x^4}{3}\right) + \ldots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + x^4 + \ldots$$

c $\sqrt{(1 + x^2)}\, e^{-x} = (1 + x^2)^{\frac{1}{2}}e^{-x}$

$$= \left\{1 + \frac{1}{2}x^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(x^2)^2}{2!} + \ldots\right\}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$

$$= \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \ldots\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

$$= \left\{1 - x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{1}{4} - \frac{1}{8}\right)x^4 + \ldots\right\}$$

$$= 1 - x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 + \ldots$$

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Exercise C, Question 10

Question:

a Write down the first five non-zero terms in the series expansions of $e^{-\frac{x^2}{2}}$.

b Using your result in **a**, find an approximate value for $\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx$, giving your answer to 3 decimal places.

Solution:

a $e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \frac{\left(-\frac{x^2}{2}\right)^3}{3!} + \frac{\left(-\frac{x^2}{2}\right)^4}{4!} + \ldots$

$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \frac{x^8}{384} - \ldots$

b Area under the curve $= \int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx = 2\int_{0}^{1} e^{-\frac{x^2}{2}}\,dx$

$= 2\left[x - \frac{x^3}{6} + \frac{x^5}{40} - \frac{x^7}{336} + \frac{x^9}{3456} - \ldots\right]_0^1$ Integrate the result from **a**.

$\approx 2\left[1 - \frac{1}{6} + \frac{1}{40} - \frac{1}{336} + \frac{1}{3456}\right]$

≈ 1.711 (3 d.p.)

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Exercise C, Question 11

Question:

a Show that $e^{px}\sin 3x = 3x + 3px^2 + \frac{3(p^2-3)}{2}x^3 + \ldots$ where p is a constant.

b Given that the first non-zero term in the expansion, in ascending powers of x, of $e^{px}\sin 3x + \ln(1+qx) - x$ is kx^3, where k is a constant, find the values of p, q and k.

Solution:

a $e^{px}\sin 3x = \left\{1 + (px) + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \ldots\right\}\left\{(3x) - \frac{(3x)^3}{3!} + \ldots\right\}$

$$= \left(1 + px + \frac{p^2x^2}{2} + \frac{p^3x^3}{6} + \ldots\right)\left(3x - \frac{9x^3}{2} + \ldots\right)$$

$$= \left(3x + 3px^2 + \frac{3p^2x^3}{2} + \ldots\right) + \left(-\frac{9x^3}{2} + \ldots\right)$$

$$= 3x + 3px^2 + \frac{3(p^2-3)x^3}{2} + \ldots$$

b $\ln(1+qx) = \left\{(qx) - \frac{(qx)^2}{2} + \frac{(qx)^3}{3} - \ldots\right\}$

So $e^{px}\sin 3x + \ln(1+qx) - x = 3x + 3px^2 + \frac{3(p^2-3)x^3}{2} + qx - \frac{q^2x^2}{2} + \frac{q^3x^3}{3} - x + \ldots$

$$= (2+q)x + \left(3p - \frac{q^2}{2}\right)x^2 + \left(\frac{3p^2}{2} + \frac{q^3}{3} - \frac{9}{2}\right)x^3 + \ldots$$

Coefficient of x is zero, so $q = -2$.

Coefficient of x^2 is zero, so $3p - 2 = 0 \Rightarrow p = \frac{2}{3}$

Coefficient of $x^3 = \frac{2}{3} - \frac{8}{3} - \frac{9}{2} = -\frac{13}{2}$, so $k = -\frac{13}{2}$

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Exercise C, Question 12

Question:

$f(x) = e^{x - \ln x} \sin x, \qquad x > 0.$

a Show that if x is sufficiently small so that x^4 and higher powers of x may be neglected,
$f(x) \approx 1 + x + \frac{x^2}{3}$.

b Show that using $x = 0.1$ in the result in **a** gives an approximation for f(0.1) which is correct to 6 significant figures.

Solution:

a $e^{x - \ln x} = e^x \times e^{-\ln x} = e^x \times e^{\ln x^{-1}}$ Using $e^{a+b} = e^a \times e^b$

$= e^x \times x^{-1}$ using $e^{\ln k} = k$

$= \frac{e^x}{x}$

$e^{x - \ln x} \sin x = \frac{e^x \sin x}{x}$, and so, using the expansions of e^x and $\sin x$,

$$f(x) = e^{x - \ln x} \sin x = \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x - \frac{x^3}{6} + \ldots\right)}{x}, x > 0$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{x^2}{6} + \ldots\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(\frac{x^2}{6} + \frac{x^3}{6}\right)$$ ignoring terms in x^4 and above.

$$= 1 + x + \frac{x^2}{3}$$ There is no term in x^3.

b $f(0.1) = \frac{e^{0.1} \sin 0.1}{0.1} = 1.103329\ldots$

The result in **a** gives an approximation for f(0.1) of $1 + 0.1 + 0.00333333 = 1.103333\ldots$ which is corect to 6 s.f.

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Exercise D, Question 1

Question:

a Find that Taylor series expansion of $\sqrt{x}$ in ascending powers of $(x - 1)$ as far as the term in $(x - 1)^4$.

b Use your answer in **a** to obtain an estimate for $\sqrt{1.2}$, giving your answer to 3 decimal places.

Solution:

a $f(x) = \sqrt{x} = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \frac{f''''(a)}{4!}(x - a)^4 + \ldots$, where $a = 1$

$f(x) = \sqrt{x}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ $\qquad f''(1) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$ $\qquad f'''(1) = \frac{3}{8}$

$f''''(x) = -\frac{15}{16}x^{-\frac{7}{2}}$ $\qquad f''''(1) = -\frac{15}{16}$

So $\sqrt{x} = 1 + \frac{1}{2}(x - 1) - \frac{1}{4 \times 2!}(x - 1)^2 + \frac{3}{8 \times 3!}(x - 1)^3 - \frac{15}{16 \times 4!}(x - 1)^4 + \ldots$

$= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3 - \frac{5}{128}(x - 1)^4 + \ldots$

b $\sqrt{1.2} \approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - \frac{5}{128}(0.2)^4$

$\approx 1 + 0.1 - 0.005 + 0.0005 - 0.0000625$

$= 1.095$ (3 d.p.)

