

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Series

Exercise A, Question 1

Question:

Write out each of the following as a sum of terms, and hence calculate the sum of the series.

$$\mathbf{a} \sum_{r=1}^{10} r$$

$$\mathbf{b} \sum_{p=3}^8 p^2$$

$$\mathbf{c} \sum_{r=1}^{10} r^3$$

$$\mathbf{d} \sum_{p=1}^{10} (2p^2 + 3)$$

$$\mathbf{e} \sum_{r=0}^5 (7r + 1)^2$$

$$\mathbf{f} \sum_{i=1}^4 2i(3 - 4i^2)$$

Solution:

$$\mathbf{a} 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$$

$$\mathbf{b} 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 199$$

$$\mathbf{c} 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 = 3025$$

{notice that this result is the square of the result for (a)}

$$\mathbf{d} 5 + 11 + 21 + 35 + 53 + 75 + 101 + 131 + 165 + 203 = 800$$

$$\mathbf{e} 1 + 64 + 225 + 484 + 841 + 1296 = 2911$$

$$\mathbf{f} -2 - 52 - 198 - 488 = -740$$

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Exercise A, Question 2

Question:

Write each of the following as a sum of terms, showing the first three terms and the last term.

a $\sum_{r=1}^n (7r - 1)$

b $\sum_{r=1}^n (2r^3 + 1)$

c $\sum_{j=1}^n (j - 4)(j + 4)$

d $\sum_{p=3}^k p(p + 3)$

Solution:

a $6 + 13 + 20 + \dots + (7n - 1)$

b $3 + 17 + 55 + \dots + (2n^3 + 1)$

c $-15 - 12 - 7 + \dots + (n - 4)(n + 4)$

d $18 + 28 + 40 + \dots + k(k + 3)$

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Exercise A, Question 3

Question:

In each part of this question write out, as a sum of terms, the two series defined by $\sum f(r)$; for example, in part **c**, write out the series $\sum_{r=1}^{10} r^2$ and $\sum_{r=1}^{10} r$. Hence, state whether the given statements relating their sums are true or not.

$$\mathbf{a} \quad \sum_{r=1}^n (3r+1) = \sum_{r=2}^{n+1} (3r-2)$$

$$\mathbf{b} \quad \sum_{r=1}^n 2r = \sum_{r=0}^n 2r$$

$$\mathbf{c} \quad \sum_{r=1}^{10} r^2 = \left(\sum_{r=1}^{10} r \right)^2$$

$$\mathbf{d} \quad \sum_{r=1}^4 r^3 = \left(\sum_{r=1}^4 r \right)^2$$

$$\mathbf{e} \quad \sum_{r=1}^n (3r^2+4) = 3 \sum_{r=1}^n r^2 + 4$$

Solution:

a The two series are exactly the same, $4 + 7 + 10 + \dots + (3n + 1)$, and so their sums are the same.

b The two series are exactly the same, $2 + 4 + 6 + \dots + 2n$, and so their sums are the same.

c The statement is not true.

$$\sum_{r=1}^{10} r^2 = 1^2 + 2^2 + 3^2 + \dots + 10^2 = 385 \text{ (using your calculator)}$$

$$\left(\sum_{r=1}^{10} r \right)^2 = (1 + 2 + 3 + \dots + 10)^2 = 55^2 = 3025.$$

[This one example is enough to prove $\sum_{r=1}^n r^2 = \left(\sum_{r=1}^n r \right)^2$ for all n is not true]

d This statement is true.

$$\sum_{r=1}^4 r^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$$

$$\left(\sum_{r=1}^4 r \right)^2 = (1 + 2 + 3 + 4)^2 = 10^2 = 100$$

[This does not prove that $\sum_{r=1}^n r^3 = \left(\sum_{r=1}^n r\right)^2$ for all n ; but it is true and this will be proved in Chapter 6]

e The statement is not true.

$$\begin{aligned}\sum_{r=1}^n (3r^2 + 4) &= \{3 \times 1^2 + 4\} + \{3 \times 2^2 + 4\} + \{3 \times 3^2 + 4\} + \dots + \{3n^2 + 4\} \\ &= 3\{1^2 + 2^2 + 3^2 + \dots + n^2\} + 4n \\ 3 \sum_{r=1}^n r^2 + 4 &= 3\{1^2 + 2^2 + 3^2 + \dots + n^2\} + 4\end{aligned}$$

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Exercise A, Question 4

Question:

Express these series using Σ notation.

a $3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

b $1 + 8 + 27 + 64 + 125 + 216 + 243 + 512$

c $11 + 21 + 35 + \dots + (2n^2 + 3)$

d $11 + 21 + 35 + \dots + (2n^2 - 4n + 5)$

e $3 \times 5 + 5 \times 7 + 7 \times 9 + \dots + (2r - 1)(2r + 1) + \dots$ to k terms.

Solution:

Answers are not unique (two examples are given, and any letter may be used for r)

a $\sum_{r=3}^{10} r, \sum_{r=1}^8 (r+2)$

b $\sum_{r=1}^8 r^3, \sum_{r=2}^9 (r-1)^3$

c $\sum_{r=2}^n (2r^2 + 3), \sum_{r=3}^{n+1} (2r^2 - 4r + 5)$

d $\sum_{r=3}^n (2r^2 - 4r + 5), \sum_{r=2}^{n-1} (2r^2 + 3)$

e $\sum_{r=2}^{k+1} (2r-1)(2r+1), \sum_{r=1}^k (2r+1)(2r+3)$

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Exercise B, Question 1

Question:

Use the result for $\sum_{r=1}^n r$ to calculate

a $\sum_{r=1}^{36} r$

b $\sum_{r=1}^{99} r$

c $\sum_{p=10}^{55} p$

d $\sum_{r=100}^{200} r$

e $\sum_{r=1}^k r + \sum_{r=k+1}^{80} r$, where $k < 80$.

Solution:

a $\frac{36 \times 37}{2} = 666$

b $\frac{99 \times 100}{2} = 4950$

c $\sum_{p=1}^{55} p - \sum_{p=1}^9 p = \frac{55 \times 56}{2} - \frac{9 \times 10}{2} = 1540 - 45 = 1495$

d $\sum_{r=1}^{200} r - \sum_{r=1}^{99} r = \frac{200 \times 201}{2} - \frac{99 \times 100}{2} = 20100 - 4950 = 15150$

e $\sum_{r=1}^k r + \sum_{r=k+1}^{80} r = \sum_{r=1}^{80} r = \frac{80 \times 81}{2} = 3240$

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Exercise B, Question 2

Question:

Given that $\sum_{r=1}^n r = 528$,

a show that $n^2 + n - 1056 = 0$

b find the value of n .