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Exercise D, Question 2

Question:

Use Taylor's expansion to express each of the following as a series in ascending powers of $(x - a)$ as far as the term in $(x - a)^k$, for the given values of a and k.

a $\ln x$ $(a = \mathrm{e}, k = 2)$ **b** $\tan x$ $\left(a = \frac{\pi}{3}, k = 3\right)$ **c** $\cos x$ $(a = 1, k = 4)$

Solution:

All solutions use the Taylor expansion in the form:

$$\mathbf{f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots + \frac{f^{(r)}(a)}{r!}(x - a)^r + \ldots,}$$

a Let $f(x) = \ln x$ then $f(a) = f(e) = \ln e = 1$

$f'(x) = \frac{1}{x}$ $\quad f'(a) = f'(e) = \frac{1}{e}$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(a) = f''(e) = -\frac{1}{e^2}$

$$\text{So } f(x) = \ln x = 1 + \frac{1}{e}(x - e) + \frac{\left(-\frac{1}{e^2}\right)}{2!}(x - e)^2 + \ldots$$

$$= 1 + \frac{(x - e)}{e} - \frac{(x - e)^2}{2e^2} + \ldots$$

b Let $f(x) = \tan x$ then $f(a) = f\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$ $\quad f'(a) = f'\left(\frac{\pi}{3}\right) = 4$

$f''(x) = 2\sec^2 x \tan x$ $\quad f''(a) = f''\left(\frac{\pi}{3}\right) = 2(4)(\sqrt{3}) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 2\tan x(2\sec^2 x \tan x)$ $\quad f'''(a) = f'''\left(\frac{\pi}{3}\right) = 2(16) + 4(4)(3) = 80$

$$\text{So } \quad f(x) = \tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{80}{3!}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

$$= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \ldots$$

c Let $f(x) = \cos x$ then $f(a) = f(1) = \cos 1$

$f'(x) = -\sin x$ $\quad f'(a) = f'(1) = -\sin 1$

$f''(x) = -\cos x$ $\quad f''(a) = f''(1) = -\cos 1$

$f'''(x) = \sin x$ $\quad f'''(a) = f'''(1) = \sin 1$

$f''''(x) = \cos x$ $\quad f''''(a) = f''''(1) = \cos 1$

$$\text{So } \quad f(x) = \cos x = \cos 1 - \sin 1\,(x - 1) - \frac{(\cos 1)}{2}(x - 1)^2 + \frac{(\sin 1)}{6}(x - 1)^3 + \frac{(\cos 1)}{24}(x - 1)^4 + \ldots$$

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Exercise D, Question 3

Question:

a Use Taylor's expansion to express each of the following as a series in ascending powers of x as far as the term in x^4.

i $\cos\left(x + \frac{\pi}{4}\right)$ **ii** $\ln(x + 5)$ **iii** $\sin\left(x - \frac{\pi}{3}\right)$

b Use your result in **ii** to find an approximation for $\ln 5.2$, giving your answer to 6 significant figures.

Solution:

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Exercise D, Question 4

Question:

Given that $y = xe^x$,

a Show that $\frac{d^n y}{dx^n} = (n + x)e^x$.

b Find the Taylor expansion of xe^x in ascending powers of $(x + 1)$ up to and including the term in $(x + 1)^4$.

Solution:

a $y = xe^x$, $\frac{dy}{dx} = xe^x + e^x = e^x(x + 1)$ — Product rule.

$$\frac{d^2y}{dx^2} = xe^x + e^x + e^x = e^x(x + 2)$$

$$\frac{d^3y}{dx^3} = xe^x + 2e^x + e^x = e^x(x + 3)$$

Each differentiation adds another e^x, so $\frac{d^n y}{dx^n} = (n + x)e^x$.

So for $f(x) = xe^x$, $f^{(n)}(x) = (n + x)e^x$.

b Using the Taylor series with $a = -1$, $f(-1) = -e^{-1}$, $f'(-1) = 0$, $f''(-1) = e^{-1}$
$f'''(-1) = 2e^{-1}$, $f''''(-1) = 3e^{-1}$

So $$xe^x = e^{-1}\left\{-1 + 0(x + 1) + \frac{1}{2!}(x + 1)^2 + \frac{2}{3!}(x + 1)^3 + \frac{3}{4!}(x + 1)^4 + \ldots\right\}$$

$$= e^{-1}\left\{-1 + \frac{1}{2}(x + 1)^2 + \frac{1}{3}(x + 1)^3 + \frac{1}{8}(x + 1)^4 + \ldots\right\}$$

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Exercise D, Question 5

Question:

a Find the Taylor series for $x^3 \ln x$ in ascending powers of $(x-1)$ up to and including the term in $(x-1)^4$.

b Using your series in **a**, find an approximation for ln 1.5, giving your answer to 4 decimal places.

Solution:

a Let $f(x) = x^3 \ln x$ then as $a = 1$ $f(a) = f(1) = 0$

$f'(x) = 3x^2 \ln x + x^3 \times \frac{1}{x} = x^2(1 + 3\ln x)$ $f'(a) = f'(1) = 1$

$f''(x) = x^2 \times \frac{3}{x} + 2x(1 + 3\ln x) = x(5 + 6\ln x)$ $f''(a) = f''(1) = 5$

$f'''(x) = x \times \frac{6}{x} + (5 + 6\ln x) = 11 + 6\ln x$ $f'''(a) = f'''(1) = 11$

$f''''(x) = \frac{6}{x}$ $f''''(a) = f''''(1) = 6$

Using Taylor, form **ii**

$$f(x) = x^3 \ln x = 0 + 1(x-1) + \frac{5}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 + \frac{6}{4!}(x-1)^4 + \ldots$$

$$= (x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 + \ldots$$

b Substituting $x = 1.5$ in series in **a**, gives

$$\frac{27}{8}\ln 1.5 \approx 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^3 + \frac{1}{4}(0.5)^4 + \ldots$$

$$\approx 0.5 + 0.625 + 0.22916\ldots + 0.015625 \; (= 1.369791\ldots)$$

So this gives an approximation for ln 1.5 of $\frac{8}{27}(1.369791\ldots) = 0.4059$ (4 d.p.)

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Exercise D, Question 6

Question:

Find the Taylor expansion of $\tan(x - \alpha)$, where $\alpha = \arctan\left(\frac{3}{4}\right)$, in ascending powers of x up to and including the term in x^2.

Solution:

Let $f(x + a) = \tan(x - \alpha)$, so that $f(x) = \tan x$ and $a = -\alpha$

As $\alpha = \arctan\left(\frac{3}{4}\right)$, $\tan\alpha = \frac{3}{4}$ and $\cos\alpha = \frac{4}{5}$

$f(x) = \tan x$ $\qquad f(a) = f(-\alpha) = \tan(-\alpha) = -\frac{3}{4}$

$f'(x) = \sec^2 x$ $\qquad f'(a) = f'(-\alpha) = \frac{25}{16}$

$f''(x) = 2\sec^2 x \tan x$ $\qquad f''(a) = f''(-\alpha) = 2\left(\frac{25}{16}\right)\left(-\frac{3}{4}\right) = -\left(\frac{75}{32}\right)$

Using the form **ii** of the Taylor expansion gives

$$f(x + a) = \tan\left(x - \arctan\left(\frac{3}{4}\right)\right) = -\frac{3}{4} + \frac{25}{16}x + \frac{\left(-\frac{75}{32}\right)}{2!}x^2 + \ldots$$

$$= -\frac{3}{4} + \frac{25}{16}x - \frac{75}{64}x^2 + \ldots$$

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Exercise D, Question 7

Question:

Find the Taylor expansion of $\sin 2x$ in ascending powers of $\left(x - \frac{\pi}{6}\right)$ up to and including the term in $\left(x - \frac{\pi}{6}\right)^4$.

Solution:

a $f(x) = \sin 2x$ and $a = \frac{\pi}{6}$

$f(x) = \sin 2x$ $\quad f(a) = f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f'(x) = 2\cos 2x$ $\quad f'(a) = f'\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{3}\right) = 1$

$f''(x) = -4\sin 2x$ $\quad f''(a) = f''\left(\frac{\pi}{6}\right) = -4\sin\left(\frac{\pi}{3}\right) = -2\sqrt{3}$

$f'''(x) = -8\cos 2x$ $\quad f'''(a) = f'''\left(\frac{\pi}{6}\right) = -8\cos\left(\frac{\pi}{3}\right) = -4$

$f''''(x) = +16\sin 2x$ $\quad f''''(a) = f''''\left(\frac{\pi}{6}\right) = 16\sin\left(\frac{\pi}{3}\right) = 8\sqrt{3}$

$$\text{So } f(x) = \sin 2x = \frac{\sqrt{3}}{2} + 1\left(x - \frac{\pi}{6}\right) + \frac{(-2\sqrt{3})}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{(-4)}{3!}\left(x - \frac{\pi}{6}\right)^3 + \frac{(8\sqrt{3})}{4!}\left(x - \frac{\pi}{6}\right)^4 + \ldots$$

$$= \frac{\sqrt{3}}{2} + 1\left(x - \frac{\pi}{6}\right) - \sqrt{3}\left(x - \frac{\pi}{6}\right)^2 - \frac{2}{3}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^4 + \ldots$$

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Exercise D, Question 8

Question:

Given that $y = \frac{1}{\sqrt{(1+x)}}$,

a find the values of $\left(\frac{dy}{dx}\right)_3$ and $\left(\frac{d^2y}{dx^2}\right)_3$.

b Find the Taylor expansion of $\frac{1}{\sqrt{(1+x)}}$, in ascending powers of $(x-3)$ up to and including the the term in $(x-3)^2$.

Solution:

a Given $y = \frac{1}{\sqrt{(1+x)}} = (1+x)^{-\frac{1}{2}}$ $\qquad$ $y_3(=\text{value of } y \text{ when } x = 3) = \frac{1}{2}$

$$\frac{dy}{dx} = -\frac{1}{2}(1+x)^{-\frac{3}{2}} \qquad \left(\frac{dy}{dx}\right)_3 = -\frac{1}{2} \times \frac{1}{8} = -\frac{1}{16}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4}(1+x)^{-\frac{5}{2}} \qquad \left(\frac{d^2y}{dx^2}\right)_3 = \frac{3}{4} \times \frac{1}{32} = \frac{3}{128}$$

b So using

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \ldots \qquad \text{with } f^{(n)}(3) \equiv \left(\frac{d^ny}{dx^n}\right)_3$$

$$y = \frac{1}{\sqrt{(1+x)}} = \frac{1}{2} - \frac{1}{16}(x-3) + \frac{3}{256}(x-3)^2 + \ldots$$

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Exercise E, Question 1

Question:

Find a series solution, in ascending powers of x up to and including the term in x^4, for the differential equation $\frac{d^2y}{dx^2} = x + 2y$, given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = \frac{1}{2}$.

Solution:

Differentiating $\frac{d^2y}{dx^2} = x + 2y$, with respect to x, gives $\frac{d^3y}{dx^3} = 1 + 2\frac{dy}{dx}$ ①

Differentiating ① gives $\frac{d^4y}{dx^4} = 2\frac{d^2y}{dx^2}$ ②

Substituting $x_0 = 0$, $y_0 = 1$ into $\frac{d^2y}{dx^2} = x + 2y$, gives

$$\left(\frac{d^2y}{dx^2}\right)_0 = 0 + 2(1), \text{ so } \left(\frac{d^2y}{dx^2}\right)_0 = 2$$

Substituting $\left(\frac{dy}{dx}\right)_0 = \frac{1}{2}$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0 = 1 + 2\left(\frac{1}{2}\right) = 2$

Substituting $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives $\left(\frac{d^4y}{dx^4}\right)_0 = 2(2) = 4$

So using the Taylor expansion in the form where $x_0 = 0$, i.e. **ii**

$$y = 1 + \left(\frac{1}{2}\right)x + \frac{(2)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(4)}{4!}x^4 + \ldots = 1 + \frac{x}{2} + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \ldots$$

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Exercise E, Question 2

Question:

The variable y satisfies $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$ and at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 1$.
Use Taylor's method to find a series expansion for y in powers of x up to and including the term in x^3.

Solution:

Differentiating $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$, gives

$$(1+x^2)\frac{dy^3}{dx^3} + 2x\frac{d^2y}{dx^2} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \quad ① \qquad \text{i.e. } (1+x^2)\frac{dy^3}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Substituting $x = 0$ and $\left(\frac{dy}{dx}\right)_0 = 1$ into $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$, gives $\left(\frac{d^2y}{dx^2}\right)_0 = 0$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = 0$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

So using the Taylor expansion in the form **ii**,

$$y = 0 + 1x + \frac{(0)}{2!}x^2 + \frac{(-1)}{3!}x^3 + \ldots = x - \frac{x^3}{6} + \ldots$$

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Exercise E, Question 3

Question:

Given that y satisfies the differential equation $\frac{dy}{dx} + y - e^x = 0$, and that $y = 2$ at $x = 0$, find a series solution for y in ascending powers of x up to and including the term in x^3.