Solution:

a $\frac{n}{2}(n+1) = 528 \Rightarrow n(n+1) = 1056 \Rightarrow n^2 + n - 1056 = 0$

b Factorising: $(n-32)(n+33) = 0$ (or use “the formula”) $\Rightarrow n = 32$, as n cannot be negative.

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Exercise B, Question 3

Question:

a Find $\sum_{k=1}^{2n-1} k$.

b Hence show that $\sum_{k=n+1}^{2n-1} k = \frac{3n}{2}(n-1), n \geq 2$.

Solution:

a $\frac{(2n-1)\{(2n-1)+1\}}{2} = \frac{(2n-1)(2n)}{2} = n(2n-1)$

b

$$\begin{aligned} \sum_{k=1}^{2n-1} k - \sum_{k=1}^n k &= n(2n-1) - \frac{n}{2}(n+1) = \frac{n}{2}\{2(2n-1) - (n+1)\} = \frac{n}{2}(3n-3) \\ &= \frac{3n}{2}(n-1) \end{aligned}$$

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Exercise B, Question 4

Question:

Show that $\sum_{r=k-1}^{2k} r = \frac{(k+2)(3k-1)}{2}, k \geq 1$

Solution:

$$\begin{aligned}\sum_{r=1}^{2k} r - \sum_{r=1}^{k-2} r &= \frac{2k}{2}(2k+1) - \frac{(k-2)}{2}(k-1) = \frac{(4k^2+2k)-(k^2-3k+2)}{2} \\ &= \frac{3k^2+5k-2}{2} = \frac{(3k-1)(k+2)}{2}\end{aligned}$$

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Exercise B, Question 5

Question:

a Show that $\sum_{r=1}^{n^2} r - \sum_{r=1}^n r = \frac{n(n^3 - 1)}{2}$.

b Hence evaluate $\sum_{r=10}^{81} r$.

Solution:

a $\frac{n^2(n^2 + 1)}{2} - \frac{n(n + 1)}{2} = \frac{n}{2} \{n(n^2 + 1) - (n + 1)\} = \frac{n}{2}(n^3 - 1)$

b $\sum_{r=10}^{81} r = \sum_{r=1}^{9^2} r - \sum_{r=1}^9 r = \frac{9}{2}(9^3 - 1)$ [using part (a)] = 3276.

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Exercise C, Question 1

Question:

(In this exercise use the results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n 1$.)

Calculate the sum of the series:

$$\mathbf{a} \sum_{r=1}^{55} (3r - 1)$$

$$\mathbf{b} \sum_{r=1}^{90} (2 - 7r)$$

$$\mathbf{c} \sum_{r=1}^{46} (9 + 2r)$$

Solution:

$$\mathbf{a} \quad 3 \sum_{r=1}^{55} r - \sum_{r=1}^{55} 1 = 3 \times \frac{55 \times 56}{2} - 55 = 4565$$

$$\mathbf{b} \quad 2 \sum_{r=1}^{90} 1 - 7 \sum_{r=1}^{90} r = 2 \times 90 - 7 \times \frac{90 \times 91}{2} = -28485$$

$$\mathbf{c} \quad 9 \sum_{r=1}^{46} 1 + 2 \sum_{r=1}^{46} r = 9 \times 46 + 2 \times \frac{46 \times 47}{2} = 2576$$

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Exercise C, Question 2

Question:

Show that

$$\mathbf{a} \sum_{r=1}^n (3r+2) = \frac{n}{2}(3n+7)$$

$$\mathbf{b} \sum_{i=1}^{2n} (5i-4) = n(10n-3)$$

$$\mathbf{c} \sum_{r=1}^{n+2} (2r+3) = (n+2)(n+6)$$

d

$$\sum_{p=3}^n (4p+5) = (2n+11)(n-2)$$

Solution:

$$\mathbf{a} \quad 3 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1 = 3 \times \frac{n}{2}(n+1) + 2n = \frac{n}{2}(3n+3+4) = \frac{n}{2}(3n+7)$$

$$\mathbf{b} \quad 5 \sum_{i=1}^{2n} i - 4 \sum_{i=1}^{2n} 1 = 5 \times \frac{2n}{2}(2n+1) - 4(2n) = n(10n+5-8) = n(10n-3)$$

$$\mathbf{c} \quad 2 \sum_{r=1}^{n+2} r + 3 \sum_{r=1}^{n+2} 1 = 2 \times \frac{(n+2)}{2}(n+3) + 3(n+2) = (n+2)(n+3+3) = (n+2)(n+6)$$

d

$$\left\{ 4 \sum_{p=1}^n p + 5 \sum_{p=1}^n 1 \right\} - \sum_{p=1}^2 (4p+5) = \left\{ 4 \times \frac{n}{2}(n+1) + 5n \right\} - (9+13)$$

$$= 2n^2 + 7n - 22 = (2n+11)(n-2)$$

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Exercise C, Question 3

Question:

a Show that $\sum_{r=1}^k (4r - 5) = 2k^2 - 3k$.

b Find the smallest value of k for which $\sum_{r=1}^k (4r - 5) > 4850$.

Solution:

a $4 \sum_{r=1}^k r - 5 \sum_{r=1}^k 1 = 4 \times \frac{k}{2}(k+1) - 5k = 2k^2 - 3k$

b $2k^2 - 3k > 4850 \Rightarrow 2k^2 - 3k - 4850 > 0 \Rightarrow (2k + 97)(k - 50) > 0$,

so $k > 50$ [k is positive] $\Rightarrow k = 51$

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Exercise C, Question 4

Question:

Given that $u_r = ar + b$ and $\sum_{r=1}^n u_r = \frac{n}{2}(7n + 1)$, find the constants a and b .

Solution:

$$\sum_{r=1}^n (ar + b) = \frac{an}{2}(n + 1) + bn = \frac{an^2 + (a + 2b)n}{2}$$

Comparing with $\frac{7n^2 + n}{2} \Rightarrow a = 7$ and $a + 2b = 1$

So $a = 7, b = -3$

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Exercise C, Question 5

Question:

a Show that $\sum_{r=1}^{4n-1} (1+3r) = 24n^2 - 2n - 1$ $n \geq 1$.

b Hence calculate $\sum_{r=1}^{99} (1+3r)$.