Solution:

Differentiating $\frac{dy}{dx} + y - e^x = 0$, gives $\frac{d^2y}{dx^2} + \frac{dy}{dx} - e^x = 0$ ①

Differentiating ① gives $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - e^x = 0$ ②

Substituting $x_0 = 0$ and $y_0 = 2$ into $\frac{dy}{dx} + y - e^x = 0$, gives $\left(\frac{dy}{dx}\right)_0 + 2 - 1 = 0$, so $\left(\frac{dy}{dx}\right)_0 = -1$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = -1$ into ① gives $\left(\frac{d^2y}{dx^2}\right)_0 + (-1) - (1) = 0$ so $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting $x = 0$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives $\left(\frac{d^3y}{dx^3}\right)_0 + (2) - (1) = 0$ so $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

Substituting into the Taylor series with $x_0 = 0$, gives

$$y = 2 + (-1)x + \frac{(2)}{2!}x^2 + \frac{(-1)}{3!}x^3 + \dots$$

$$= 2 - x + x^2 - \frac{x^3}{6} \dots$$

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Exercise E, Question 4

Question:

Use the Taylor method to find a series solution for
$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, given that $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$,
giving your answer in ascending powers of x up to and including the term in x^4.

Solution:

Differentiating $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ with respect to x gives

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0 \quad ①, \qquad \text{i.e. } \frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

Differentiating ① gives

$$\frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{d^2y}{dx^2} = 0 \quad ②, \qquad \text{i.e. } \frac{d^4y}{dx^4} + x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 0$$

Substituting $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$ into $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 0(2) + 1 = 0 \Rightarrow \left(\frac{d^2y}{dx^2}\right)_0 = -1$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -1$ into ① gives

$$\left(\frac{d^3y}{dx^3}\right)_0 + 0(-1) + 2(2) = 0, \text{ so } \left(\frac{d^3y}{dx^3}\right)_0 = -4$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 2$, $\left(\frac{d^2y}{dx^2}\right)_0 = -1$ and $\left(\frac{d^3y}{dx^3}\right)_0 = -4$ into ② gives

$$\left(\frac{d^4y}{dx^4}\right)_0 + 0(-4) + 3(-1) = 0, \text{ so } \left(\frac{d^4y}{dx^4}\right)_0 = 3$$

Substituting into the Taylor series with form **ii**, gives

$$y = 1 + 2x + \frac{(-1)}{2!}x^2 + \frac{(-4)}{3!}x^3 + \frac{(3)}{4!}x^4 + \ldots$$

$$= 1 + 2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{8}x^4 + \ldots$$

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Exercise E, Question 5

Question:

The variable y satisfies the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$, and $y = 1$ and $\frac{dy}{dx} = -1$ at $x = 1$.

Express y as a series in powers of $(x - 1)$ up to and including the term in $(x - 1)^3$.

Solution:

Differentiating $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$ gives $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 3x\frac{dy}{dx} + 3y$ ①

Substituting $x_0 = 1$, $y_0 = 1$ and $\left(\frac{dy}{dx}\right)_1 = -1$ into $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 3xy$ gives $\left(\frac{d^2y}{dx^2}\right)_1 = 5$

Substituting $x_0 = 1$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_1 = -1$ and $\left(\frac{d^2y}{dx^2}\right)_1 = 5$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_1 = -10$

Substituting into the form of the Taylor series form **i**, with $x_0 = 1$, gives

$$y = 1 + (-1)(x-1) + \frac{(5)}{2!}(x-1)^2 + \frac{(-10)}{3!}(x-1)^3 + \ldots$$

$$= 1 - (x-1) + \frac{5}{2}(x-1)^2 - \frac{5}{3}(x-1)^3 + \ldots$$

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Exercise E, Question 6

Question:

Find a series solution, in ascending powers of x up to and including the term x^4, to the differential equation $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$, given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$.

Solution:

Differentiating $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$, twice with respect to x, gives

$$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 3y^2\frac{dy}{dx} = 1 \qquad ①$$

$$\frac{d^4y}{dx^4} + 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 = 0 \qquad ②$$

Substituting $x = 0$, $y = 1$ and $\frac{dy}{dx} = 1$ into $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y^3 = 1 + x$ gives $\left(\frac{d^2y}{dx^2}\right)_0 = -2$

Substituting $y = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ into ① gives $\left(\frac{d^3y}{dx^3}\right)_0 = 0$

Substituting $y = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = -2$, $\left(\frac{d^3y}{dx^3}\right)_0 = 0$ into ② gives $\left(\frac{d^4y}{dx^4}\right)_0 = 12$

So, using the Taylor series form **ii**, $y = 1 + 1x + \frac{(-2)}{2!}x^2 + \frac{(0)}{3!}x^3 + \frac{(12)}{4!}x^4 + \ldots$

so $y = 1 + x - x^2 + \frac{1}{2}x^4 + \ldots$

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Exercise E, Question 7

Question:

$(1 + 2x)\frac{dy}{dx} = x + 2y^2$

a Show that $(1 + 2x)\frac{d^3y}{dx^3} + 4(1 - y)\frac{d^2y}{dx^2} = 4\left(\frac{dy}{dx}\right)^2$

b Given that $y = 1$ at $x = 0$, find a series solution of $(1 + 2x)\frac{dy}{dx} = x + 2y^2$, in ascending powers of x up to and including the term in x^3.

Solution:

a Differentiating $(1 + 2x)\frac{dy}{dx} = x + 2y^2$ with respect to x

$$\left\{(1 + 2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right\} = 1 + 4y\frac{dy}{dx} \qquad ①$$

Differentiating ① gives

$$\left\{(1 + 2x)\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\right\} + \left\{2\frac{d^2y}{dx^2}\right\} = \left\{4y\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow (1 + 2x)\frac{d^3y}{dx^3} + 4(1 - y)\frac{d^2y}{dx^2} = 4\left(\frac{dy}{dx}\right)^2 \qquad ②$$

b Substituting $x_0 = 0$ and $y_0 = 1$ into $(1 + 2x)\frac{dy}{dx} = x + 2y^2$ gives $\left(\frac{dy}{dx}\right)_0 = 2(1) = 2$

Substituting known values into ① gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 2(2) = 1 + 4(1)(2) \Rightarrow \left(\frac{d^2y}{dx^2}\right)_0 = 5$$

Substituting known values into ② gives $\left(\frac{d^3y}{dx^3}\right)_0 = 4(2)^2 = 16$

So using $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \dots$

$$y = 1 + 2x + \frac{5}{2!}x^2 + \frac{16}{3!}x^3 + \dots = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \dots$$

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Exercise E, Question 8

Question:

Find the series solution in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^2$ for the differential equation $\sin x \frac{dy}{dx} + y \cos x = y^2$ given that $y = \sqrt{2}$ at $x = \frac{\pi}{4}$.