Solution:

a $\sum_{r=1}^{4n-1} 1 + 3 \sum_{r=1}^{4n-1} r = (4n-1) + 3 \times \frac{(4n-1)(4n)}{2} = (4n-1)(1+6n) = 24n^2 - 2n - 1$

b Substituting $n = 25$ into above result gives 14949

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Exercise C, Question 6

Question:

Show that $\sum_{r=1}^{2k+1} (4 - 5r) = -(2k + 1)(5k + 1), k \geq 0$

Solution:

$$\begin{aligned} 4 \sum_{r=1}^{2k+1} 1 - 5 \sum_{r=1}^{2k+1} r &= 4(2k + 1) - 5 \frac{(2k + 1)(2k + 2)}{2} = (2k + 1)\{4 - 5(k + 1)\} \\ &= (2k + 1)(-1 - 5k) = -(2k + 1)(5k + 1) \end{aligned}$$

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Exercise D, Question 1

Question:

Verify that $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ is true for $n = 1, 2$ and 3 .

Solution:

$$\text{For } n = 1, \quad \sum_{r=1}^n r^2 = 1^2 = 1, \quad \frac{n}{6}(n+1)(2n+1) = \frac{1}{6}(1+1)(2+1) = 1$$

$$\text{For } n = 2, \quad \sum_{r=1}^n r^2 = 1^2 + 2^2 = 5, \quad \frac{n}{6}(n+1)(2n+1) = \frac{2}{6}(2+1)(4+1) = 5$$

$$\text{For } n = 3, \quad \sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 = 14, \quad \frac{n}{6}(n+1)(2n+1) = \frac{3}{6}(3+1)(6+1) = 14$$

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Exercise D, Question 2

Question:

a By writing out each series, evaluate $\sum_{r=1}^n r$ for $n = 1, 2, 3$ and 4 .

b By writing out each series, evaluate $\sum_{r=1}^n r^3$ for $n = 1, 2, 3$ and 4 .

c What do you notice about the corresponding results for each value of n ?

Solution:

$$\mathbf{a} \quad \sum_{r=1}^1 r = 1; \quad \sum_{r=1}^2 r = 1 + 2 = 3; \quad \sum_{r=1}^3 r = 1 + 2 + 3 = 6; \quad \sum_{r=1}^4 r = 1 + 2 + 3 + 4 = 10$$

$$\mathbf{b} \quad \sum_{r=1}^1 r^3 = 1; \quad \sum_{r=1}^2 r^3 = 1^3 + 2^3 = 9; \quad \sum_{r=1}^3 r^3 = 1^3 + 2^3 + 3^3 = 36; \quad \sum_{r=1}^4 r^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$$

c The results for (b) are the square of the results for (a)

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Exercise D, Question 3

Question:

Using the appropriate formula, evaluate

$$\mathbf{a} \sum_{r=1}^{100} r^2$$

$$\mathbf{b} \sum_{r=20}^{40} r^2$$

$$\mathbf{c} \sum_{r=1}^{30} r^3$$

$$\mathbf{d} \sum_{r=25}^{45} r^3$$

Solution:

$$\mathbf{a} \frac{100}{6} \times 101 \times 201 = 338350$$

$$\mathbf{b} \sum_{r=1}^{40} r^2 - \sum_{r=1}^{19} r^2 = \frac{40}{6} \times 41 \times 81 - \frac{19}{6} \times 20 \times 39 = 22140 - 2470 = 19670$$

$$\mathbf{c} \frac{30^2 \times 31^2}{4} = 216225$$

$$\mathbf{d} \sum_{r=1}^{45} r^3 - \sum_{r=1}^{24} r^3 = \frac{45^2 \times 46^2}{4} - \frac{24^2 \times 25^2}{4} = 1071225 - 90000 = 981225$$

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Exercise D, Question 4

Question:

Use the formula for $\sum_{r=1}^n r^2$ or $\sum_{r=1}^n r^3$ to find the sum of

a $1^2 + 2^2 + 3^2 + 4^2 + \dots + 52^2$

b $2^3 + 3^3 + 4^3 + \dots + 40^3$

c $26^2 + 27^2 + 28^2 + 29^2 + \dots + 100^2$

d $1^2 + 2^2 + 3^2 + \dots + (k+1)^2$

e $1^3 + 2^3 + 3^3 + \dots + (2n-1)^3$

Solution:

a $\sum_{r=1}^{52} r^2 = \frac{52}{6} \times 53 \times 105 = 48230$

b $\sum_{r=1}^{40} r^3 - 1 = \frac{40^2 \times 41^2}{4} - 1 = 672399$

c $\sum_{r=1}^{100} r^2 - \sum_{r=1}^{25} r^2 = \frac{100}{6} \times 101 \times 201 - \frac{25}{6} \times 26 \times 51 = 338350 - 5525 = 332825$

d $\sum_{r=1}^{k+1} r^2 = \frac{(k+1)}{6} (k+2)(2k+3)$

e $\sum_{r=1}^{2n-1} r^3 = \frac{(2n-1)^2 (2n)^2}{4} = n^2 (2n-1)^2$

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Exercise D, Question 5

Question:

For each of the following series write down, in terms of n , the sum, giving the result in its simplest form

$$\mathbf{a} \sum_{r=1}^{2n} r^2$$

$$\mathbf{b} \sum_{r=1}^{n^2-1} r^2$$

$$\mathbf{c} \sum_{i=1}^{2n-1} i^2$$

$$\mathbf{d} \sum_{r=1}^{n+1} r^3$$

$$\mathbf{e} \sum_{k=n+1}^{3n} k^3, n > 0.$$

Solution:

$$\mathbf{a} \frac{(2n)}{6}(2n+1)(4n+1) = \frac{n}{3}(2n+1)(4n+1)$$

$$\mathbf{b} \frac{(n^2-1)n^2(2n^2-1)}{6}$$

$$\mathbf{c} \frac{(2n-1)}{6}(2n)[2(2n-1)+1] = \frac{(2n-1)}{6}(2n)(4n-1) = \frac{n}{3}(2n-1)(4n-1)$$

$$\mathbf{d} \frac{(n+1)^2(n+2)^2}{4}$$

e

$$\begin{aligned} \sum_{r=1}^{3n} k^3 - \sum_{r=1}^n k^3 &= \frac{(3n)^2(3n+1)^2}{4} - \frac{n^2(n+1)^2}{4} = \frac{n^2}{4}\{9(3n+1)^2 - (n+1)^2\} \\ &= \frac{n^2}{4}\{3(3n+1) - (n+1)\}\{3(3n+1) + (n+1)\} \text{ [using } a^2 - b^2 = (a-b)(a+b)\text{]} \\ &= \frac{n^2}{4}\{(8n+2)(10n+4)\} \\ &= n^2(4n+1)(5n+2) \end{aligned}$$