Solution:

Differentiating $\sin x \frac{dy}{dx} + y \cos x = y^2$ with respect to x, gives

$$\left(\sin x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx}\right) + \left(-y \sin x + \cos x \frac{dy}{dx}\right) = 2y \frac{dy}{dx} \quad ①$$

or $\sin x \frac{d^2y}{dx^2} + 2\cos x \frac{dy}{dx} - y \sin x = 2y \frac{dy}{dx}$

Substituting $x_0 = \frac{\pi}{4}$, $y_0 = \sqrt{2}$ into $\sin x \frac{dy}{dx} + y \cos x = y^2$ gives $\frac{1}{\sqrt{2}}\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} + \sqrt{2} \times \frac{1}{\sqrt{2}} = 2$

so $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$

Substituting $x_0 = \frac{\pi}{4}$, $y_0 = \sqrt{2}$, $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$ into ① gives

$$\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) - (\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = 2(\sqrt{2})(\sqrt{2})\right\}$$

So $\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2 - 1 = 4\right\} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 3\sqrt{2}$

Substituting all values into $y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \dots$

gives the series solution $y = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \dots$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 9

Question:

The variable y satisfies the differential equation $\frac{dy}{dx} - x^2 - y^2 = 0$.

a Show that

i $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - 2x = 0$, **ii** $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 2$.

b Derive a similar equation involving $\frac{d^4y}{dx^4}, \frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}$, and y.

c Given also that at $x = 0$, $y = 1$, express y as a series in ascending powers of x in powers of x up to and including the term in x^4.

Solution:

a i Differentiating $\frac{dy}{dx} - x^2 - y^2 = 0$ with respect to x, gives $\frac{d^2y}{dx^2} - 2y\frac{dy}{dx} - 2x = 0$ ①

ii Differentiating ① gives $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - 2 = 0$

So $\frac{d^3y}{dx^3} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 = 2$ ②

b Differentiating ② gives $\frac{d^4y}{dx^4} - 2y\frac{d^3y}{dx^3} - 2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$

so $\frac{d^4y}{dx^4} - 2y\frac{d^3y}{dx^3} - 6\frac{dy}{dx} \times \frac{d^2y}{dx^2} = 0$ ③

c Substituting $x_0 = 0$, $y_0 = 1$, into $\frac{dy}{dx} - x^2 - y^2 = 0$ gives

$\left(\frac{dy}{dx}\right)_0 - 0 - 1 = 0$, so $\left(\frac{dy}{dx}\right)_0 = 1$

Substituting $x_0 = 0$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ into ① gives

$\left(\frac{d^2y}{dx^2}\right)_0 - 2(1)(1) - 2(0) = 0$, so $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ into ② gives

$\left(\frac{d^3y}{dx^3}\right)_0 - 2(1)(2) - 2(1)^2 = 2$, so $\left(\frac{d^3y}{dx^3}\right)_0 = 8$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = 2$ and $\left(\frac{d^3y}{dx^3}\right)_0 = 8$ into ③ gives

$\left(\frac{d^4y}{dx^4}\right)_0 - 2(1)(8) - 6(1)(2) = 0$, so $\left(\frac{d^4y}{dx^4}\right)_0 = 28$

Substituting these values into the form of Taylor's series form **ii**, gives

$$y = 1 + (1)x + \frac{(2)}{2!}x^2 + \frac{(8)}{3!}x^3 + \frac{(28)}{4!}x^4 + \ldots = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$$

Solutionbank FP2
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 10

Question:

Given that $\cos x \frac{dy}{dx} + y \sin x + 2y^3 = 0$, and that $y = 1$ at $x = 0$, use Taylor's method to show that, close to $x = 0$, so that terms in x^4 and higher power can be ignored,
$y \approx 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3$.

Solution:

Differentiating $\cos x \frac{dy}{dx} + y \sin x + 2y^3 = 0$, ① with respect to x, gives

$$\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} + y \cos x + \sin x \frac{dy}{dx} + 6y^2 \frac{dy}{dx} = 0, \qquad ②$$

Differentiating again

$$\cos x \frac{d^3y}{dx^3} - \sin x \frac{d^2y}{dx^2} - y \sin x + \cos x \frac{dy}{dx} + 6y^2 \frac{d^2y}{dx^2} + 12y\left(\frac{dy}{dx}\right)^2 = 0, \qquad ③$$

Substituting $x_0 = 0$, $y_0 = 1$ into ① gives $\left(\frac{dy}{dx}\right)_0 + 2(1) = 0$, so $\left(\frac{dy}{dx}\right)_0 = -2$

Substituting $x_0 = 0$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = -2$ into ② gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 1 + 6(1)(-2) = 0, \text{ so } \left(\frac{d^2y}{dx^2}\right)_0 = 11$$

Substituting $x = 0$, $y = 1$, $\left(\frac{dy}{dx}\right)_0 = -2$, $\left(\frac{d^2y}{dx^2}\right)_0 = 11$ into ③ gives

$$\left(\frac{d^3y}{dx^3}\right)_0 + (1)(-2) + 6(1)(11) + 12(1)(-2)^2, \text{ so } \left(\frac{d^3y}{dx^3}\right)_0 = -112$$

Substituting these values into the form of Taylor's series form **ii**,

$$\text{gives } y = 1 + (-2)x + \frac{11}{2!}x^2 + \frac{(-112)}{3!}x^3 + \ldots$$

$$y = 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3 + \ldots$$

Ignoring terms in x^4 and higher powers, $y \approx 1 - 2x + \frac{11}{2}x^2 - \frac{56}{3}x^3$.

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Exercise F, Question 1

Question:

Using Taylor's series show that the first three terms in the expansion of $\left(x - \frac{\pi}{4}\right)\cot x$, in powers of $\left(x - \frac{\pi}{4}\right)$, are $\left(x - \frac{\pi}{4}\right) - 2\left(x - \frac{\pi}{4}\right)^2 + 2\left(x - \frac{\pi}{4}\right)^3$.

Solution:

$f(x) = \cot x$ and $a = \frac{\pi}{4}$.

$f(x) = \cot x$ so $f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = -\text{cosec}^2 x$ $f'\left(\frac{\pi}{4}\right) = -2$

$f''(x) = -2\,\text{cosec}\,x(-\text{cosec}\,x\cot x)$

$= 2\,\text{cosec}\,2x\cot x$ $f''\left(\frac{\pi}{4}\right) = 4$

Substituting in the form of Taylor

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots$$

$$\cot x = 1 + (-2)\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \ldots$$

So $\left(x - \frac{\pi}{4}\right)\cot x = \left(x - \frac{\pi}{4}\right) - 2\left(x - \frac{\pi}{4}\right)^2 + 2\left(x - \frac{\pi}{4}\right)^3 + \ldots$

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Exercise F, Question 2

Question:

a For the functions $f(x) = \ln(1 + e^x)$, find the values of $f'(0)$ and $f''(0)$.

b Show that $f'''(0) = 0$.

c Find the series expansion of $\ln(1 + e^x)$, in ascending powers of x up to and including the term in x^2, and state the range of values of x for which the expansion is valid.

Solution:

a $f(x) = \ln(1 + e^x)$ so $f(0) = \ln 2$

$f'(x) = \dfrac{e^x}{1 + e^x}$ $\quad f'(0) = \dfrac{1}{2}$

$= 1 - \dfrac{1}{1 + e^x} = 1 - (1 + e^x)^{-1}$

So $f''(x) = \dfrac{e^x}{(1 + e^x)^2}$ or use the quotient rule $\quad f''(0) = \dfrac{1}{4}$

b $f'''(x) = \dfrac{(1 + e^x)^2 e^x - e^x 2(1 + e^x)e^x}{(1 + e^x)^4}$

Use the quotient rule and chain rule.

$= \dfrac{(1 + e^x)e^x\{(1 + e^x) - 2e^x\}}{(1 + e^x)^4} = \dfrac{e^x(1 - e^x)}{(1 + e^x)^3}$ $\quad f'''(0) = 0$

c Using Maclaurin's expansion:

$$\ln(1 + e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$$

The expansion is valid for $-1 < e^x \leqslant 1 \Rightarrow 0 , e^x \leqslant 1$ so for $x \leqslant 0$.

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Exercise F, Question 3

Question:

a Write down the series for $\cos 4x$ in ascending powers of x, up to and including the term in x^6.

b Hence, or otherwise, show that the first three non-zero terms in the series expansion of $\sin^2 2x$ are $4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6$.

Solution:

a $\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \ldots$

$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \ldots$

b $\cos 4x = 1 - 2\sin^2 2x$,

so $2\sin^2 2x = 1 - \cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \ldots$

$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \ldots$

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Exercise F, Question 4

Question:

Given that terms in x^5 and higher power may be neglected, use the series for e^x and $\cos x$, to show that $e^{\cos x} \approx e\left(1 - \frac{x^2}{2} + \frac{x^4}{6}\right)$.

Solution:

Using $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$

$$e^{\cos x} = e^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = e \times e^{-\frac{x^2}{2}} \times e^{\frac{x^4}{24}}$$

$$= e\left\{1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\} \quad \text{no other terms required}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$$

$$= e\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \ldots\right\} = e\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + \ldots\right\}$$

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Exercise F, Question 5

Question:

$\frac{dy}{dx} = 2 + x + \sin y$ with $y = 0$ at $x = 0$.

Use the Taylor series method to obtain y as a series in ascending powers of x up to and including the term in x^3, and hence obtain an approximate value for y at $x = 0.1$.

Solution:

$\frac{dy}{dx} = 2 + x + \sin y$ and $x_0 = 0, y_0 = 0$ ① so $\left(\frac{dy}{dx}\right)_0 = 2$

Differentiating ① gives $\frac{d^2y}{dx^2} = 1 + \cos y \frac{dy}{dx}$ ②

Substituting $x_0 = 0, y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2$ into ② gives $\left(\frac{d^2y}{dx^2}\right)_0 = 3$

Differentiating ② gives $\frac{d^3y}{dx^3} = \cos y \frac{d^2y}{dx^2} - \sin y \left(\frac{dy}{dx}\right)^2$ ③

Substituting $y_0 = 0, \left(\frac{dy}{dx}\right)_0 = 2, \left(\frac{d^2y}{dx^2}\right)_0 = 3$ into ③ gives $\left(\frac{d^3y}{dx^3}\right)_0 = 3$

Substituting found values into $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 2x + \tfrac{3}{2}x^2 + \tfrac{1}{2}x^3 + \ldots$$

At $x = 0.1$, $y \approx 2(0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 = 0.2155$

Solutionbank FP2

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Exercise F, Question 6

Question:

Given that $|2x| < 1$, find the first two non-zero terms in the expansion of $\ln[(1 + x)^2(1 - 2x)]$ in a series of ascending powers of x.

Solution:

$$\ln[(1 + x)^2(1 - 2x)] = 2\ln(1 + x) + \ln(1 - 2x)$$

$$= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots\right\}$$

$$= 2x - x^2 + \tfrac{2}{3}x^3 - \tfrac{1}{2}x^4 - 2x - 2x^2 - \tfrac{8}{3}x^3 - 4x^4 + \ldots$$

$$= -3x^2 - 2x^3 - \ldots$$

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Exercise F, Question 7

Question:

Find the solution, in ascending powers of x up to and including the term in x^3, of the differential equation $\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} + 3y = 0$, given that at $x = 0$, $y = 2$ and $\frac{dy}{dx} = 4$.

Solution:

$$\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} + 3y = 0 \qquad ①$$

Differentiating ① gives $\frac{d^3y}{dx^2} - (x + 2)\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\frac{dy}{dx} = 0 \qquad ②$

Substituting initial data in ① gives $\left(\frac{d^2y}{dx^2}\right)_0 = 2$

Substituting known data in ② gives $\left(\frac{d^3y}{dx^3}\right)_0 = -4$

So $y = 2 + 4x + \frac{2x^2}{2!} - \frac{4x^3}{3!} + \ldots$

$= 2 + 4x + x^2 - \frac{2}{3}x^3$

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Exercise F, Question 8

Question:

Use differentiation and the Maclaurin expansion, to express $\ln(\sec x + \tan x)$ as a series in ascending powers of x up to and including the term in x^3.

Solution:

$f(x) = \ln(\sec x + \tan x)$ $\qquad f(0) = \ln 1 = 0$

$f'(x) = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \dfrac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \sec x$ $\qquad f'(0) = 1$

$f''(x) = \sec x \tan x$ $\qquad f''(0) = 0$

$f'''(x) = \sec x \sec^2 x + \sec x \tan x \tan x$ $\qquad f'''(0) = 1$

Substituting into Maclaurin's expansion gives $y = x + \dfrac{x^3}{6} + \ldots$

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Exercise F, Question 9

Question:

Show that the results of differentiating the following series expansions

$$e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots,$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots$$

agree with the results

a $\frac{d}{dx}(e^x) = e^x$ **b** $\frac{d}{dx}(\sin x) = \cos x$ **c** $\frac{d}{dx}(\cos x) = -\sin x$

Solution:

a $\frac{d}{dx}(e^x) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^r}{r!} + \frac{x^{r+1}}{(r+1)!} + \ldots\right)$

$$= 1 + x + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \ldots + \frac{(r+1)x^r}{(r+1)!} + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$$

$$= e^x$$

b $\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots\right)$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \ldots + (-1)^r\frac{(2r+1)x^{2r}}{(2r+1)!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots = \cos x$$

c $\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + (-1)^{r+1}\frac{x^{2r+2}}{(2r+2)!} + \ldots\right)$

$$= \left(-\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \ldots + (-1)^r\frac{2rx^{2r-1}}{(2r)!} + (-1)^{r+1}\frac{(2r+2)x^{2r+1}}{(2r+2)!} + \ldots\right)$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots + (-1)^{r+1}\frac{x^{2r+1}}{(2r+1)!} + \ldots$$

$$= -\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots + \frac{(-1)^r}{(2r+1)!}x^{2r+1} + \ldots\right) = -\sin x$$

Solutionbank FP2

Edexcel AS and A Level Modular Mathematics

Exercise F, Question 10

Question:

$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$, at $x = 1$, $y = 0$, $\frac{dy}{dx} = 2$.