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Exercise D, Question 6

Question:

Show that

$$\mathbf{a} \sum_{r=2}^n r^2 = \frac{1}{6}(n-1)(2n^2+5n+6)$$

$$\mathbf{b} \sum_{r=n}^{2n} r^2 = \frac{n}{6}(n+1)(14n+1)$$

Solution:

$$\mathbf{a} \frac{n}{6}(n+1)(2n+1) - 1 = \frac{2n^3+3n^2+n-6}{6} = \frac{(n-1)(2n^2+5n+6)}{6} \text{ [use factor theorem]}$$

b

$$\begin{aligned} \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 &= \frac{2n}{6}(2n+1)(4n+1) - \frac{(n-1)}{6}n(2n-1) \\ &= \frac{n}{6}\{2(2n+1)(4n+1) - (n-1)(2n-1)\} \\ &= \frac{n}{6}\{(16n^2+12n+2) - (2n^2-3n+1)\} = \frac{n}{6}(14n^2+15n+1) \\ &= \frac{n}{6}(14n+1)(n+1) \end{aligned}$$

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Exercise D, Question 7

Question:

a Show that $\sum_{k=n}^{2n} k^3 = \frac{3n^2(n+1)(5n+1)}{4}$

b Find $\sum_{k=30}^{60} k^3$.

Solution:

a

$$\begin{aligned} \sum_{k=1}^{2n} k^3 - \sum_{k=1}^{n-1} k^3 &= \frac{(2n)^2(2n+1)^2}{4} - \frac{(n-1)^2n^2}{4} \\ &= \frac{n^2}{4} \{4(2n+1)^2 - (n-1)^2\} \\ &= \frac{n^2}{4} \{[2(2n+1) + (n-1)]\{2(2n+1) - (n-1)\}\} \text{ "Difference of two squares"} \\ &= \frac{n^2}{4} (5n+1)(3n+3) = \frac{3n^2}{4} (5n+1)(n+1) \end{aligned}$$

b Substituting $n = 30$ into (a) gives 3 159 675

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Exercise D, Question 8

Question:

a Show that $\sum_{r=1}^{2n} r^3 = n^2(2n+1)^2$.

b By writing out the series for $\sum_{r=1}^n (2r)^3$, show that $\sum_{r=1}^n (2r)^3 = 8 \sum_{r=1}^n r^3$.

c Show that $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$ can be written as $\sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (2r)^3$.

d Hence show that the sum of the cubes of the first n odd natural numbers, $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3$, is $n^2(2n^2-1)$.

Solution:

a $\sum_{r=1}^{2n} r^3 = \frac{(2n)^2(2n+1)^2}{4} = n^2(2n+1)^2$.

b $\sum_{r=1}^n (2r)^3 = 2^3 + 4^3 + 6^3 + \dots + (2n)^3 = 2^3\{1^3 + 2^3 + 3^3 + \dots + n^3\} = 8 \sum_{r=1}^n r^3$.

c

$$\begin{aligned} 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 &= \{1^3 + 2^3 + 3^3 + \dots + (2n-1)^3 + (2n)^3\} - \{2^3 + 4^3 + 6^3 + \dots + (2n)^3\} \\ &= \sum_{r=1}^{2n} r^3 - \sum_{r=1}^n (2r)^3. \end{aligned}$$

d Using the results in parts (b) and (c), $1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = \sum_{r=1}^{2n} r^3 - 8 \sum_{r=1}^n r^3$

$$\begin{aligned} &= n^2(2n+1)^2 - 8 \sum_{r=1}^n r^3 \text{ (using(a))} \\ &= n^2(2n+1)^2 - \frac{8n^2(n+1)^2}{4} \\ &= n^2[(2n+1)^2 - 2(n+1)^2] \\ &= n^2[(4n^2 + 4n + 1) - 2(n^2 + 2n + 1)] \\ &= n^2(2n^2 - 1) \end{aligned}$$

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Exercise E, Question 1

Question:

Use the formulae for $\sum_{r=1}^n r^3$, $\sum_{r=1}^n r^2$, $\sum_{r=1}^n r$ and $\sum_{r=1}^n 1$, where appropriate, to find

a $\sum_{m=1}^{30} (m^2 - 1)$

b $\sum_{r=1}^{40} r(r+4)$

c $\sum_{r=1}^{80} r(r^2+3)$

d $\sum_{r=11}^{35} (r^3 - 2)$.

Solution:

a $\sum_{m=1}^{30} m^2 - 30 = \frac{30 \times 31 \times 61}{6} - 30 = 9425$

b $\sum_{r=1}^{40} r^2 + 4 \sum_{r=1}^{40} r = \frac{40 \times 41 \times 81}{6} + 4 \times \frac{40 \times 41}{2} = 22140 + 3280 = 25420$

c $\sum_{r=1}^{80} r^3 + 3 \sum_{r=1}^{80} r = \frac{80^2 \times 81^2}{4} + 3 \times \frac{80 \times 81}{2} = 10497600 + 9720 = 10507320$

d $\sum_{r=1}^{35} (r^3 - 2) - \sum_{r=1}^{10} (r^3 - 2) = \sum_{r=1}^{35} r^3 - 2(35) - \left[\sum_{r=1}^{10} r^3 - 2(10) \right]$

$\sum_{r=1}^{35} r^3 - \sum_{r=1}^{10} r^3 - 2(35 - 10) = \frac{35^2 \times 36^2}{4} - \frac{10^2 \times 11^2}{4} - 50 = 396900 - 3025 - 50 = 393825.$

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Exercise E, Question 2

Question:

Use the formulae for $\sum_{r=1}^n r^3$, $\sum_{r=1}^n r^2$, and $\sum_{r=1}^n r$, where appropriate, to find

a $\sum_{r=1}^n (r^2 + 4r)$

b $\sum_{r=1}^n r(2r^2 - 1)$

c $\sum_{r=1}^{2n} r^2(1+r)$, giving your answer in its simplest form.