Find a series solution of the differential equation, in ascending powers of $(x - 1)$ up to and including the term in $(x - 1)^3$.

Solution:

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x \qquad ①$$

Differentiating $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$, gives $\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1 \qquad ②$

Substituting initial values into ① gives $\left(\frac{d^2y}{dx^2}\right)_1 = 1$

Substituting $\left(\frac{dy}{dx}\right)_1 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_1 = 1$ into ② gives $\left(\frac{d^3y}{dx^3}\right) = -3$.

Using Taylor's expansion in the form with $x_0 = 1$

$$y = 0 + 2(x - 1) + \frac{(1)}{2!}(x - 1)^2 + \frac{(-3)}{3!}(x - 1)^3 + \ldots$$

$$= 2(x - 1) + \frac{1}{2}(x - 1)^2 - \frac{1}{2}(x - 1)^3 + \ldots$$

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Exercise F, Question 11

Question:

a Given that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$, show that $\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$.

b Using the result found in **a**, and given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, find the first three non-zero terms in the series expansion, in ascending powers of x, for $\tan x$.

Solution:

a You can write $\cos x = 1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)$; it is not necessary to have higher powers

$$\sec x = \frac{1}{\cos x} = \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \ldots\right)\right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\sec x = 1 + (-1)\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\} + \frac{(-1)(-2)}{2!}\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\}^2 + \ldots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{x^4}{4} + \text{higher powers of } x$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots$$

b $\tan x = \frac{\sin x}{\cos x} = \sin x \times \sec x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \ldots\right)$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \ldots$$

$$= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

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Exercise F, Question 12

Question:

By using the series expansions of e^x and $\cos x$, or otherwise, find the expansion of $e^x \cos 3x$ in ascending powers of x up to and including the term in x^3.

Solution:

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\cos 3x = 1 - \frac{(3x)^2}{2!} + \ldots$

$$e^x \cos 3x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{9x^2}{2} + \ldots\right)$$

$$= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \ldots\right\}$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

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Exercise F, Question 13

Question:

$\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$ with $y = 2$ at $x = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$.

a Use the Taylor series method to express y as a polynomial in x up to and including the term in x^3.

b Show that at $x = 0$, $\frac{d^4y}{dx^4} = 0$.

Solution:

a Differentiating $\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + y = 0$ ① with respect to x, gives

$$\frac{d^3y}{dx^3} + 2x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} = 0 \qquad ②$$

Substituting given data $x_0 = 0$, $y_0 = 2$ and $\left(\frac{dy}{dx}\right)_0 = 1$ into ① gives $\left(\frac{d^2y}{dx^2}\right)_0 = -2$

Substituting $x_0 = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ into ② gives $\left(\frac{d^3y}{dx^3}\right)_0 = -1$

So using Taylor series $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 2 + x - x^2 - \frac{x^3}{6} + \ldots$$

b Differentiating ② with respect to x gives

$$\frac{d^4y}{dx^4} + 2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x^2\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = 0 \qquad ③$$

Substituting $x = 0$, $\left(\frac{dy}{dx}\right)_0 = 1$, $\left(\frac{d^2y}{dx^2}\right)_0 = -2$ and $\left(\frac{d^3y}{dx^3}\right)_0 = -1$ into ③ gives,

at $x = 0$, $\frac{d^4y}{dx^4} + 2(1) + (-2) = 0$, so $\frac{d^4y}{dx^4} = 0$

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Exercise F, Question 14

Question:

Find the first three derivatives of $(1+x)^2 \ln(1+x)$. Hence, or otherwise, find the expansion of $(1+x)^2 \ln(1+x)$ in ascending powers of x up to and including the term in x^3.

Solution:

$f(x) = (1+x)^2 \ln(1+x)$.

$f'(x) = (1+x)^2 \frac{1}{1+x} + 2(1+x)\ln(1+x) = (1+x)\{1 + 2\ln(1+x)\}$

$f''(x) = (1+x)\left(\frac{2}{1+x}\right) + \{1 + 2\ln(1+x)\} = 3 + 2\ln(1+x)$

$f'''(x) = \left(\frac{2}{1+x}\right)$

$f(0) = 0,\ f'(0) = 1,\ f''(0) = 3,\ f'''(0) = 2$

Using Maclaurin's expansion

$$(1+x)^2 \ln(1+x) = 0 + (1)x + \frac{3}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$

$$= x + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \ldots$$

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Exercise F, Question 15

Question:

a Expand $\ln(1 + \sin x)$ in ascending powers of x up to and including the term in x^4.

b Hence find an approximation for $\int_0^{\frac{\pi}{6}} \ln(1 + \sin x)\,dx$ giving your answer to 3 decimal places.

Solution:

a $\ln(1 + \sin x) = \ln\left\{1 + \left(x - \frac{x^3}{3!} + \ldots\right)\right\}$

$$= \left(x - \frac{x^3}{3!} + \ldots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \ldots\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{3!} + \ldots\right)^3 - \frac{1}{4}\left(x - \frac{x^3}{3!} + \ldots\right)^4 + \ldots$$

$$= \left(x - \frac{x^3}{6} + \ldots\right) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right) + \frac{1}{3}(x^3 + \ldots) - \frac{1}{4}(x^4 + \ldots) \quad \text{no other terms necessary}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \ldots$$

b $\int_0^{\frac{\pi}{6}} \ln(1 + \sin x)dx \approx \int_0^{\frac{\pi}{6}} \left(x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}\right)dx$

$$\approx \left[\frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{60}\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72} - \frac{\pi^3}{1296} + \frac{\pi^4}{31104} - \frac{\pi^5}{466560} = 0.116 \text{ (3 d.p.)}$$

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Exercise F, Question 16

Question:

a Using the first two terms, $x + \frac{x^3}{3}$, in the expansion of $\tan x$, show that

$e^{\tan x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \ldots$

b Deduce the first four terms in the expansion of $e^{-\tan x}$, in ascending powers of x.