Solution:

a $\sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r = \frac{n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} = \frac{n(n+1)\{(2n+1)+12\}}{6} = \frac{n}{6}(n+1)(2n+13)$

b $2 \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{2n^2(n+1)^2}{4} - \frac{n(n+1)}{2} = \frac{n(n+1)\{n(n+1)-1\}}{2} = \frac{n}{2}(n+1)(n^2+n-1)$

c

$$\begin{aligned} \sum_{r=1}^{2n} r^2 + \sum_{r=1}^{2n} r^3 &= \frac{2n(2n+1)(4n+1)}{6} + \frac{(2n)^2(2n+1)^2}{4} = \frac{n(2n+1)\{(4n+1)+3n(2n+1)\}}{3} \\ &= \frac{n}{3}(2n+1)(6n^2+7n+1) = \frac{n}{3}(n+1)(2n+1)(6n+1) \end{aligned}$$

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Exercise E, Question 3

Question:

a Write out $\sum_{r=1}^n r(r+1)$ as a sum, showing at least the first three terms and the final term.

b Use the results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$ to calculate

$$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6 + \dots + 60 \times 61.$$

Solution:

a $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1)$

b Putting $n = 60$: $\sum_{r=1}^{60} r^2 + \sum_{r=1}^{60} r = \frac{60 \times 61 \times 121}{6} + \frac{60 \times 61}{2} = 73810 + 1830 = 75640$

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Exercise E, Question 4

Question:

a Show that $\sum_{r=1}^n (r+2)(r+5) = \frac{n}{3}(n^2 + 12n + 41)$.

b Hence calculate $\sum_{r=10}^{50} (r+2)(r+5)$.

Solution:

a

$$\begin{aligned} \sum_{r=1}^n (r^2 + 7r + 10) &= \sum_{r=1}^n r^2 + 7 \sum_{r=1}^n r + 10 \sum_{r=1}^n 1 \\ &= \frac{n}{6}(n+1)(2n+1) + 7 \frac{n}{2}(n+1) + 10n \\ &= \frac{n}{6} \{ (2n^2 + 3n + 1) + 21(n+1) + 60 \} \\ &= \frac{n}{6} (2n^2 + 24n + 82) = \frac{n}{3} (n^2 + 12n + 41) \end{aligned}$$

b Substituting $n = 50$ and $n = 9$ in the formula in (a), and subtracting, gives 51 660.

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Exercise E, Question 5

Question:

a Show that $\sum_{r=2}^n (r-1)r(r+1) = \frac{(n-1)n(n+1)(n+2)}{4}$.

b Hence find the sum of the series $13 \times 14 \times 15 + 14 \times 15 \times 16 + 15 \times 16 \times 17 + \dots + 44 \times 45 \times 46$.

Solution:

a

$$\begin{aligned} \sum_{r=2}^n (r^3 - r) &= \sum_{r=1}^n (r^3 - r) = \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{n^2(n+1)^2}{4} - \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{4}(n^2 + n - 2) \\ &= \frac{n}{4}(n+1)\{n^2 + n - 2\} \\ &= \frac{n}{4}(n+1)(n+2)(n-1) = \frac{(n-1)n(n+1)(n+2)}{4} \end{aligned}$$

b $\sum_{r=14}^{45} (r-1)r(r+1) = \sum_{r=2}^{45} (r-1)r(r+1) - \sum_{r=2}^{13} (r-1)r(r+1) = \frac{44 \times 45 \times 46 \times 47}{4} - \frac{12 \times 13 \times 14 \times 15}{4}$

$= 1\,062\,000$

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Exercise E, Question 6

Question:

Find the following sums, and check your results for the cases $n = 1, 2$ and 3 .

$$\mathbf{a} \sum_{r=1}^n (r^3 - 1)$$

$$\mathbf{b} \sum_{r=1}^n (2r - 1)^2$$

$$\mathbf{c} \sum_{r=1}^n r(r+1)^2$$

Solution:

$$\mathbf{a} \sum_{r=1}^n r^3 - \sum_{r=1}^n 1 = \frac{n^2(n+1)^2}{4} - n = \frac{n}{4} \{n(n+1)^2 - 4\} = \frac{n}{4} (n^3 + 2n^2 + n - 4)$$

$$\text{When } n = 1 : \sum_{r=1}^1 (r^3 - 1) = 0; \quad \frac{n}{4} (n^3 + 2n^2 + n - 4) = \frac{1 \times 0}{4} = 0$$

$$\text{When } n = 2 : \sum_{r=1}^2 (r^3 - 1) = 0 + 7 = 7; \quad \frac{n}{4} (n^3 + 2n^2 + n - 4) = \frac{2 \times 14}{4} = 7$$

$$\text{When } n = 3 : \sum_{r=1}^3 (r^3 - 1) = 0 + 7 + 26 = 33; \quad \frac{n}{4} (n^3 + 2n^2 + n - 4) = \frac{3 \times 44}{4} = 33$$

b

$$\begin{aligned} \sum_{r=1}^n (4r^2 - 4r + 1) &= 4 \sum_{r=1}^n r^2 - 4 \sum_{r=1}^n r + \sum_{r=1}^n 1 = \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\ &= \frac{n}{3} \{2(2n^2 + 3n + 1) - 6(n+1) + 3\} = \frac{n}{3} (4n^2 - 1) \end{aligned}$$

$$\text{When } n = 1 : \sum_{r=1}^1 (4r^2 - 4r + 1) = 1; \quad \frac{n}{3} (4n^2 - 1) = \frac{1 \times 3}{3} = 1$$

$$\text{When } n = 2 : \sum_{r=1}^2 (4r^2 - 4r + 1) = 1 + 9 = 10; \quad \frac{n}{3} (4n^2 - 1) = \frac{2 \times 15}{3} = 10$$

$$\text{When } n = 3 : \sum_{r=1}^3 (4r^2 - 4r + 1) = 1 + 9 + 25 = 35; \quad \frac{n}{3} (4n^2 - 1) = \frac{3 \times 35}{3} = 35$$

c

$$\begin{aligned}\sum_{r=1}^n (r^3 + 2r^2 + r) &= \sum_{r=1}^n r^3 + 2 \sum_{r=1}^n r^2 + \sum_{r=1}^n r = \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{12} \{3n(n+1) + 4(2n+1) + 6\} = \frac{n(n+1)}{12} \{3n^2 + 11n + 10\} \\ &= \frac{n}{12} (n+1)(n+2)(3n+5)\end{aligned}$$

When $n = 1$: $\sum_{r=1}^1 r(r+1)^2 = 1 \times 4 = 4$; $\frac{n}{12} (n+1)(n+2)(3n+5) = \frac{1 \times 2 \times 3 \times 8}{12} = 4$

When $n = 2$: $\sum_{r=1}^2 r(r+1)^2 = 4 + 2 \times 9 = 22$; $\frac{n}{12} (n+1)(n+2)(3n+5) = \frac{2 \times 3 \times 4 \times 11}{12} = 22$

When $n = 3$: $\sum_{r=1}^3 r(r+1)^2 = 22 + 3 \times 16 = 70$; $\frac{n}{12} (n+1)(n+2)(3n+5) = \frac{3 \times 4 \times 5 \times 14}{12} = 70$

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Exercise E, Question 7

Question:

a Show that $\sum_{r=1}^n r^2(r-1) = \frac{n}{12}(n^2-1)(3n+2)$.

b Deduce the sum of $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + 30 \times 31^2$.

Solution:

a

$$\begin{aligned} \sum_{r=1}^n r^3 - \sum_{r=1}^n r^2 &= \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{12} \{3n(n+1) - 2(2n+1)\} \\ &= \frac{n(n+1)}{12} (3n^2 - n - 2) \\ &= \frac{n(n+1)(n-1)(3n+2)}{12} = \frac{n(n^2-1)(3n+2)}{12} \end{aligned}$$

b As $\sum_{r=2}^{31} r^2(r-1) = \sum_{r=1}^{31} r^2(r-1)$, substitute $n = 31$ in (a); sum = 235 600

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Exercise E, Question 8

Question:

a Show that $\sum_{r=2}^n (r-1)(r+1) = \frac{n}{6}(2n+5)(n-1)$.

b Hence sum the series $1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + 35 \times 37$.

Solution:

a [$\sum_{r=2}^n (r^2 - 1) = \sum_{r=1}^n (r^2 - 1)$ as when $r = 1$ the term is zero]

$$\begin{aligned} \sum_{r=1}^n (r^2 - 1) &= \sum_{r=1}^n r^2 - \sum_{r=1}^n 1 = \frac{n}{6}(n+1)(2n+1) - n \\ &= \frac{n}{6}\{(2n^2 + 3n + 1) - 6\} \\ &= \frac{n}{6}(2n^2 + 3n - 5) \\ &= \frac{n}{6}(2n+5)(n-1) \end{aligned}$$

b $1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + 35 \times 37 = \sum_{r=1}^{36} (r-1)(r+1)$

Substituting $n = 36$ into result in (a) gives 16 170

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Series

Exercise E, Question 9

Question:

a Write out the series defined by $\sum_{r=7}^{12} r(2+3r)$, and hence find its sum.

b Show that $\sum_{r=n+1}^{2n} r(2+3r) = \frac{n}{2}(14n^2 + 15n + 3)$.

c By substituting the appropriate value of n into the formula in **b**, check that your answer for **a** is correct.

Solution:

a $7 \times 23 + 8 \times 26 + 9 \times 29 + 10 \times 32 + 11 \times 35 + 12 \times 38 = 1791$.

$$\mathbf{b} \quad \sum_{r=n+1}^{2n} (2r+3r^2) = \sum_{r=1}^{2n} (2r+3r^2) - \sum_{r=1}^n (2r+3r^2)$$

$$\begin{aligned} \sum_{r=1}^n (2r+3r^2) &= 2 \sum_{r=1}^n r + 3 \sum_{r=1}^n r^2 = n(n+1) + \frac{n}{2}(n+1)(2n+1) \\ &= \frac{n}{2}(n+1)\{2+(2n+1)\} \\ &= \frac{n}{2}(n+1)(2n+3) \end{aligned}$$

$$\Rightarrow \sum_{r=1}^{2n} (2r+3r^2) = n(2n+1)(4n+3)$$

$$\begin{aligned} \sum_{r=n+1}^{2n} (2r+3r^2) &= n(2n+1)(4n+3) - \frac{n}{2}(n+1)(2n+3) \\ &= \frac{n}{2}\{2(2n+1)(4n+3) - (n+1)(2n+3)\} \\ &= \frac{n}{2}\{(16n^2+20n+6) - (2n^2+5n+3)\} \\ &= \frac{n}{2}(14n^2+15n+3) \end{aligned}$$

c Substituting $n = 6$ gives 1791

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Series

Exercise E, Question 10

Question:

Find the sum of the series $1 \times 1 + 2 \times 3 + 3 \times 5 + \dots$ to n terms.

Solution:

Series can be written as $\sum_{r=1}^n r(2r-1)$

$$\begin{aligned}\sum_{r=1}^n r(2r-1) &= 2 \sum_{r=1}^n r^2 - \sum_{r=1}^n r = 2 \times \frac{n}{6}(n+1)(2n+1) - \frac{n}{2}(n+1) \\ &= \frac{n(n+1)\{2(2n+1)-3\}}{6} \\ &= \frac{n(n+1)(4n-1)}{6}\end{aligned}$$

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Exercise F, Question 1

Question:

a Write down the first three terms and the last term of the series given by $\sum_{r=1}^n (2r + 3^r)$.

b Find the sum of this series.

c Verify that your result in **b** is correct for the cases $n = 1, 2$ and 3 .

Solution:

a $(2 + 3) + (4 + 3^2) + (6 + 3^3) + \dots + (2n + 3^n)$ [= $5 + 13 + 33 + \dots + (2n + 3^n)$]

b $\sum_{r=1}^n (2r + 3^r) = 2 \sum_{r=1}^n r + \sum_{r=1}^n 3^r = n(n+1) + \frac{3}{2}(3^n - 1)$ [AP + GP]

c

$n = 1$: (b) gives $2 + 3 = 5$, agrees with (a)

$n = 2$: (b) gives $6 + 12 = 18$, agrees with (a)

$n = 3$: (b) gives $12 + 39 = 51$, agrees with (a)

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Exercise F, Question 2

Question:

Find

$$\mathbf{a} \sum_{r=1}^{50} (7r+5)$$

$$\mathbf{b} \sum_{r=1}^{40} (2r^2 - 1)$$

$$\mathbf{c} \sum_{r=33}^{75} r^3.$$

Solution:

$$\mathbf{a} \quad 7 \sum_{r=1}^{50} r + 5 \sum_{r=1}^{50} 1 = \frac{7 \times 50 \times 51}{2} + 5(50) = 9175$$

$$\mathbf{b} \quad 2 \sum_{r=1}^{40} r^2 - \sum_{r=1}^{40} 1 = \frac{40(41)(81)}{3} - 40 = 44\,240$$

$$\mathbf{c} \quad \sum_{r=1}^{75} r^3 - \sum_{r=1}^{32} r^3 = \frac{75^2 \times 76^2}{4} - \frac{32^2 \times 33^2}{4} = 7\,843\,716$$

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Exercise F, Question 3

Question:

Given that $\sum_{r=1}^n U_r = n^2 + 4n$,

a find $\sum_{r=1}^{n-1} U_r$.

b Deduce an expression for U_n .

c Find $\sum_{r=n}^{2n} U_r$.