Solution:

a $f(x) = e^{\tan x} = e^{x + \frac{x^3}{3} + \ldots} = e^x \times e^{\frac{x^3}{3}}$ (As only terms up to x^3 are required, only first two terms of $\tan x$ are needed.)

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(1 + \frac{x^3}{3} + \ldots\right)$$ no other terms required.

$$= \left(1 + \frac{x^3}{3} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \ldots$$

b $e^{-\tan x} = e^{\tan(-x)}$, so replacing x by $-x$ in **a** gives

$$e^{-\tan x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \ldots$$

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Exercise F, Question 17

Question:

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0.$$

a Find an expression for $\frac{d^3y}{dx^3}$.

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

b find the series solution for y, in ascending powers of x, up to an including the term in x^3.

c Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

Solution:

a Differentiating the given differential equation with respect to x gives

$$y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

So $$\frac{d^3y}{dx^3} = -\frac{1}{y}\left\{\frac{dy}{dx}\left(3\frac{d^2y}{dx^2} + 1\right)\right\}$$

b Given that $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 1$ at $x = 0$,

$$\left(\frac{d^2y}{dx^2}\right)_0 + (1)^2 + (1) = 0, \text{ so } \left(\frac{d^2y}{dx^2}\right)_0 = -2,$$

And $$\left(\frac{d^3y}{dx^3}\right)_0 = -\frac{1}{(1)}\{(1)[3(-2) + 1]\}, \text{ so } \left(\frac{d^3y}{dx^3}\right)_0 = 5$$

So $$y = 1 + (1)x + \frac{(-2)}{2!}x^2 + \frac{5}{3!}x^3 + \ldots = 1 + x - x^2 + \frac{5x^3}{6} + \ldots$$

c The approximation is best for small values of x (close to 0): $x = 0.2$, therefore, would be acceptable, but not $x = 50$.

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Exercise F, Question 18

Question:

a Using the Maclaurin expansion, and differentiation, show that $\ln \cos x = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots$

b Using $\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1$, and the result in **a**, show that

$\ln(1 + \cos x) = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} + \ldots$

Solution:

a $f(x) = \ln \cos x$ $\quad f(0) = 0$

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$ $\quad f'(0) = 0$

$f''(x) = -\sec^2 x$ $\quad f''(0) = -1$

$f'''(x) = -2\sec^2 x \tan x$ $\quad f'''(0) = 0$

$f''''(x) = -2\sec^4 x - 4\sec^2 x \tan^2 x$ $\quad f''''(0) = -2$

Substituting into Maclaurin:

$$\ln \cos x = (-1)\frac{x^2}{2!} + (-2)\frac{x^4}{4!} + \ldots = -\frac{x^2}{2} - \frac{x^4}{12} - \ldots$$

b Using $1 + \cos x = 2\cos^2\left(\frac{x}{2}\right)$, $\ln(1 + \cos x) = \ln 2\cos^2\left(\frac{x}{2}\right) = \ln 2 + 2\ln \cos\left(\frac{x}{2}\right)$

so $\ln(1 + \cos x) = \ln 2 + 2\left\{-\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \ldots\right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \ldots$

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Exercise F, Question 19

Question:

a Show that $3^x = e^{x \ln 3}$.

b Hence find the first four terms in the series expansion of 3^x.

c Using your result in **b**, with a suitable value of x, find an approximation for $\sqrt{3}$, giving your answer to 3 significant figures.

Solution:

a Let $y = 3^x$, then $\ln y = \ln 3^x = x \ln 3 \Rightarrow y = e^{x \ln 3}$ so $3^x = e^{x \ln 3}$

b $3^x = e^{x \ln 3} = 1 + (x \ln 3) + \frac{(x \ln 3)^2}{2!} + \frac{(x \ln 3)^3}{3!} + \ldots$

$$= 1 + x \ln 3 + \frac{x^2 (\ln 3)^2}{2} + \frac{x^3 (\ln 3)^3}{6} + \ldots$$

c Put $x = \frac{1}{2}$: $\sqrt{3} \approx 1 + \frac{\ln 3}{2} + \frac{(\ln 3)^2}{8} + \frac{(\ln 3)^3}{48} = 1.73$ (3 s.f.)

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Exercise F, Question 20

Question:

Given that $f(x) = \operatorname{cosec} x$,

a show that

i $f''(x) = \operatorname{cosec} x(2\operatorname{cosec}^2 x - 1)$

ii $f'''(x) = -\operatorname{cosec} x \cot x(6\operatorname{cosec}^2 x - 1)$

b Find the Taylor expansion of $\operatorname{cosec} x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term $\left(x - \frac{\pi}{4}\right)^3$.

Solution:

a $f(x) = \operatorname{cosec} x$

$f'(x) = -\operatorname{cosec} x \cot x$

i
$$\begin{aligned} f''(x) &= -\operatorname{cosec} x(-\operatorname{cosec}^2 x) + \cot x(\operatorname{cosec} x \cot x) \\ &= \operatorname{cosec} x(\operatorname{cosec}^2 x + \cot^2 x) \\ &= \operatorname{cosec} x\{\operatorname{cosec}^2 x + (\operatorname{cosec}^2 x - 1)\} \\ &= \operatorname{cosec} x\{2\operatorname{cosec}^2 x - 1\} \end{aligned}$$

ii
$$\begin{aligned} f'''(x) &= \operatorname{cosec} x(-4\operatorname{cosec}^2 x \cot x) - \operatorname{cosec} x \cot x(2\operatorname{cosec}^2 x - 1) \\ &= -\operatorname{cosec} x \cot x(6\operatorname{cosec}^2 x - 1) \end{aligned}$$

b $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, $f'\left(\frac{\pi}{4}\right) = -\sqrt{2}$, $f''\left(\frac{\pi}{4}\right) = 3\sqrt{2}$, $f'''\left(\frac{\pi}{4}\right) = -11\sqrt{2}$.

Substituting all values into $y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x - x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \ldots$ with $x_0 = \frac{\pi}{4}$

$$\begin{aligned} \operatorname{cosec} x &= \sqrt{2} + (-\sqrt{2})\left(x - \frac{\pi}{4}\right) + \frac{(3\sqrt{2})}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{(-11\sqrt{2})}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots \\ &= \sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 - \frac{11\sqrt{2}}{6}\left(x - \frac{\pi}{4}\right)^3 + \ldots \end{aligned}$$