Solution:

a Replacing n with $(n - 1)$ gives $(n - 1)^2 + 4(n - 1) = n^2 + 2n - 3$

b $U_n = \sum_{r=1}^n U_r - \sum_{r=1}^{n-1} U_r = n^2 + 4n - (n^2 + 2n - 3) = 2n + 3$

c $\sum_{r=1}^{2n} U_r - \sum_{r=1}^{n-1} U_r = (4n^2 + 8n) - (n^2 + 2n - 3) = 3n^2 + 6n + 3 = 3(n + 1)^2$

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Exercise F, Question 4

Question:

Evaluate $\sum_{r=1}^{30} r(3r-1)$

Solution:

$$3 \sum_{r=1}^{30} r^2 - \sum_{r=1}^{30} r = \frac{3 \times 30 \times 31 \times 61}{6} - \frac{30 \times 31}{2} = 28365 - 465 = 27900$$

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Exercise F, Question 5

Question:

Find $\sum_{r=1}^n r^2(r-3)$.

Solution:

$$\begin{aligned}\sum_{r=1}^n r^3 - 3 \sum_{r=1}^n r^2 &= \frac{n^2}{4}(n+1)^2 - \frac{n}{2}(n+1)(2n+1) \\ &= \frac{n}{4}(n+1)\{n(n+1) - 2(2n+1)\} \\ &= \frac{n}{4}(n+1)(n^2 - 3n - 2)\end{aligned}$$

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Exercise F, Question 6

Question:

Show that $\sum_{r=1}^{2n} (2r-1)^2 = \frac{2n}{3}(16n^2-1)$.

Solution:

$$\begin{aligned} 4 \sum_{r=1}^{2n} r^2 - 4 \sum_{r=1}^{2n} r + \sum_{r=1}^{2n} 1 &= \frac{4}{3}n(2n+1)(4n+1) - 4n(2n+1) + 2n \\ &= \frac{n}{3}\{4(2n+1)(4n+1) - 12(2n+1) + 6\} \\ &= \frac{n}{3}\{32n^2 + 24n + 4 - 12(2n+1) + 6\} \\ &= \frac{n}{3}(32n^2 - 2) \\ &= \frac{2n}{3}(16n^2 - 1) \end{aligned}$$

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Exercise F, Question 7

Question:

a Show that $\sum_{r=1}^n r(r+2) = \frac{n}{6}(n+1)(2n+7)$.

b Using this result, or otherwise, find in terms of n , the sum of $3\log 2 + 4\log 2^2 + 5\log 2^3 + \dots + (n+2)\log 2^n$.

Solution:

a

$$\begin{aligned} \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r &= \frac{n}{6}(n+1)(2n+1) + 2 \frac{n}{2}(n+1) \\ &= \frac{n}{6}(n+1)\{(2n+1) + 6\} \\ &= \frac{n}{6}(n+1)(2n+7) \end{aligned}$$

b

$$\begin{aligned} \text{The series is : } \sum_{r=1}^n (r+2)\log 2^r &= \sum_{r=1}^n r(r+2)\log 2 \quad \text{as } \log 2^r = r \cdot \log 2 \\ &= \log 2 \sum_{r=1}^n r(r+2) \quad \text{as } \log 2 \text{ is a constant} \\ &= \frac{n}{6}(n+1)(2n+7) \log 2 \end{aligned}$$

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Exercise F, Question 8

Question:

Show that $\sum_{r=n}^{2n} r^2 = \frac{n}{6}(n+1)(an+b)$, where a and b are constants to be found.

Solution:

$$\begin{aligned}\sum_{r=n}^{2n} r^2 &= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 = \frac{(2n)(2n+1)(4n+1)}{6} - \frac{(n-1)n(2n-1)}{6} \\ &= \frac{n}{6}\{2(8n^2+6n+1) - (2n^2-3n+1)\} \\ &= \frac{n}{6}(14n^2+15n+1) \\ &= \frac{n}{6}(n+1)(14n+1) \quad a=14, \quad b=1\end{aligned}$$

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Exercise F, Question 9

Question:

a Show that $\sum_{r=1}^n (r^2 - r - 1) = \frac{n}{3}(n-2)(n+2)$.

b Hence calculate $\sum_{r=10}^{40} (r^2 - r - 1)$.

Solution:

a

$$\begin{aligned} \sum_{r=1}^n r^2 - \sum_{r=1}^n r - \sum_{r=1}^n 1 &= \frac{n}{6}(n+1)(2n+1) - \frac{n}{2}(n+1) - n \\ &= \frac{n}{6}\{(n+1)(2n+1) - 3(n+1) - 6\} \\ &= \frac{n}{6}(2n^2 - 8) \\ &= \frac{n}{3}(n^2 - 4) \\ &= \frac{n}{3}(n-2)(n+2) \end{aligned}$$

b $\sum_{r=10}^{40} (r^2 - r - 1) = \sum_{r=1}^{40} (r^2 - r - 1) - \sum_{r=1}^9 (r^2 - r - 1)$

Substitute $n = 40$ and $n = 9$ into the result for part (a), and subtract.

The result is $21280 - 230 = 21049$

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Exercise F, Question 10

Question:

a Show that $\sum_{r=1}^n r(2r^2 + 1) = \frac{n}{2}(n+1)(n^2 + n + 1)$.

b Hence calculate $\sum_{r=26}^{58} r(2r^2 + 1)$.

Solution:

a

$$\begin{aligned} 2 \sum_{r=1}^n r^3 + \sum_{r=1}^n r &= \frac{n^2(n+1)^2}{2} + \frac{n}{2}(n+1) \\ &= \frac{n}{2}(n+1)\{n(n+1) + 1\} \\ &= \frac{n}{2}(n+1)(n^2 + n + 1) \end{aligned}$$

b Substitute $n = 58$ and $n = 25$ into the result for (a), and subtract. The result = 5654178.

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Exercise F, Question 11

Question:

Find

$$\mathbf{a} \sum_{r=1}^n r(3r-1)$$

$$\mathbf{b} \sum_{r=1}^n (r+2)(3r+5)$$

$$\mathbf{c} \sum_{r=1}^n (2r^3 - 2r + 1).$$

Solution:

$$\mathbf{a} \quad 3 \sum_{r=1}^n r^2 - \sum_{r=1}^n r = \frac{n(n+1)(2n+1)}{2} - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}(2n+1-1) = n^2(n+1)$$

b

$$\begin{aligned} 3 \sum_{r=1}^n r^2 + 11 \sum_{r=1}^n r + 10 \sum_{r=1}^n 1 &= \frac{n(n+1)(2n+1)}{2} + \frac{11n(n+1)}{2} + 10n \\ &= \frac{n}{2} \{ (2n^2 + 3n + 1) + 11(n+1) + 20 \} \\ &= \frac{n}{2} (2n^2 + 14n + 32) = n(n^2 + 7n + 16) \end{aligned}$$

c

$$\begin{aligned} 2 \sum_{r=1}^n r^3 - 2 \sum_{r=1}^n r + \sum_{r=1}^n 1 &= \frac{n^2(n+1)^2}{2} - n(n+1) + n \\ &= \frac{n}{2} \{ n(n+1)^2 - 2(n+1) + 2 \} \\ &= \frac{n}{2} \{ n(n+1)^2 - 2n \} = \frac{n^2}{2} (n^2 + 2n - 1) \end{aligned}$$

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Exercise F, Question 12

Question:

a Show that $\sum_{r=1}^n r(r+1) = \frac{n}{3}(n+1)(n+2)$.

b Hence calculate $\sum_{r=31}^{60} r(r+1)$.

Solution:

a

$$\begin{aligned}\sum_{r=1}^n r^2 + \sum_{r=1}^n r &= \frac{n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1) = \frac{n}{6}(n+1)\{2n+1+3\} \\ &= \frac{n}{3}(n+1)(n+2)\end{aligned}$$

b Substitute $n = 60$ and $n = 30$ into the result for part (a), and subtract. The result = 65720.

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Exercise F, Question 13

Question:

a Show that $\sum_{r=1}^n r(r+1)(r+2) = \frac{n}{4}(n+1)(n+2)(n+3)$.

b Hence evaluate $3 \times 4 \times 5 + 4 \times 5 \times 6 + 5 \times 6 \times 7 + \dots + 40 \times 41 \times 42$.

Solution:

a

$$\begin{aligned} \sum_{r=1}^n r^3 + 3 \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r &= \frac{n^2}{4}(n+1)^2 + \frac{n}{2}(n+1)(2n+1) + n(n+1) \\ &= \frac{n}{4}(n+1)\{(n(n+1) + 2(2n+1) + 4)\} \\ &= \frac{n}{4}(n+1)(n+2)(n+3) \end{aligned}$$

b $3 \times 4 \times 5 + 4 \times 5 \times 6 + 5 \times 6 \times 7 + \dots + 40 \times 41 \times 42 = \sum_{r=3}^{40} r(r+1)(r+2)$

$$\begin{aligned} \sum_{r=3}^{40} r(r+1)(r+2) &= \sum_{r=1}^{40} r(r+1)(r+2) - \sum_{r=1}^2 r(r+1)(r+2) \\ &= \frac{40 \times 41 \times 42 \times 43}{4} - \frac{2 \times 3 \times 4 \times 5}{4} = 740430 \end{aligned}$$

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Series

Exercise F, Question 14

Question:

a Show that $\sum_{r=1}^n r\{2(n-r)+1\} = \frac{n}{6}(n+1)(2n+1)$.

b Hence sum the series $(2n-1) + 2(2n-3) + 3(2n-5) + \dots + n$

Solution:

a Series can be written as $(2n+1) \sum_{r=1}^n r - 2 \sum_{r=1}^n r^2$ as n is a constant.

$$= (2n+1) \frac{n}{2}(n+1) - \frac{n}{3}(n+1)(2n+1)$$

$$= \frac{n}{6}(n+1)(2n+1)$$

b $\sum_{r=1}^n r[2(n-r)+1] = (2n-1) + 2[(2n-4)+1] + 3[(2n-6)+1] + \dots + n[2(n-n)+1]$

$$= (2n-1) + 2(2n+3) + 3(2n+5) + \dots + n, \text{ the series in part (b).}$$

The sum, therefore, is $\frac{n}{6}(n+1)(2n+1)$

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Exercise F, Question 15

Question:

a Show that when n is even,

$$1^3 - 2^3 + 3^3 - \dots - n^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 - 16 \left(1^3 + 2^3 + 3^3 + \dots + \left(\frac{n}{2}\right)^3 \right)$$

$$= \sum_{r=1}^n r^3 - 16 \sum_{r=1}^{\frac{n}{2}} r^3.$$

b Hence show that, for n even, $1^3 - 2^3 + 3^3 - \dots - n^3 = -\frac{n^2}{4}(2n+3)$

c Deduce the sum of $1^3 - 2^3 + 3^3 - \dots - 40^3$.

Solution:

a

$$1^3 - 2^3 + 3^3 - \dots - n^3 = (1^3 + 2^3 + 3^3 + \dots + n^3) - 2(2^3 + 4^3 + 6^3 + \dots + n^3)$$

$$= (1^3 + 2^3 + 3^3 + \dots + n^3) - 2 \left\{ 2^3(1^3 + 2^3 + 3^3 + \dots + \left(\frac{n}{2}\right)^3) \right\} \text{ as } n \text{ is even}$$

$$= (1^3 + 2^3 + 3^3 + \dots + n^3) - 16 \left\{ 1^3 + 2^3 + 3^3 + \dots + \left(\frac{n}{2}\right)^3 \right\}$$

$$= \sum_{r=1}^n r^3 - 16 \sum_{r=1}^{\frac{n}{2}} r^3 \text{ [As } n \text{ is even, } \frac{n}{2} \text{ is an integer]}$$

b

$$\sum_{r=1}^n r^3 - 16 \sum_{r=1}^{\frac{n}{2}} r^3 = \frac{n^2}{4}(n+1)^2 - 16 \frac{\left(\frac{n}{2}\right)^2 \left(\frac{n}{2}+1\right)^2}{4}$$

$$= \frac{n^2}{4}(n+1)^2 - 4 \frac{n^2}{4} \frac{(n+2)^2}{4}$$

$$= \frac{n^2}{4}(n+1)^2 - \frac{n^2}{4}(n+2)^2$$

$$= \frac{n^2}{4} \{ (n+1)^2 - (n+2)^2 \}$$

$$= \frac{n^2}{4} (-2n-3) = -\frac{n^2}{4}(2n+3)$$

c Substituting $n = 40$, gives -33200 .